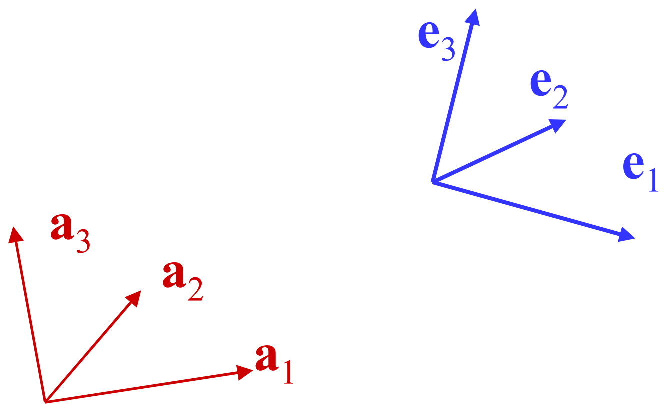
Kinematics

Scalar and Vector Functions

$\phi(q_1, q_2,\ldots,q_n)$ is a scalar function of n variables

$\phi(q_1, q_2,\ldots,q_n)$ is independent of reference frames – scalar invariant



NB: No assumptions of orthogonality!

$\mathbf{v}(q_1, q_2,\ldots,q_n)$ is a vector function $\mathbf{v}$ of n variables.

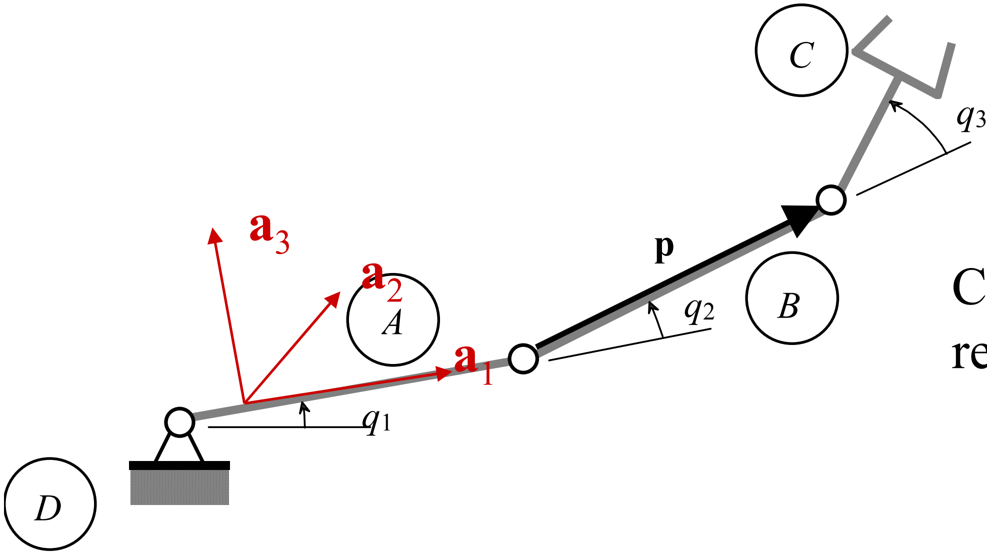
In any reference frame $\{A\}$, we can find three independent vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ that are basis vectors.

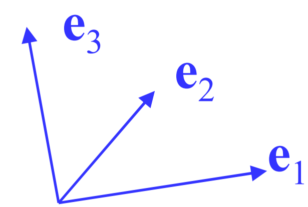
The vector function $\mathbf{v}(q_1, q_2,\ldots,q_n)$ can be expressed as a linear combination of the three vectors:

$$\mathbf{v}(q_1, q_2,\ldots,q_n) = v_1(q_1, q_2,\ldots,q_n)\,\mathbf{a}_1 + v_2(q_1, q_2,\ldots,q_n)\,\mathbf{a}_2 + v_3(q_1, q_2,\ldots,q_n)\,\mathbf{a}_3$$

The three coefficients are the three scalar functions v_1, v_2, and v_3. They are called *components* and these three functions are unique once the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are specified.

Reference Frames





Components of vectors depend on the reference triad

$$\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3\,,$$

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3\,,$$

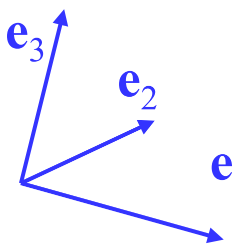
$$^{E}[\mathbf{p}] = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \qquad {}^{E}[\mathbf{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

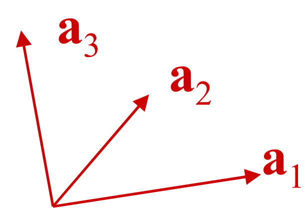
A robotic arm is a system of rigid bodies (reference frames) *A*, *B*, and *C*. *D* is the inertial or the laboratory reference frame that is considered fixed.

$$^{E}[\mathbf{p}] \neq {}^{A}[\mathbf{p}]$$

Standard Reference Triad

Three vectors rigidly attached to a reference frame satisfying the equations below constitute a standard reference triad or simply a reference triad.





$$\mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{a}_3,$$

$$\mathbf{a}_2 \times \mathbf{a}_3 = \mathbf{a}_1,$$

$$\mathbf{a}_3 \times \mathbf{a}_1 = \mathbf{a}_2$$

- Projection rule

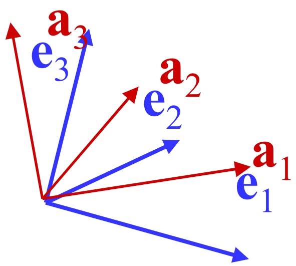
$$\mathbf{u} \cdot \mathbf{e}_i = u_i$$

- Composition rule

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = \sum_{i=1}^{3} u_i \mathbf{e}_i$$

$$\mathbf{u} = \sum_{i=1}^{3} (\mathbf{u} \cdot \mathbf{e}_i) \mathbf{e}_i$$

Transformation of Vectors



Components in $\{A\}$ and in $\{E\}$ are related by:

$$^{E}[\mathbf{p}]_i = \sum_j (\mathbf{e}_i \cdot \mathbf{a}_j)\, ^{A}[\mathbf{p}]_j$$

$$\sum_i {}^{E}[\mathbf{p}]_i \mathbf{e}_i = \sum_j {}^{A}[\mathbf{p}]_j \mathbf{a}_j$$

$$^{E}[\mathbf{p}] = \left[{}^{E}\mathbf{R}_A\right] {}^{A}[\mathbf{p}]$$

$$= \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{a}_1 & \mathbf{e}_1 \cdot \mathbf{a}_2 & \mathbf{e}_1 \cdot \mathbf{a}_3 \\ \mathbf{e}_2 \cdot \mathbf{a}_1 & \mathbf{e}_2 \cdot \mathbf{a}_2 & \mathbf{e}_2 \cdot \mathbf{a}_3 \\ \mathbf{e}_3 \cdot \mathbf{a}_1 & \mathbf{e}_3 \cdot \mathbf{a}_2 & \mathbf{e}_3 \cdot \mathbf{a}_3 \end{bmatrix} {}^{A}[\mathbf{p}]$$

Rotation matrix that transforms components in $\{A\}$ into components in $\{E\}$

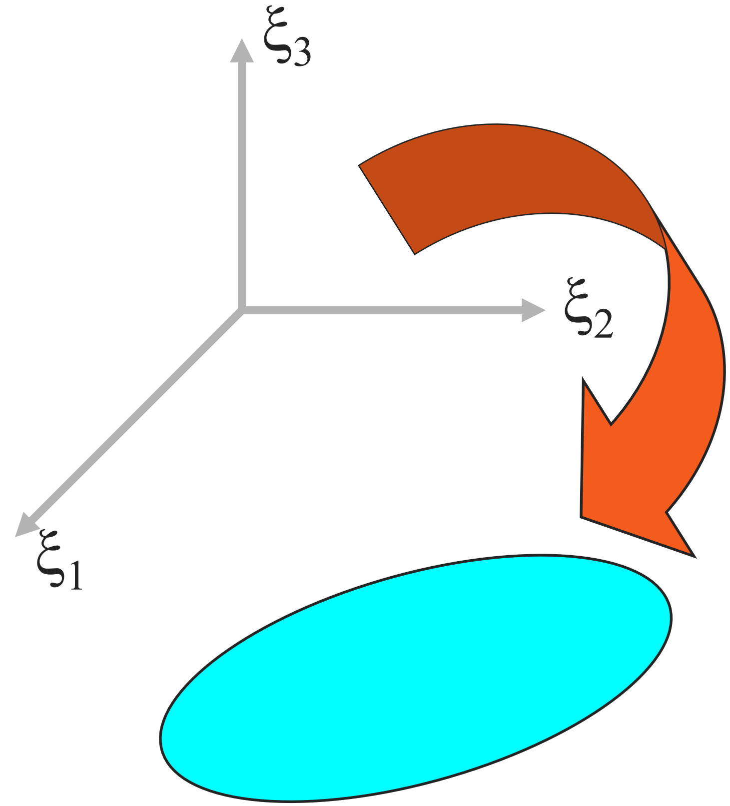
Rotation Matrices

Properties of Rotation Matrices

- Orthogonal
  - Matrix times its transpose equals 1
- Special orthogonal
  - Determinant is +1
- Closed under multiplication
- Composition

$$ {}^{A}R_{C} = {}^{A}R_{B} \times {}^{B}R_{C} $$

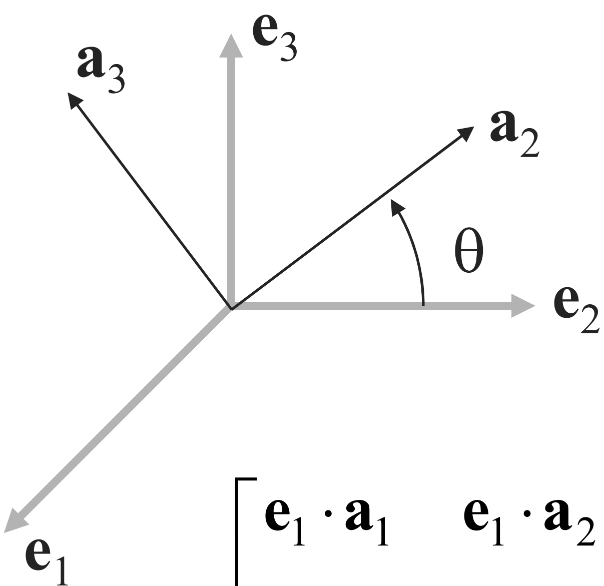
- The inverse of a rotation matrix is also a rotation matrix
- Composition and inverse operations are "continuous functions"
- The set of all rotations is a *Lie Group*, *SO*(3)
- *SO*(3) can be parameterized by 3 coordinates



