Dynamics of a Single Particle

Review

- Newton's Second Law
- Momentum
  - Linear momentum
  - Angular momentum
- Work
- Energy
  - Kinetic Energy
  - Potential Energy

Basics

Newton's Second Law

- In an **inertial frame *A*** (non accelerating)
- **F** is the force acting on the particle, *P*

$$\mathbf{F} = m \frac{{}^{A}d\, {}^{A}\mathbf{v}^{P}}{dt}$$

Linear momentum **in *A***

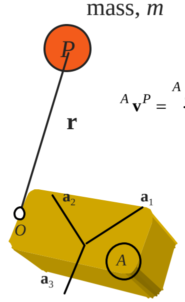
$$\mathbf{p} = m\ {}^{A}\mathbf{v}^{P}$$

$$\mathbf{F} = \frac{{}^{A}d\,\mathbf{p}}{dt}$$

Impulse

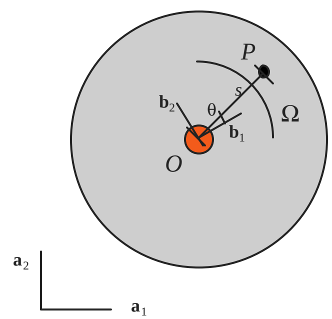
$$\int_{t_0}^{t} \mathbf{F}\, dt = \mathbf{p}(t) - \mathbf{p}(t_0)$$

mass, *m*



$${}^{A}\mathbf{v}^{P} = \frac{{}^{A}d\mathbf{r}}{dt}$$

Example 1



The bug is moving radially outward with a uniform speed (the rate of change of *s* is constant). What are the forces at its feet?

Choose $\theta = 0$.

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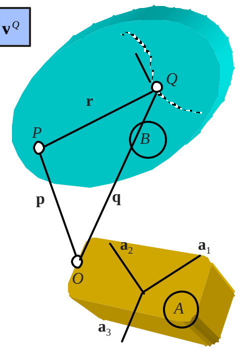
Acceleration of P and Q

$${}^A\mathbf{a}^Q = {}^A\mathbf{a}^P + {}^B\mathbf{a}^Q + {}^A\alpha^B \times \mathbf{r} + {}^A w^B \times \left({}^A w^B \times \mathbf{r}\right) + 2\,{}^A w^B \times {}^B\mathbf{v}^Q$$

tangential acceleration

centripetal (normal) acceleration

Coriolis acceleration



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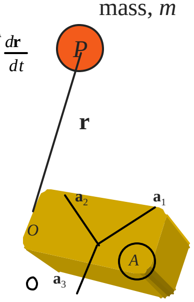
Simulation

General setting

- Know inertial properties
- $\mathbf{F}$ is known force acting on the particle, P
  - $\mathbf{F}(\mathbf{r}, \mathbf{v}, t)$ can be a known function

$${}^A\mathbf{v}^P = \frac{{}^A d\mathbf{r}}{dt}$$

mass, m



State space formulation

$$\mathbf{x}_{6\times1} = \begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} = \begin{bmatrix}\mathbf{r}\\ \mathbf{v}\end{bmatrix} \qquad \dot{\mathbf{x}}_{6\times1} = \begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\end{bmatrix} = \begin{bmatrix}\mathbf{x}_2\\ \dfrac{\mathbf{F}(\mathbf{x}_1, \mathbf{x}_2, t)}{m}\end{bmatrix}$$

Or

$$\mathbf{x} = \begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} = \begin{bmatrix}\mathbf{r}\\ \mathbf{p}\end{bmatrix} \qquad \dot{\mathbf{x}} = \begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\end{bmatrix} = \begin{bmatrix}\frac{1}{m}\mathbf{x}_2\\ \mathbf{F}(\mathbf{x}_1, \mathbf{x}_2, t)\end{bmatrix}$$

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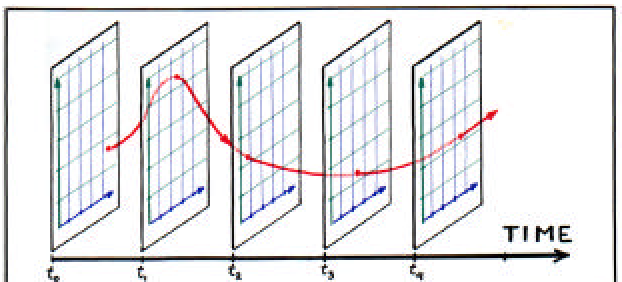
State Space and Time



1.1.10. Here the vertical plane represents the state space, and the horizontal axis represents the time of observation. The parameters observed at a given time are plotted in the vertical plane passing through the appropriate point on the time axis.