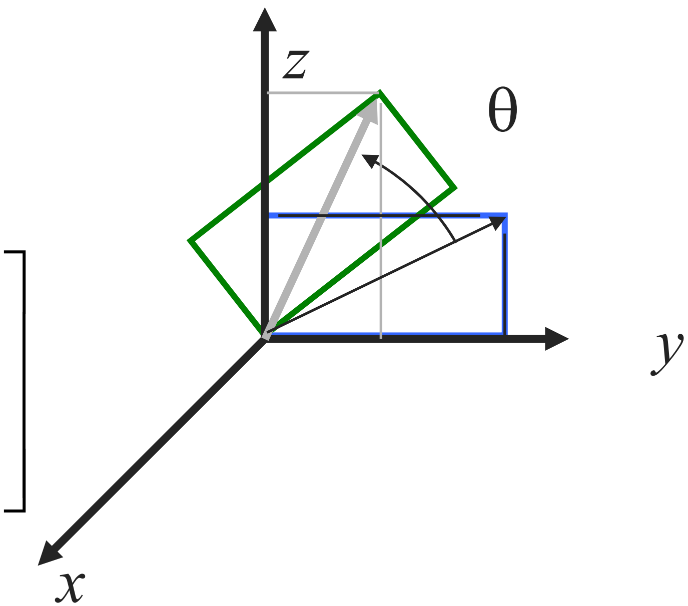
Example: "Simple" Rotations

- Rotation about the *x*-axis through θ

$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$$

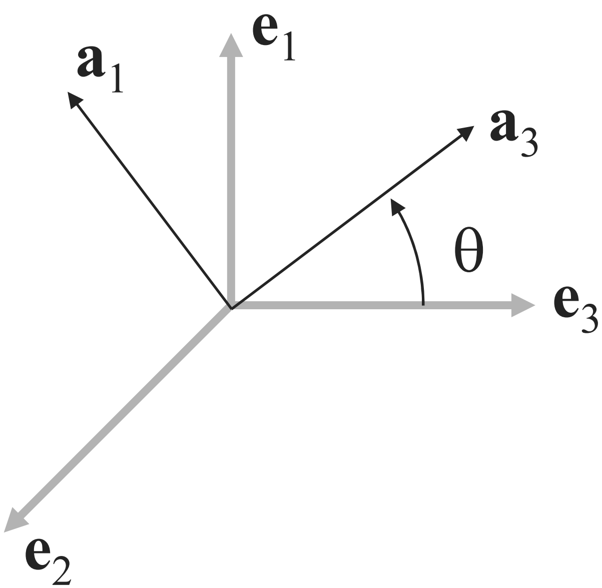


$$\begin{bmatrix}\mathbf{e}_1\cdot\mathbf{a}_1 & \mathbf{e}_1\cdot\mathbf{a}_2 & \mathbf{e}_1\cdot\mathbf{a}_3\\ \mathbf{e}_2\cdot\mathbf{a}_1 & \mathbf{e}_2\cdot\mathbf{a}_2 & \mathbf{e}_2\cdot\mathbf{a}_3\\ \mathbf{e}_3\cdot\mathbf{a}_1 & \mathbf{e}_3\cdot\mathbf{a}_2 & \mathbf{e}_3\cdot\mathbf{a}_3\end{bmatrix}$$



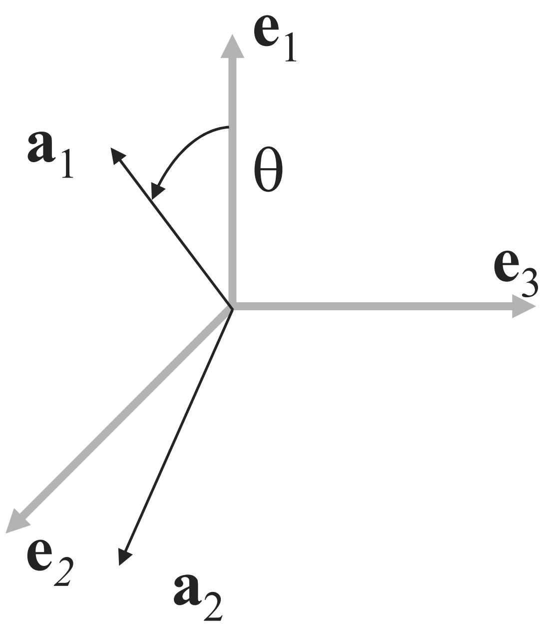
Example: Rotation

Rotation about the *y*-axis through θ



$$\begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{a}_1 & \mathbf{e}_1 \cdot \mathbf{a}_2 & \mathbf{e}_1 \cdot \mathbf{a}_3 \\ \mathbf{e}_2 \cdot \mathbf{a}_1 & \mathbf{e}_2 \cdot \mathbf{a}_2 & \mathbf{e}_2 \cdot \mathbf{a}_3 \\ \mathbf{e}_3 \cdot \mathbf{a}_1 & \mathbf{e}_3 \cdot \mathbf{a}_2 & \mathbf{e}_3 \cdot \mathbf{a}_3 \end{bmatrix}$$

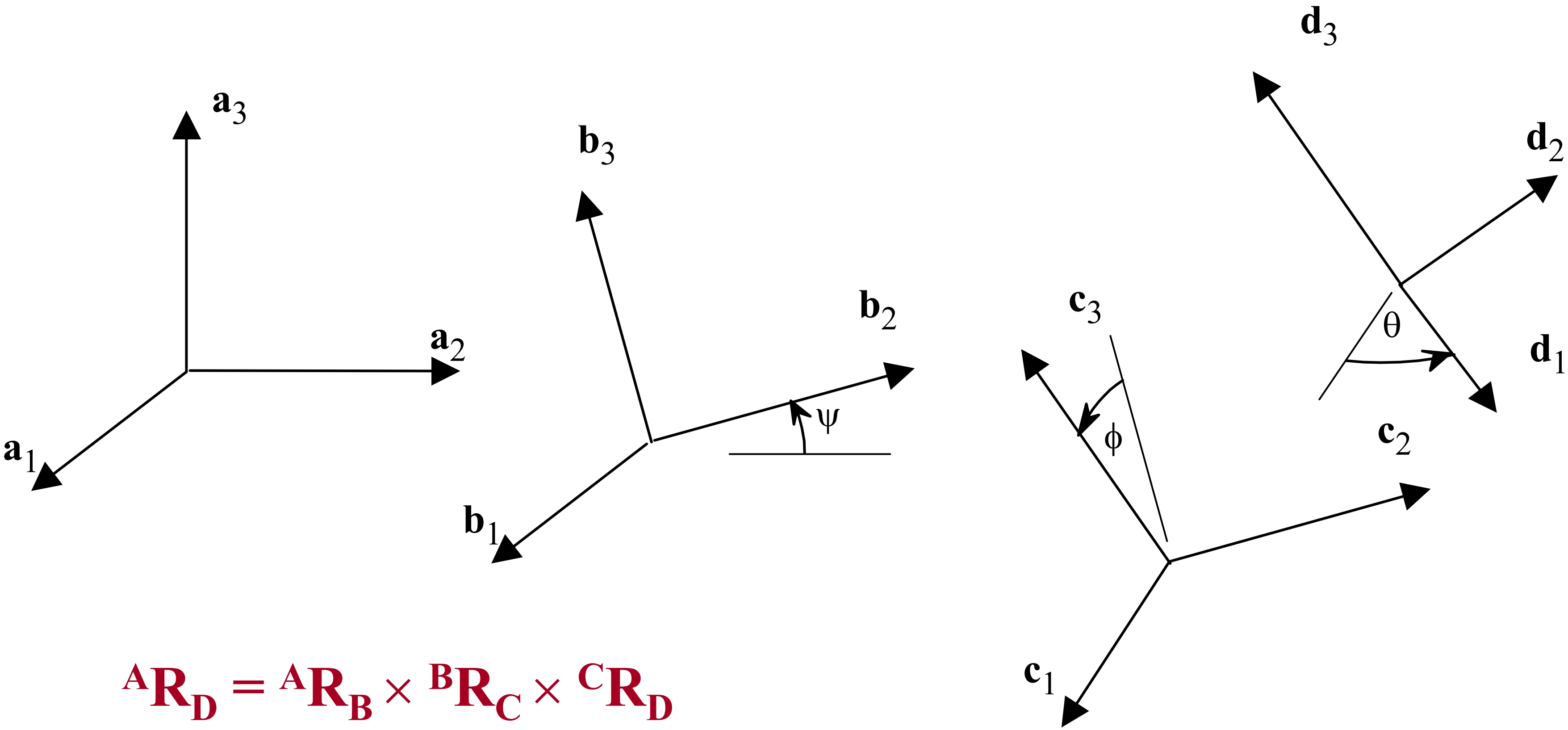
Rotation about the *z*-axis through θ



$$Rot(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$Rot(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

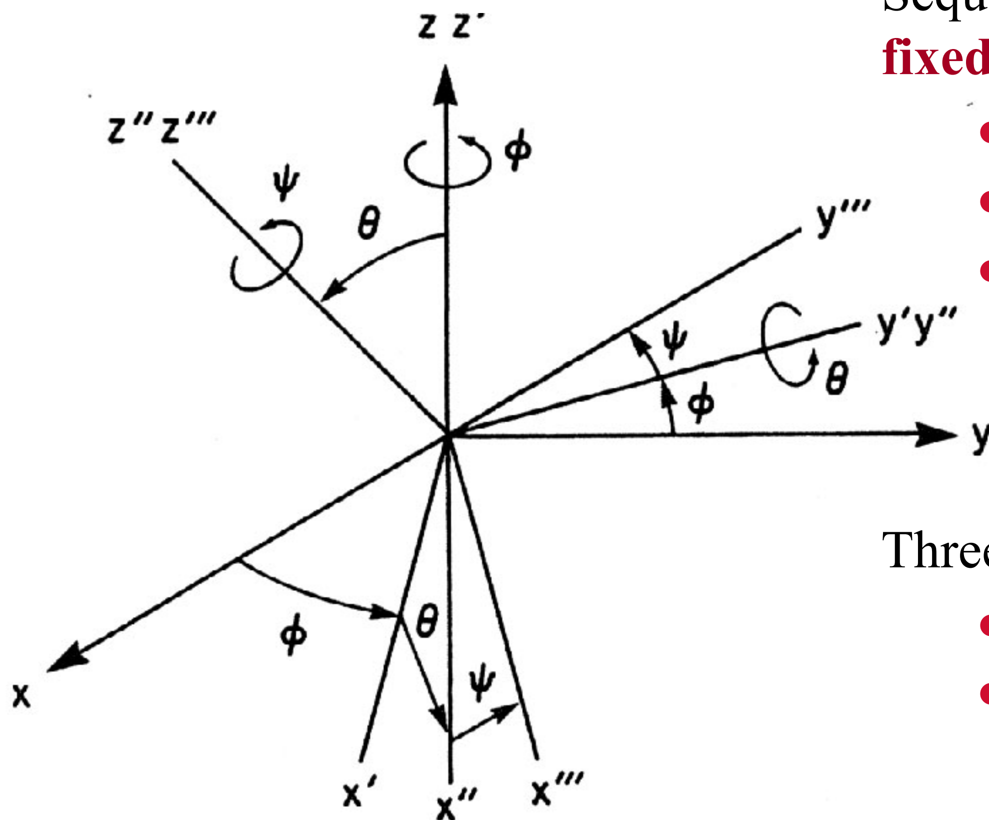
Composition of Three Rotations



$$ {}^{A}\mathbf{R}_{D} = {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} \times {}^{C}\mathbf{R}_{D} $$

$$ {}^{A}\mathbf{R}_{D} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi) $$

Euler Angles: Parameterization of Rotation Matrices



Sequence of three rotations about **body-fixed** axes

- Rot(z, ϕ)
- Rot(y, θ)
- Rot(z, ψ)

Three Euler Angles

- ϕ, θ, and ψ
- Parameterize rotations

Note

- $\theta = 0$ is a special (singular) case

$$\mathbf{R} = \text{Rot}(z, \phi) \times \text{Rot}(y, \theta) \times \text{Rot}(z, \psi)$$

Determination of Euler Angles

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

$$R := \begin{bmatrix} \cos(\phi)\cos(\theta)\cos(\psi) - \sin(\phi)\sin(\psi) & -\cos(\phi)\cos(\theta)\sin(\psi) - \sin(\phi)\cos(\psi) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta)\cos(\psi) + \cos(\phi)\sin(\psi) & -\sin(\phi)\cos(\theta)\sin(\psi) + \cos(\phi)\cos(\psi) & \sin(\phi)\sin(\theta) \\ -\sin(\theta)\cos(\psi) & \sin(\theta)\sin(\psi) & \cos(\theta) \end{bmatrix}$$

$$R_{13} = -\sin\theta\cos\psi$$

$$R_{23} = \sin\theta\sin\psi$$

$$\begin{bmatrix} R_{1,1} & R_{1,2} & R_{1,3} \\ R_{2,1} & R_{2,2} & R_{2,3} \\ R_{3,1} & R_{3,2} & R_{3,3} \end{bmatrix}$$

$$R_{33} = \cos\theta$$

$$R_{13} = \sin\theta\cos\phi$$

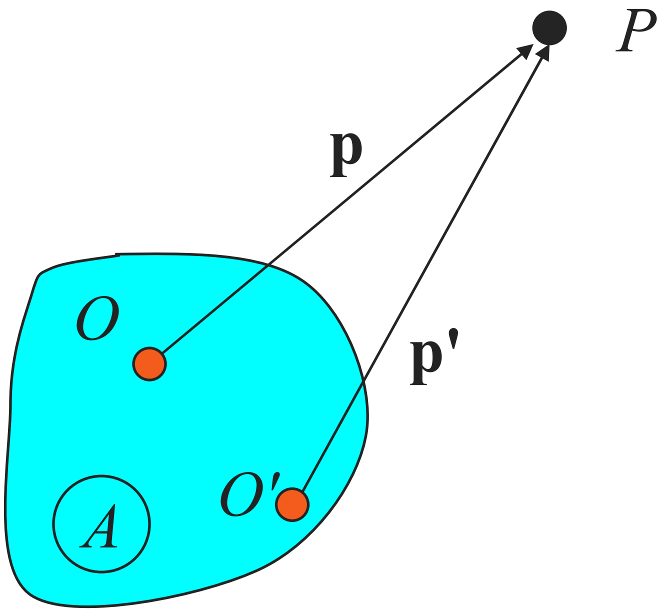
$$R_{23} = \sin\theta\sin\phi$$

Position, Velocity, Angular Velocity and Acceleration Vectors

Position, Velocity and Acceleration Vectors

p is a position vector of *P* in *A*

- Emanates from a point fixed to *A*
- Ends up at *P*



${}^{A}\mathbf{v}^{P}$ is the velocity of *P* in *A*

$$ {}^{A}\mathbf{v}^{P} = \frac{{}^{A}d\mathbf{p}}{dt} $$

${}^{A}\mathbf{a}^{P}$ is the acceleration of *P* in *A*

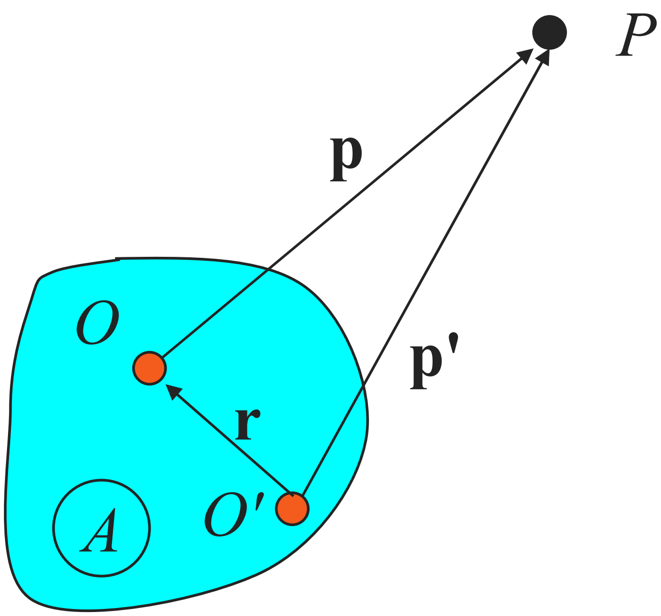
$$ {}^{A}\mathbf{a}^{P} = \frac{{}^{A}d\left({}^{A}\mathbf{v}^{P}\right)}{dt} $$

What if a different position vector were chosen?