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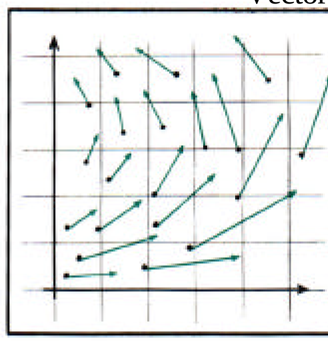
Vector Field

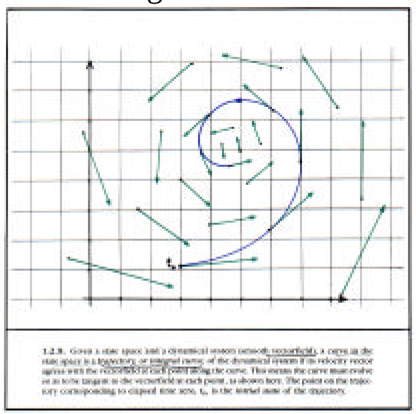


$$\dot{\mathbf{x}} = f(\mathbf{x})$$

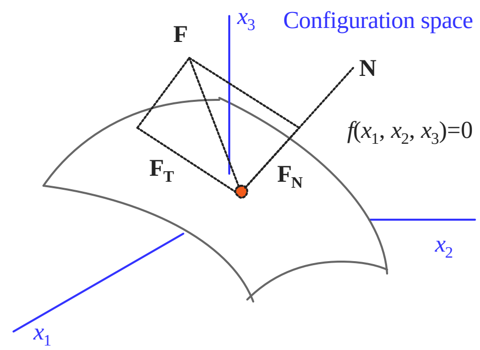
1.2.4. A *vectorfield* is a field of bound vectors, one defined at (and bound to) each and every point of the state space. Here only a few of the vectors are drawn, to suggest the full field.

- Vector field
- Trajectories
- State space + trajectories in state space = phase portrait

Integral Curve



Holonomic Constraint



$$\mathbf{x}_{4\times1} = \begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} = \begin{bmatrix}\mathbf{r}\\ \mathbf{v}\end{bmatrix}$$

$$\dot{\mathbf{x}}_{4\times1} = \begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\end{bmatrix} = \begin{bmatrix}\mathbf{x}_2\\ [?]\end{bmatrix}$$

Angular Momentum

Definition must include two pieces of information

- Frame, A (need not be inertial)
- Origin, O

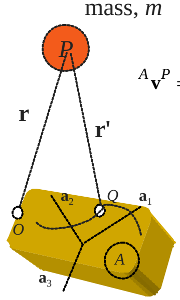
$$ {}^A\mathbf{H}_O^P = \mathbf{r}\times\left(m\frac{{}^A d\,\mathbf{r}}{dt}\right)$$

*Dependent on **the origin !!***

$$ {}^A\mathbf{H}_Q^P = \mathbf{r}'\times\, m\,{}^A\mathbf{v}^P\,; \quad {}^A\mathbf{v}^P = \frac{{}^A d\,\mathbf{r}}{dt}$$

Q need **not** be fixed in A

mass, m



$$ {}^A\mathbf{v}^P = \frac{{}^A d\mathbf{r}}{dt}$$

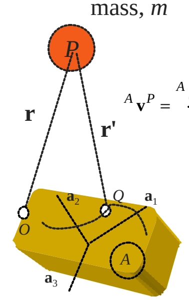
Newton’s Second Law

- Frame A is <u>**inertial**</u> (non accelerating)
- Origin O is **fixed** in A

$$\frac{{}^A d}{dt}\left({}^A\mathbf{H}_O^P\right) = \frac{{}^A d}{dt}\left(\mathbf{r}\times\, m\,{}^A\mathbf{v}^P\right)$$
$$= {}^A\mathbf{v}^P\times m\,{}^A\mathbf{v}^P + \mathbf{r}\times\, m\,{}^A\mathbf{a}^P$$
$$= \mathbf{r}\times\mathbf{F}$$

The rate of change of the angular momentum of P in an inertial frame A, relative to a fixed point O in A, is equal to the moment of the force about that fixed point.

mass, m



$$ {}^A\mathbf{v}^P = \frac{{}^A d\mathbf{r}}{dt}$$

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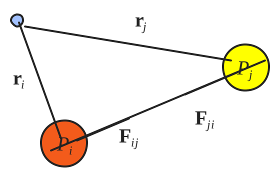
Newton's Laws: Continued

Newton's 3rd Law

$$\mathbf{F}_{ij} + \mathbf{F}_{ji} = 0$$

$$\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji} = 0$$



Principle of superposition

Two forces **F** and **G** acting simultaneously on a particle are equivalent to a single force equal to the vector sum **F**+**G**

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Conservation Laws

Linear momentum

Recall

$$\mathbf{p} = m\,{}^{A}\mathbf{v}^{P}$$

$$\mathbf{F} = \frac{{}^{A}d\,\mathbf{p}}{dt}$$

If the vector resultant of forces acting on the particle is equal to zero, the linear momentum in an inertial frame is a constant

Angular momentum

Recall

$${}^{A}\mathbf{H}_{O}^{P} = \mathbf{r} \times \left(m \frac{{}^{A}d\,\mathbf{r}}{dt} \right)$$

$$\frac{{}^{A}d}{dt}\left({}^{A}\mathbf{H}_{O}^{P}\right) = \mathbf{r} \times \mathbf{F}$$

If the vector resultant of the moments of forces acting on the particle about a point fixed in an inertial frame is equal to zero, the angular momentum relative to that fixed point is a constant in the inertial frame.

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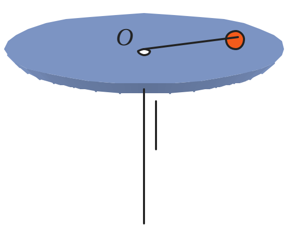
Example 2



A particle of mass m is being whirled with a string around a vertical axis passing through O. The distance from the center is R, and the speed is V. There is no friction between the particle and the table. The string is pulled gently so the particle moves toward the center until the distance from the center is r. What is the speed now?

$$mrv = mRV$$

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Work

Work done by the force **F** on the particle P over the path from Q to R is given by:

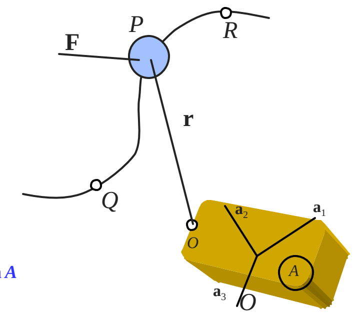
$$W_{QR} = \int_{Q}^{R} \mathbf{F} \cdot d\mathbf{r}$$

$$dW = \mathbf{F} \cdot d\mathbf{r} = m\ddot{\mathbf{r}} \cdot d\mathbf{r} = \tfrac{1}{2} m\, d\left(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right)$$

The work done by **F** is equal to the change in the *kinetic energy* of the particle

$$W_{QR} = \int_{Q}^{R} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} m\left(v_R^2 - v_Q^2\right)$$

Note A is inertial and O is a point fixed in A



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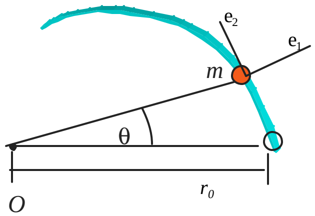
Example 3



A particle of mass m slides along a horizontal frictionless track which is shaped like a logarithmic spiral

$$r = r_0 \exp(-a\theta)$$

If the initial speed is v_0 when θ=0, find the speed of the particle and the magnitude of the track force acting on the particle.

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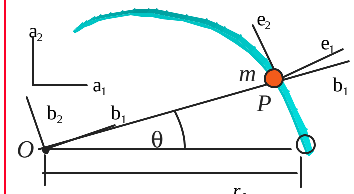
Example 3



A particle of mass m slides along a horizontal frictionless track which is shaped like a logarithmic spiral

$$r = r_0 \exp(-a\theta)$$

If the initial speed is v_0 when θ=0, find the speed of the particle and the magnitude of the track force acting on the particle.

$$\mathbf{r} = r\,\mathbf{b}_1$$

$${}^A\mathbf{v}^P = r\dot{q}\,(-a\mathbf{b}_1 + \mathbf{b}_2) = v\,\mathbf{e}_2$$

$$v = r\dot{q}\sqrt{1+a^2}$$

$$\mathbf{e}_1 = \frac{\mathbf{b}_1 + a\mathbf{b}_2}{\sqrt{1+a^2}} \quad \text{(unit normal)}$$

$$\mathbf{e}_2 = \frac{-a\mathbf{b}_1 + \mathbf{b}_2}{\sqrt{1+a^2}}$$

Track force = $N\,\mathbf{e}_1$

No force in the $\mathbf{e}_2$ direction

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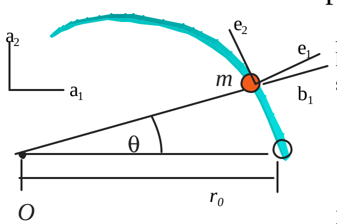
Example 3



A particle of mass m slides along a horizontal frictionless track which is shaped like a logarithmic spiral

$$r = r_0 \exp(-aq)$$

If the initial speed is v_0 when θ=0, find the speed of the particle and the magnitude of the track force acting on the particle.