Velocity of P in A is independent of choice of "origin" in A!

p is a position vector of P in A

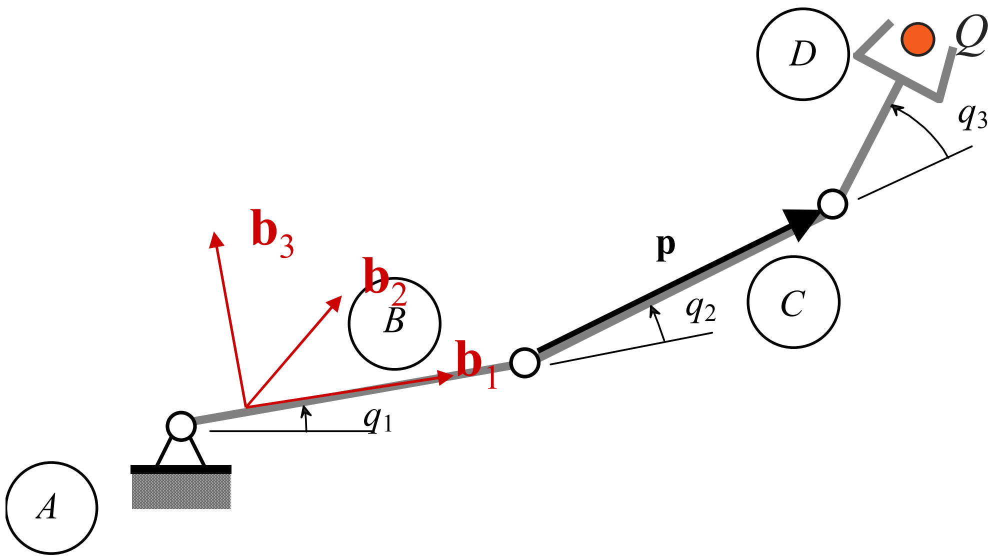
- Emanates from a point fixed to A
- Ends up at P



${}^{A}\mathbf{v}^{P}$ is the velocity of P in A

$$ {}^{A}\mathbf{v}^{P} = \frac{{}^{A}d\mathbf{p}'}{dt} = \frac{{}^{A}d(\mathbf{p}+\mathbf{r})}{dt} $$

Example



1. Are the following equal?

$$\frac{\partial \mathbf{p}}{\partial q_2} \qquad \frac{{}^{A}\partial \mathbf{p}}{\partial q_2} \qquad \frac{{}^{B}\partial \mathbf{p}}{\partial q_2} \qquad \frac{{}^{C}\partial \mathbf{p}}{\partial q_2}$$

2. If motor (joint) rates are given, calculate $\frac{{}^{A}d\mathbf{p}}{dt}$

$$\frac{{}^{A}d\mathbf{p}}{dt} = \frac{{}^{A}\partial \mathbf{p}}{\partial q_1}\dot{q}_1 + \frac{{}^{A}\partial \mathbf{p}}{\partial q_2}\dot{q}_2 + \frac{{}^{A}\partial \mathbf{p}}{\partial q_3}\dot{q}_3$$

A robotic arm is a system of rigid bodies (reference frames) *B*, *C*, and *D*. *A* is the inertial or the laboratory reference frame that is considered fixed.

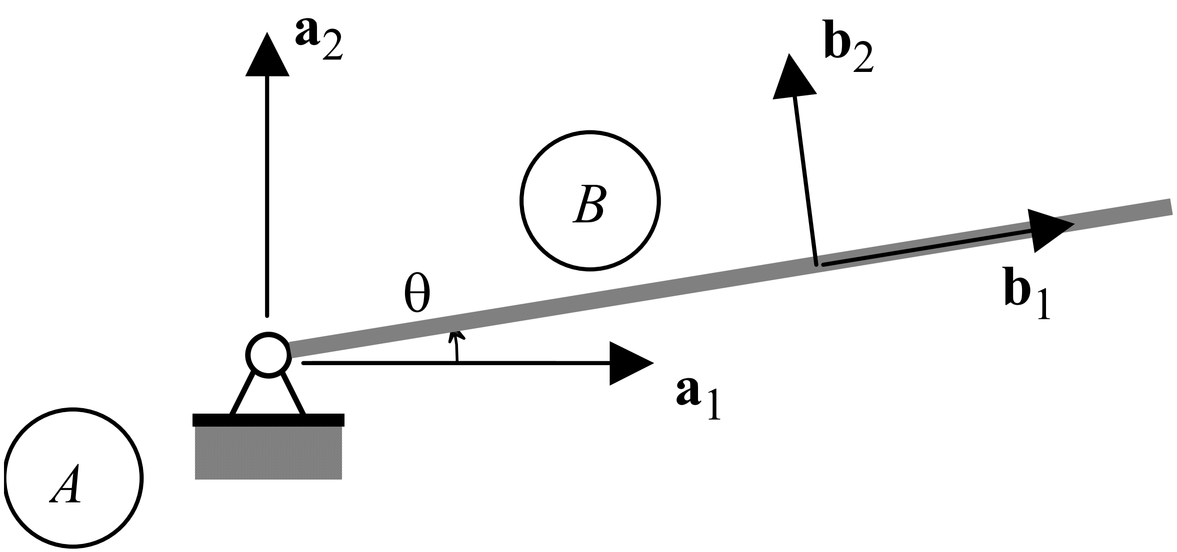
3. Find the velocity of *Q* in *A*

Angular Velocity (Kane)

The *angular velocity of B in A*, denoted by ${}^A\omega^B$, is defined as:

$${}^A\omega^B = \mathbf{b}_1\left(\frac{{}^A d\mathbf{b}_2}{dt}.\mathbf{b}_3\right) + \mathbf{b}_2\left(\frac{{}^A d\mathbf{b}_3}{dt}.\mathbf{b}_1\right) + \mathbf{b}_3\left(\frac{{}^A d\mathbf{b}_1}{dt}.\mathbf{b}_2\right)$$

- Defined in terms of a reference triad attached to *B*
- *Independent* of reference triad attached to *A*
- Generalizes to three dimensions
- Yields simple results for derivatives of vectors



Example

$$\mathbf{b}_1 = \mathbf{a}_1 \cos\theta + \mathbf{a}_2 \sin\theta$$

$$\mathbf{b}_2 = \mathbf{a}_2 \cos\theta - \mathbf{a}_1 \sin\theta$$

Differentiation of vectors
1. Vector fixed to *B*

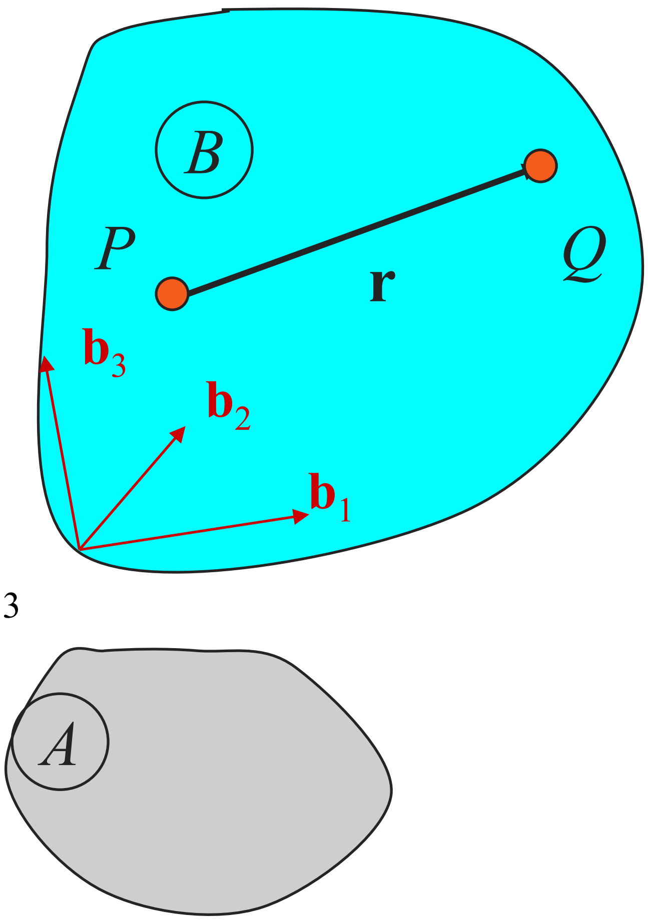
$$\mathbf{r} = r_1\,\mathbf{b}_1 + r_2\,\mathbf{b}_2 + r_3\,\mathbf{b}_3$$

$$\frac{{}^A d\mathbf{r}}{dt} = r_1\frac{{}^A d\mathbf{b}_1}{dt} + r_2\frac{{}^A d\mathbf{b}_2}{dt} + r_3\frac{{}^A d\mathbf{b}_3}{dt}$$

Composition and Projection rule

$$\frac{{}^A d\mathbf{b}_1}{dt} = \left(\frac{{}^A d\mathbf{b}_1}{dt}\cdot\mathbf{b}_1\right)\mathbf{b}_1 + \left(\frac{{}^A d\mathbf{b}_1}{dt}\cdot\mathbf{b}_2\right)\mathbf{b}_2 + \left(\frac{{}^A d\mathbf{b}_1}{dt}\cdot\mathbf{b}_3\right)\mathbf{b}_3$$

$$-\left(\mathbf{b}_1\cdot\frac{{}^A d\mathbf{b}_3}{dt}\right)$$



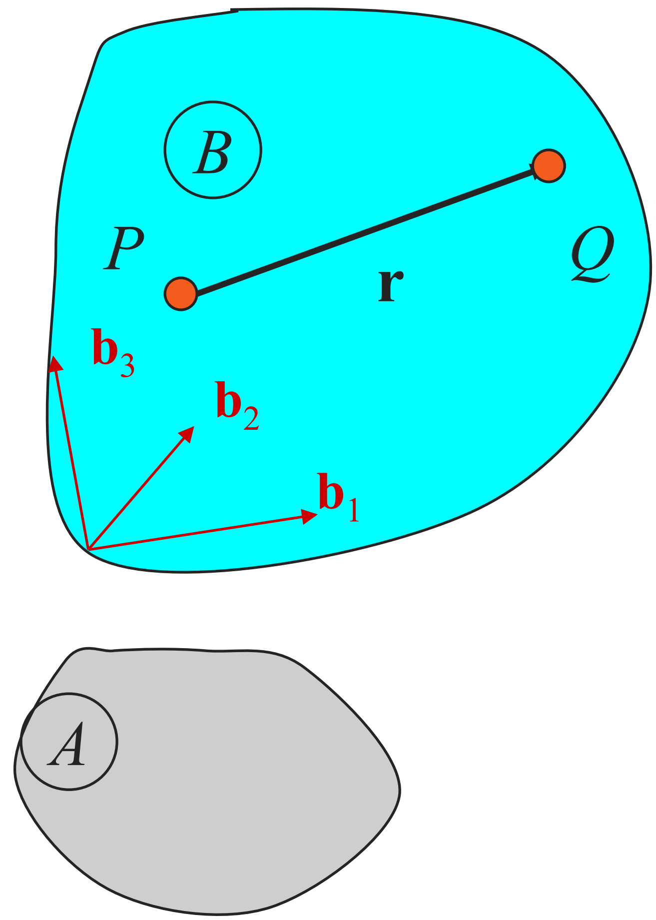
Differentiation of vectors (cont'd)

$$ {}^{A}\omega^{B} \times \mathbf{b}_1 = \left[\mathbf{b}_1 \left(\frac{{}^{A}d\mathbf{b}_2}{dt} . \mathbf{b}_3 \right) + \mathbf{b}_2 \left(\frac{{}^{A}d\mathbf{b}_3}{dt} . \mathbf{b}_1 \right) + \mathbf{b}_3 \left(\frac{{}^{A}d\mathbf{b}_1}{dt} . \mathbf{b}_2 \right) \right] \times \mathbf{b}_1 $$

$$ \frac{{}^{A}d\mathbf{b}_1}{dt} = \left(\frac{{}^{A}d\mathbf{b}_1}{dt} \cdot \mathbf{b}_2 \right) \mathbf{b}_2 - \left(\frac{{}^{A}d\mathbf{b}_3}{dt} \cdot \mathbf{b}_1 \right) \mathbf{b}_3 $$

Important Result 1

$$ \frac{{}^{A}d\mathbf{b}_i}{dt} = {}^{A}\omega^{B} \times \mathbf{b}_i $$

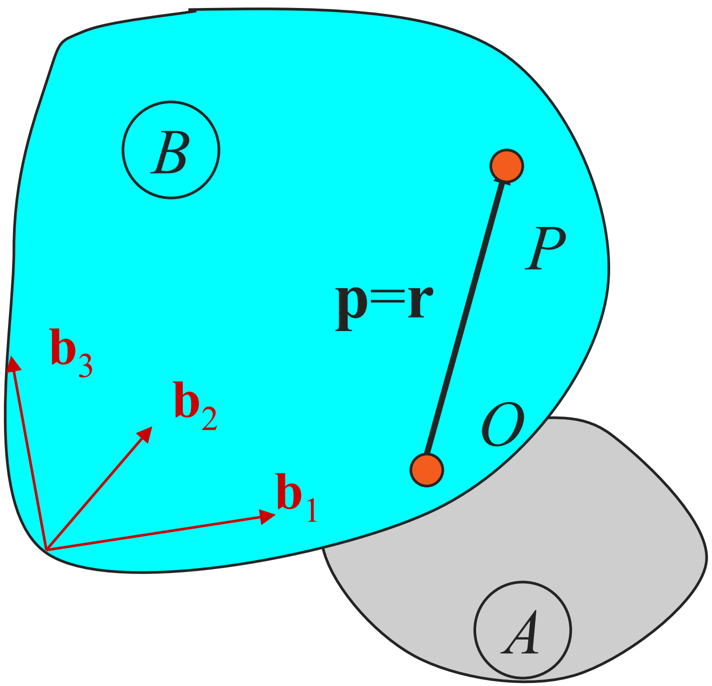


Important Result 2

$$ \frac{{}^{A}d\mathbf{r}}{dt} = r_1 \frac{{}^{A}d\mathbf{b}_1}{dt} + r_2 \frac{{}^{A}d\mathbf{b}_2}{dt} + r_3 \frac{{}^{A}d\mathbf{b}_3}{dt} $$
$$ = r_1 {}^{A}\omega^{B} \times \mathbf{b}_1 + r_2 {}^{A}\omega^{B} \times \mathbf{b}_2 + r_3 {}^{A}\omega^{B} \times \mathbf{b}_3 $$
$$ = {}^{A}\omega^{B} \times \mathbf{r} $$

Velocity of *P* (attached to *B*) in *A* when *A* and *B* have a common point *O*

Choose **p** to be a position vector of *P* in *A*



$$ {}^{A}\mathbf{v}^{P} = \frac{{}^{A}d\mathbf{p}}{dt} = {}^{A}\omega^{B} \times \mathbf{p} $$

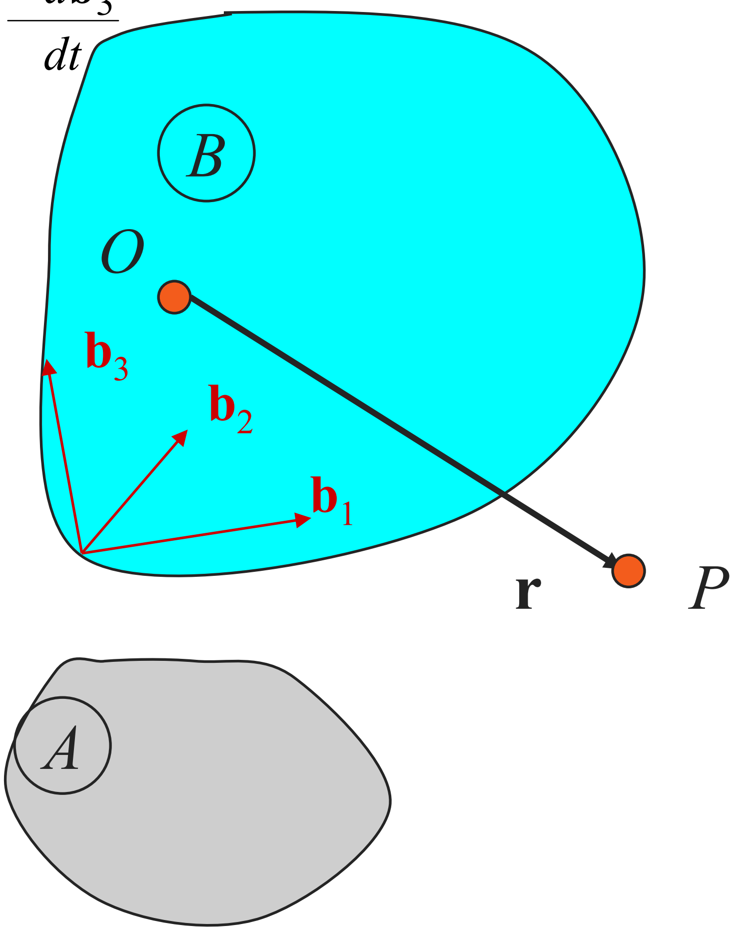
Differentiation of vectors
2. Vector not fixed to B

$$\frac{{}^{A}d\mathbf{r}}{dt} = \frac{dr_1}{dt}\mathbf{b}_1 + \frac{dr_2}{dt}\mathbf{b}_2 + \frac{dr_3}{dt}\mathbf{b}_3 + r_1\frac{{}^{A}d\mathbf{b}_1}{dt} + r_2\frac{{}^{A}d\mathbf{b}_2}{dt} + r_3\frac{{}^{A}d\mathbf{b}_3}{dt}$$

$$= \frac{{}^{B}d\mathbf{r}}{dt} + r_1\,{}^{A}\omega^{B}\times\mathbf{b}_1 + r_2\,{}^{A}\omega^{B}\times\mathbf{b}_2 + r_3\,{}^{A}\omega^{B}\times\mathbf{b}_3$$

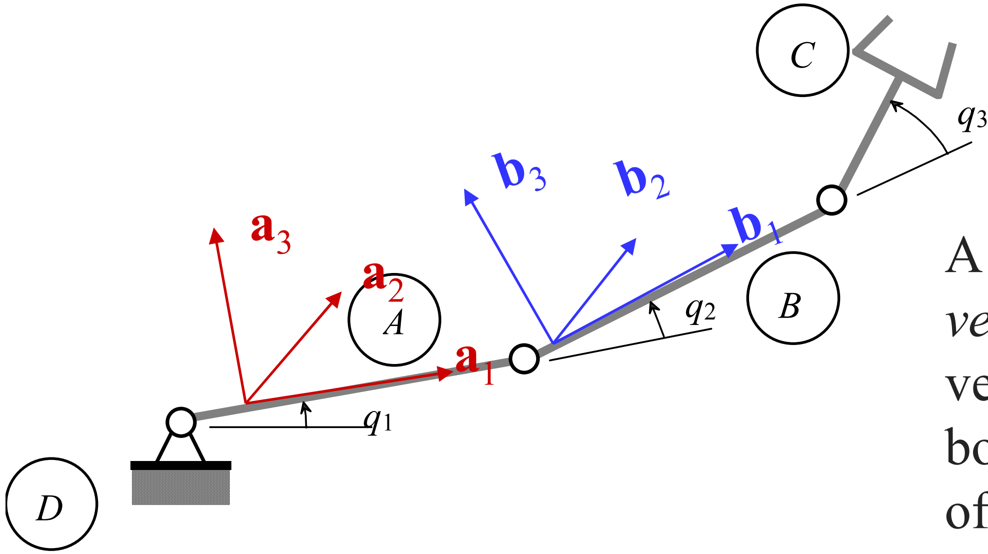
$$= \frac{{}^{B}d\mathbf{r}}{dt} + {}^{A}\omega^{B}\times\mathbf{r}$$