$$\mathbf{e}_1 = \frac{\mathbf{b}_1 + a\mathbf{b}_2}{\sqrt{1+a^2}}$$

$$\mathbf{e}_2 = \frac{-a\mathbf{b}_1 + \mathbf{b}_2}{\sqrt{1+a^2}}$$

$$\frac{{}^A d}{dt}\left(m\,{}^A\mathbf{v}^P\right) = N\mathbf{e}_1 \quad \Rightarrow \quad \frac{{}^A d}{dt}\left(mv\mathbf{e}_2\right) = N\mathbf{e}_1$$

$$v = const. \quad (Why?)$$

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Conservative Force Field

F is conservative

- **F** is a function only of the position of the particle and the work done by the force **F** on the particle P to get it from Q to R is independent of the path
- **F** is a function only of the position of the particle and the work done by the force **F** on the particle P is zero along *any* closed path
- There exists a scalar function ϕ such that

$$dW = \mathbf{F} \cdot d\mathbf{r} = -d\phi$$

These are three equivalent definitions.

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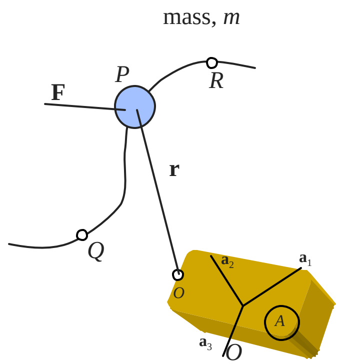
Conservation of Energy

F is conservative

- There exists a scalar function ϕ such that
  $$dW = \mathbf{F} \cdot d\mathbf{r} = -d\phi$$
- Work done by **F**
  $$W_{QR} = \int_Q^R \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} m\left(v_R^2 - v_Q^2\right)$$
  $$= \phi(Q) - \phi(R)$$
- Total energy is constant
  $$\frac{1}{2} m\left(v_R^2\right) + \phi(R) = \frac{1}{2} m\left(v_Q^2\right) + \phi(Q)$$



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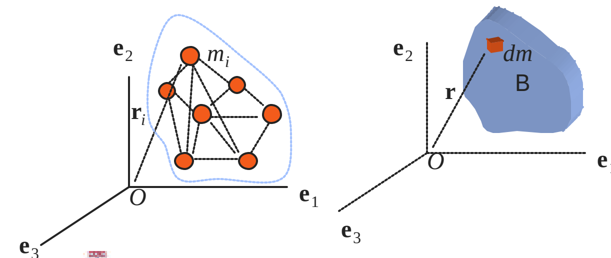
Where this is going?

Particle -> System of Particles

System of Particles -> Rigid body



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Rigid Body Dynamics

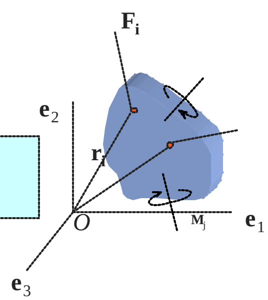
Newton-Euler Equations of Motion

For each rigid body

- Newton's second law (translation)
  $\sum \mathbf{F}_i = m\mathbf{a}$
- Euler's equations of motion (rotation)
  $$\sum \mathbf{r}_i \times \mathbf{F}_i + \sum \mathbf{M}_j = \mathbf{I}_O a$$

Inertia Matrix with respect to O

Angular Acceleration



Note: **I** depends on the reference point

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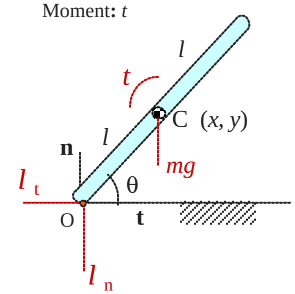
Example of Rigid Body Motion in a Plane

Forces: $-mg\,\mathbf{n}$ acting on O,
$l_n\,\mathbf{n}$ acting on C,
$l_t\,\mathbf{t}$ acting on C

$$\sum \mathbf{F}_i = m\mathbf{a}$$
$$\sum \mathbf{r}_i \times \mathbf{F}_i + \sum \mathbf{M}_j = \mathbf{I}_C a$$

Moment: t



3 DOF: $x \;\; y \;\; \theta$

2 translations

$$l_{\mathbf{t}}\,\mathbf{t} + l_{\mathbf{n}}\,\mathbf{n} - mg\,\mathbf{n} = m(\ddot{x}\,\mathbf{t} + \ddot{y}\,\mathbf{n})$$

1 rotation

$$l \sin q \; l_{\mathbf{t}} - l \cos q \; l_{\mathbf{t}} + t = I_C \ddot{q}$$

$$\begin{cases} m\ddot{x} = l_{\mathbf{t}} \\ m\ddot{y} = l_{\mathbf{n}} - mg \\ I_C \ddot{q} = l \sin q \; l_{\mathbf{t}} - l \cos q \; l_{\mathbf{n}} + t \end{cases}$$