$$\frac{{}^{A}d\mathbf{r}}{dt} = \frac{{}^{B}d\mathbf{r}}{dt} + {}^{A}\omega^{B}\times\mathbf{r}$$



r can be *any vector*

Simple Angular Velocity



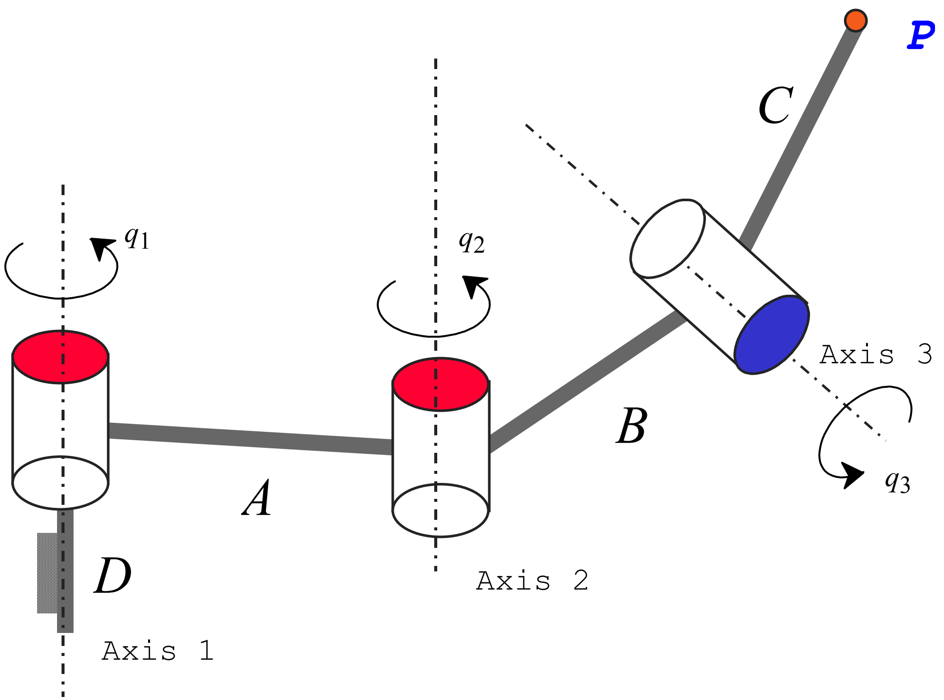
A rigid body *B* has a *simple angular velocity* in *A*, when there exists a unit vector **k** whose orientation (as seen) in both *A* and in *B* is constant (independent of time).

Angular velocity of *B* in *A*

- is along $\mathbf{a}_2$ as seen in *A*
- is along $\mathbf{b}_2$ as seen in *B*

In each frame, the angular velocity has a constant direction (magnitude may change)

Simple Angular Velocity



How about ${}^{D}\omega^{C}$?

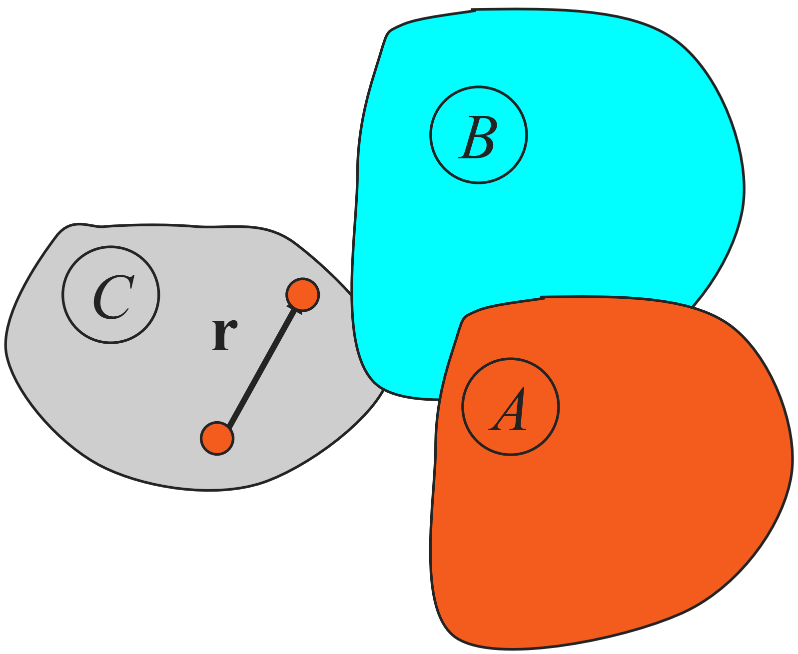
${}^{D}\omega^{A}$, ${}^{A}\omega^{B}$ and ${}^{C}\omega^{B}$ are simple angular velocities.

However, ${}^{D}\omega^{C}$ is not a simple angular velocity. The motion of *C* relative to *D* is such that there is no vector fixed in *D* that also remains fixed in *C*.

Addition Theorem for Angular Velocities

Let A, B, and C be three rigid bodies. The addition theorem for angular velocities states:

$$^{A}\omega^{C} = {}^{A}\omega^{B} + {}^{B}\omega^{C}$$



Proof

Let $\mathbf{r}$ be fixed to C.

$$\frac{^{A}d\mathbf{r}}{dt} = \frac{^{B}d\mathbf{r}}{dt} + {}^{A}\omega^{B} \times \mathbf{r}$$

$$= \frac{^{C}d\mathbf{r}}{dt} + {}^{B}\omega^{C} \times \mathbf{r} + {}^{A}\omega^{B} \times \mathbf{r}$$

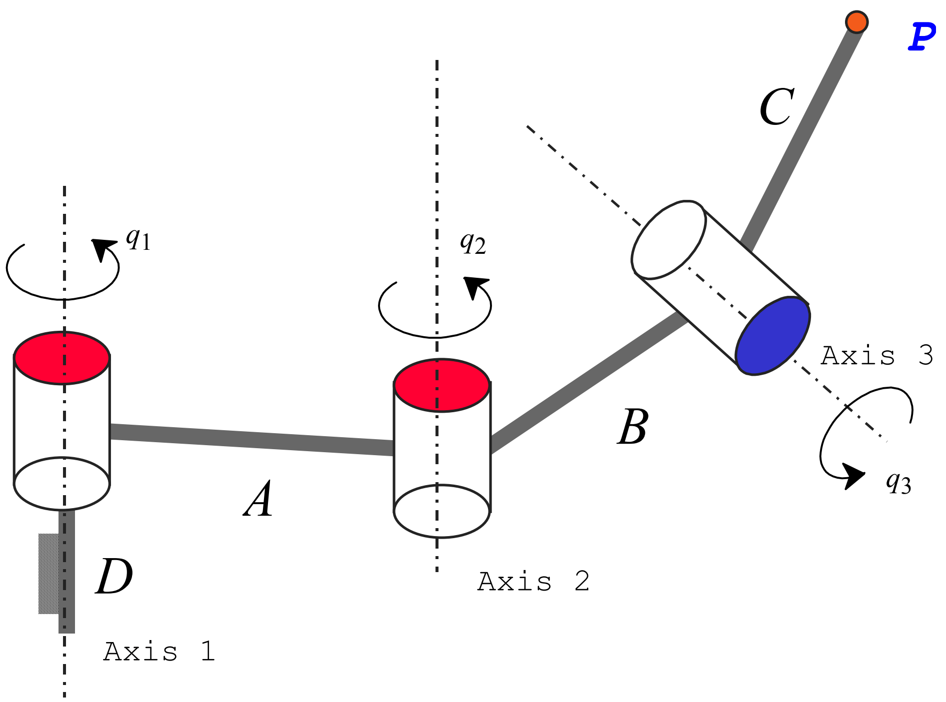
$$= \frac{^{C}d\mathbf{r}}{dt} + \left({}^{B}\omega^{C} + {}^{A}\omega^{B}\right) \times \mathbf{r}$$

$$= \left({}^{B}\omega^{C} + {}^{A}\omega^{B}\right) \times \mathbf{r}$$

And,

$$\frac{^{A}d\mathbf{r}}{dt} = {}^{A}\omega^{C} \times \mathbf{r}$$

Angular velocities can be found by adding up "simple" angular velocities



${}^{D}\omega^{C}$ is not a simple angular velocity. The motion of C relative to D is such that there is no vector fixed in D that also remains fixed in C.

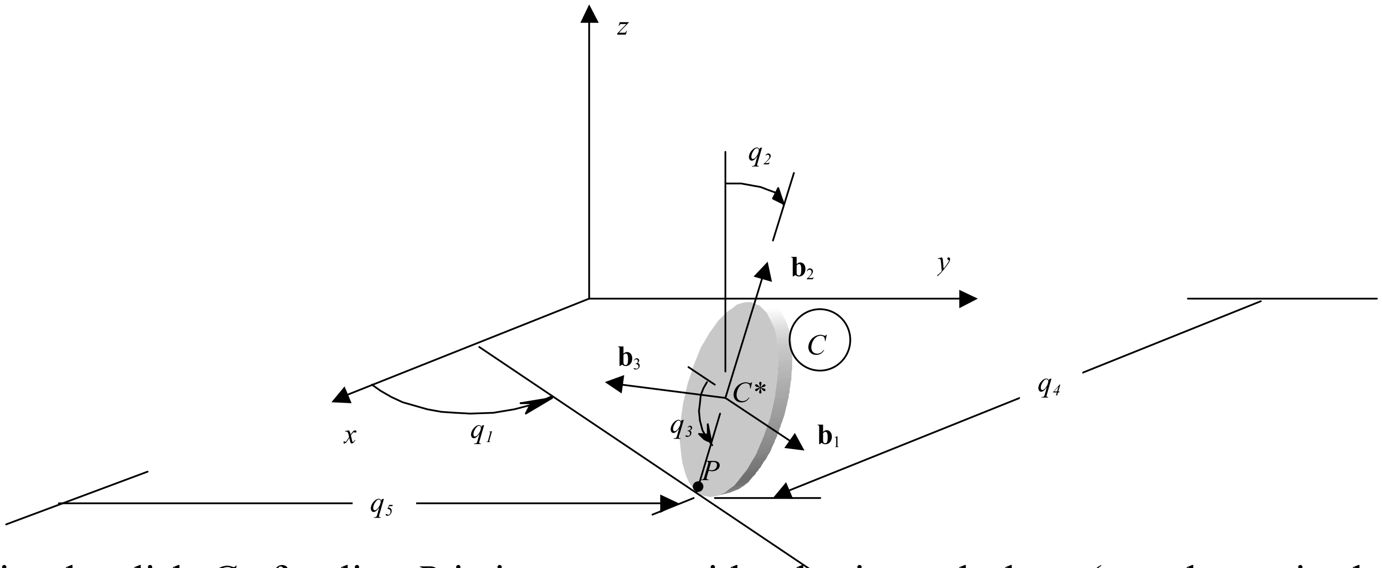
But,

$$ {}^{D}\omega^{C} = {}^{D}\omega^{A} + {}^{A}\omega^{B} + {}^{B}\omega^{C} $$

${}^{D}\omega^{A}$, ${}^{A}\omega^{B}$ and ${}^{C}\omega^{B}$ are simple angular velocities.

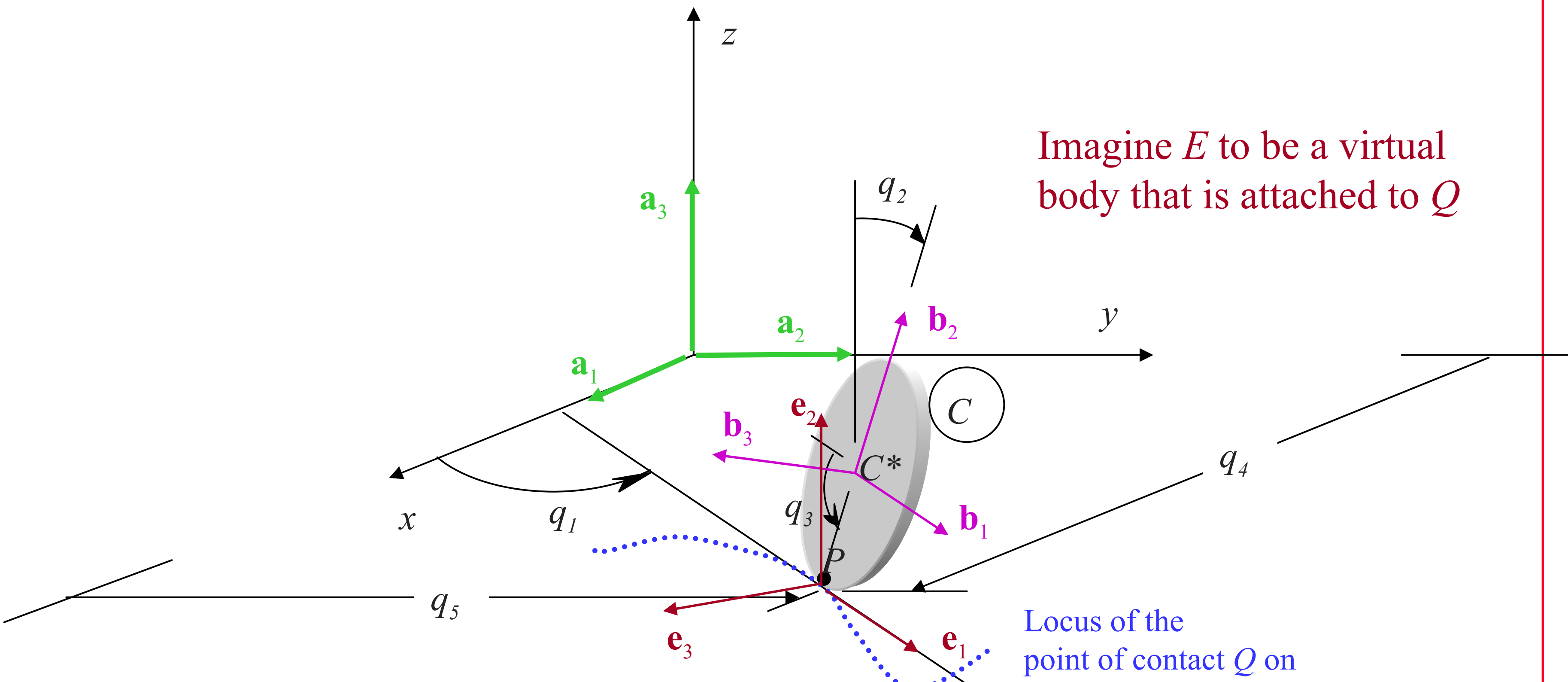
Example

The rolling (and sliding) disk on a horizontal plane



A circular disk C of radius R is in contact with a horizontal plane (not shown in the figure) at the point P. The point P is attached to the disk. The plane is the x-y plane. It is rigidly attached to the earth. The standard reference triad $\mathbf{b}_i$ is chosen so that $\mathbf{b}_1$ is along the direction of progression of the disk (parallel to the tangent to the disk at P), $\mathbf{b}_2$ is parallel to the plane of the disk, and $\mathbf{b}_3$ is normal to the disk. Note that this triad is *not* fixed to the disk. Call the earth-fixed reference frame A and choose the standard reference triad $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ in an obvious fashion along the x, y, and z axes shown in the figure.

Reference Triads



Imagine E to be a virtual body that is attached to Q

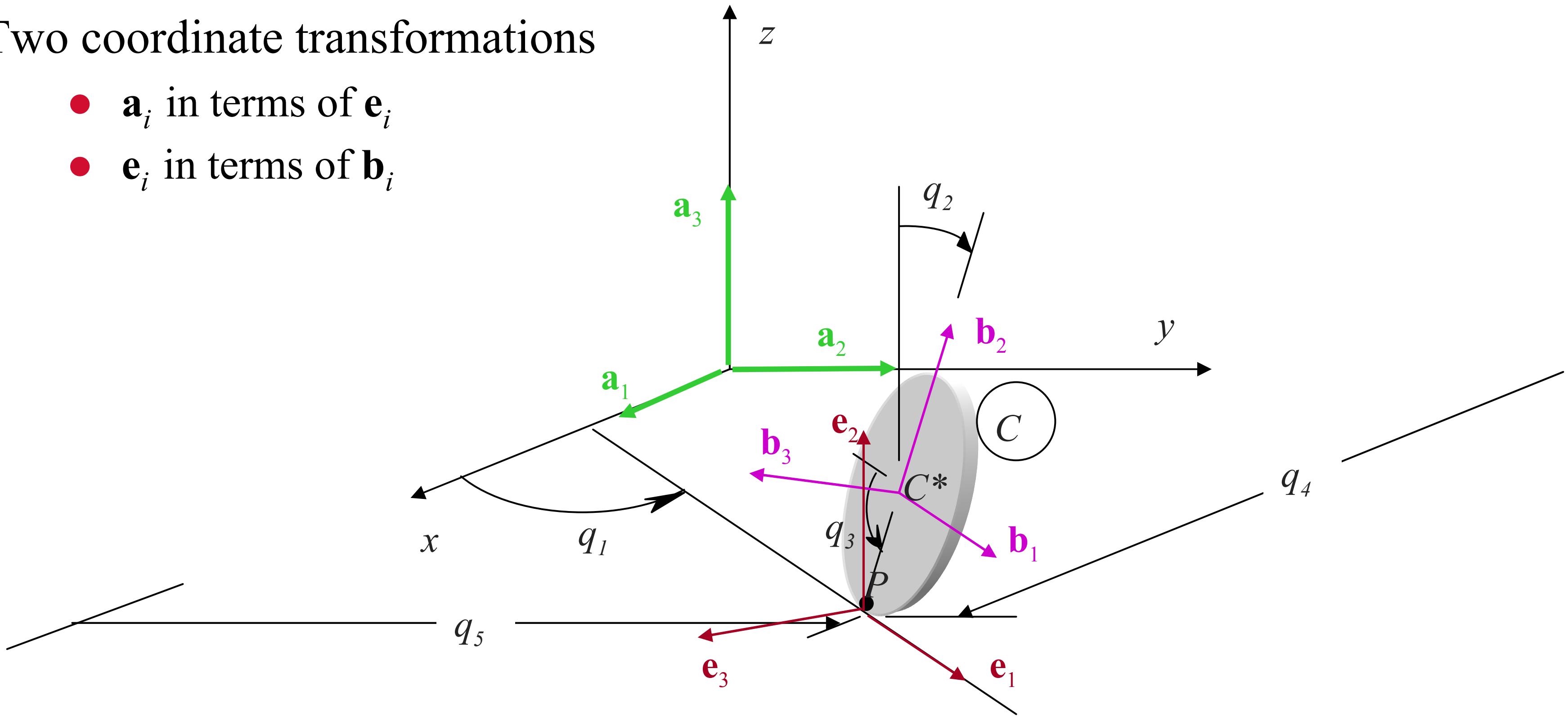
- *Rotate* triad A about z through q_1 followed by rotation about x by 90 deg to get E
- *Rotate* triad E about $-x$ through q_2 to get B
- *Rotate* triad B about z through q_3 to get C (not shown)

Imagine B to be a virtual body that is attached to C^*

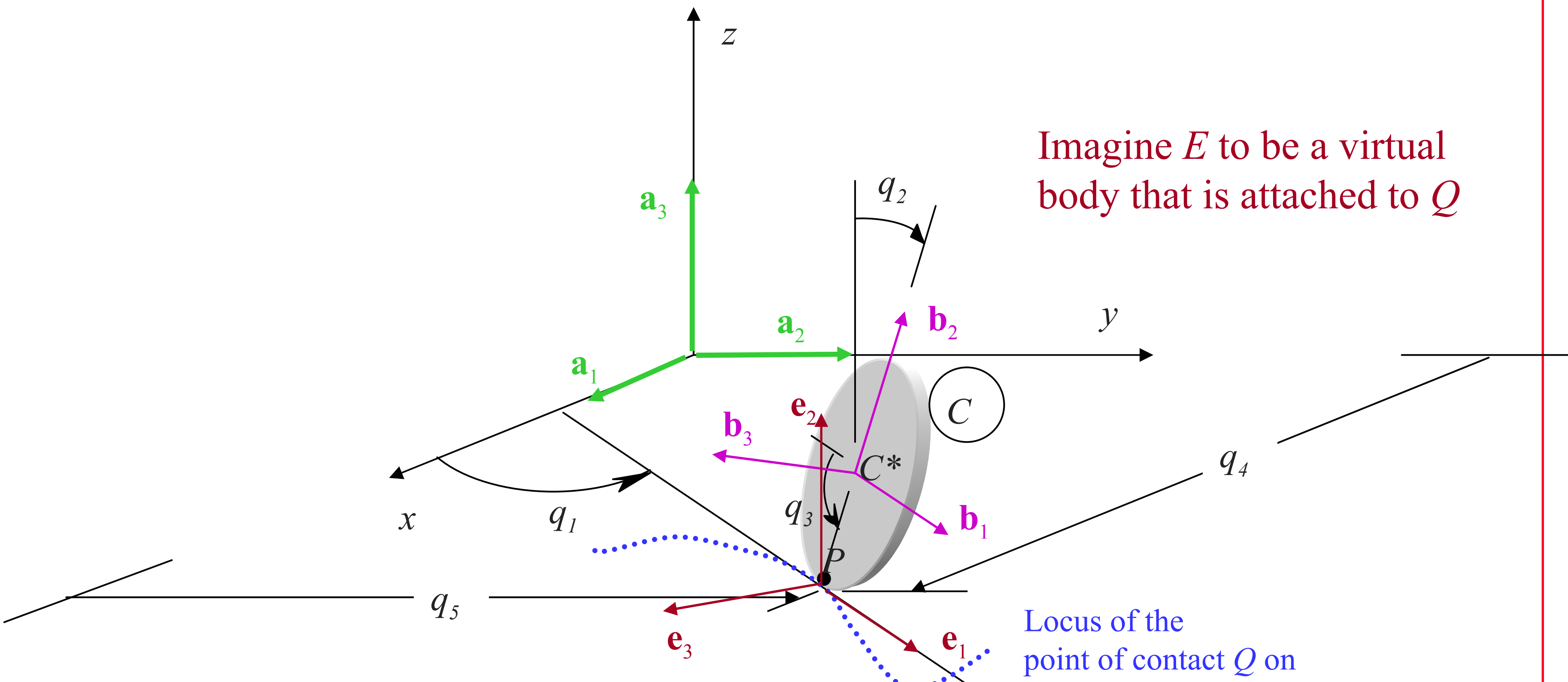
Transformations

Two coordinate transformations

- $\mathbf{a}_i$ in terms of $\mathbf{e}_i$
- $\mathbf{e}_i$ in terms of $\mathbf{b}_i$



Reference Triads



Imagine E to be a virtual body that is attached to Q

Locus of the point of contact Q on the plane A

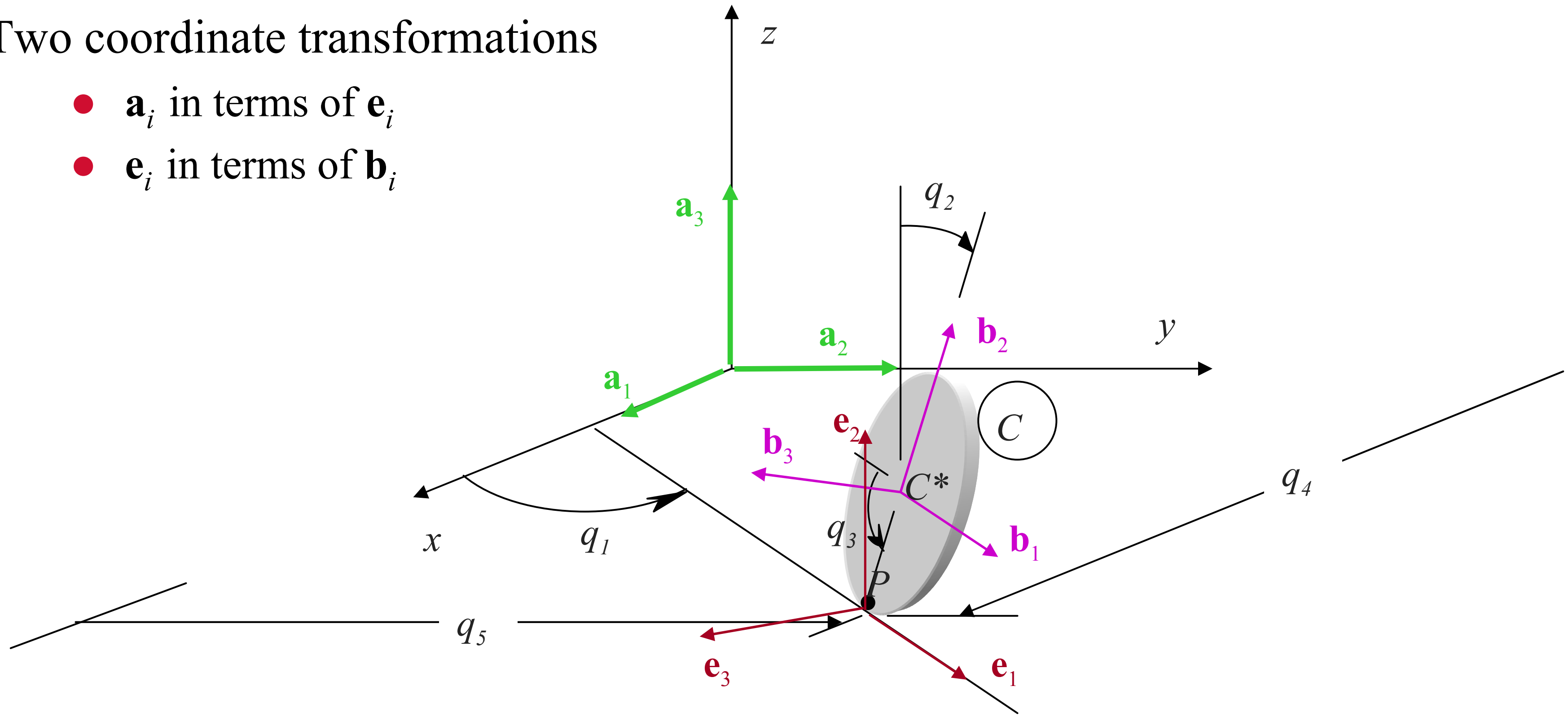
- *Rotate* triad A about z through q_1 followed by rotation about x by 90 deg to get E
- *Rotate* triad E about -x through q_2 to get B
- *Rotate* triad B about z through q_3 to get C (not shown)

Imagine B to be a virtual body that is attached to C^*

Transformations

Two coordinate transformations

- $\mathbf{a}_i$ in terms of $\mathbf{e}_i$
- $\mathbf{e}_i$ in terms of $\mathbf{b}_i$



Angular Velocity: Components

$$^{A}\omega^{C} = u_1\,\mathbf{b}_1 + u_2\,\mathbf{b}_2 + u_3\,\mathbf{b}_3$$

- u_i are the components of the angular velocity of the disk with respect to the reference triad B

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & \mathbf{X} & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix}$$

$$^{A}\omega^{C} = u_x\,\mathbf{a}_x + u_y\,\mathbf{a}_y + u_z\,\mathbf{a}_z$$

- u_α are the components of the angular velocity of the disk with respect to the reference triad A

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix} = \begin{bmatrix} & & & & \\ & & & & \\ & & \mathbf{Y} & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

Angular Acceleration

The *angular acceleration of B in A*, denoted by ${}^A\alpha^B$, is defined as the first time-derivative in A of the angular velocity of B in A:

$$ {}^A\alpha^B = \frac{{}^A d}{dt}\left({}^A\omega^B\right) $$

Addition theorem for angular accelerations?