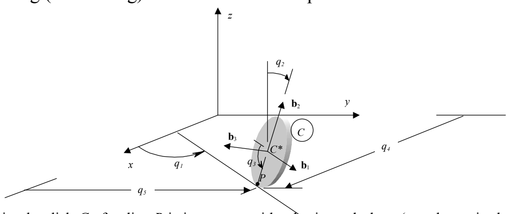
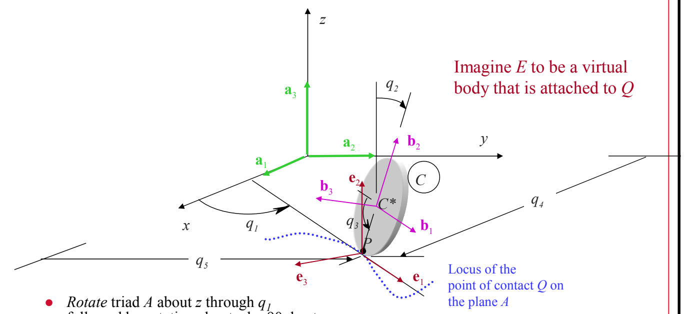
Example

The rolling (and sliding) disk on a horizontal plane



A circular disk C of radius R is in contact with a horizontal plane (not shown in the figure) at the point P. The point P is attached to the disk. The plane is the x-y plane. It is rigidly attached to the earth. The standard reference triad $\mathbf{b}_i$ is chosen so that $\mathbf{b}_1$ is along the direction of progression of the disk (parallel to the tangent to the disk at P), $\mathbf{b}_2$ is parallel to the plane of the disk, and $\mathbf{b}_3$ is normal to the disk. Note that this triad is *not* fixed to the disk. Call the earth-fixed reference frame A and choose the standard reference triad $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ in an obvious fashion along the x, y, and z axes shown in the figure.

Reference Triads



Imagine E to be a virtual body that is attached to Q

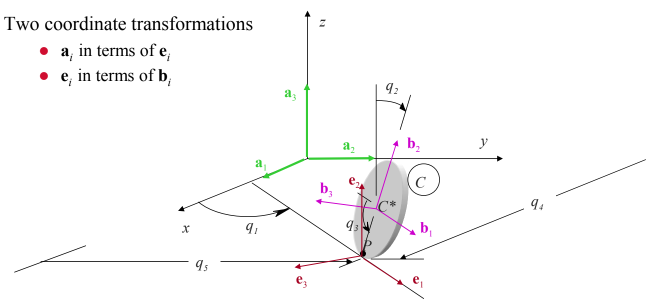
- *Rotate* triad A about z through q_1 followed by rotation about x by 90 deg to get E
- *Rotate* triad E about -x through q_2 to get B
- *Rotate* triad B about z through q_3 to get C (not shown)

Imagine B to be a virtual body that is attached to C^*

Transformations

Two coordinate transformations

- $\mathbf{a}_i$ in terms of $\mathbf{e}_i$
- $\mathbf{e}_i$ in terms of $\mathbf{b}_i$



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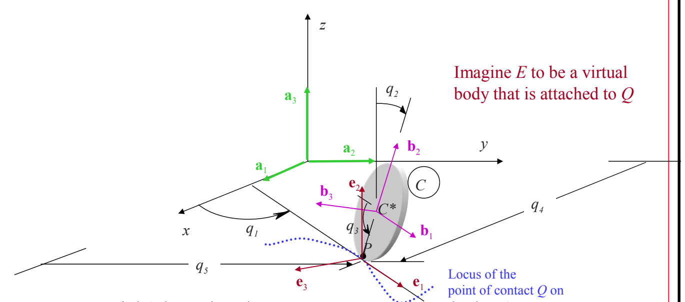

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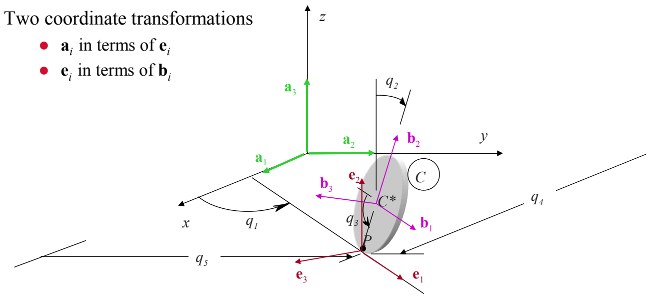

$${}^{B}\mathbf{R}_{A} = {}^{B}\mathbf{R}_{E}\ {}^{E}\mathbf{R}_{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_2 & \sin q_2 \\ 0 & -\sin q_2 & \cos q_2 \end{bmatrix} \times \begin{bmatrix} \cos q_1 & \sin q_1 & 0 \\ 0 & 0 & 1 \\ \sin q_1 & -\cos q_1 & 0 \end{bmatrix} = \begin{bmatrix} \cos q_1 & \sin q_1 & 0 \\ -\sin q_1 \sin q_2 & \cos q_1 \sin q_2 & \cos q_2 \\ \sin q_1 \cos q_2 & -\cos q_1 \cos q_2 & \sin q_2 \end{bmatrix}$$

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Angular Velocity: Components

$${}^{A}\boldsymbol{\omega}^{C} = u_1\,\mathbf{b}_1 + u_2\,\mathbf{b}_2 + u_3\,\mathbf{b}_3$$

- u_i are the components of the angular velocity of the disk with respect to the reference triad B

$${}^{A}\boldsymbol{\omega}^{C} = u_x\,\mathbf{a}_x + u_y\,\mathbf{a}_y + u_z\,\mathbf{a}_z$$

- u_α are the components of the angular velocity of the disk with respect to the reference triad A

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} & & \\ & \mathbf{X} & \\ & & \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix} = \begin{bmatrix} & & \\ & \mathbf{Y} & \\ & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

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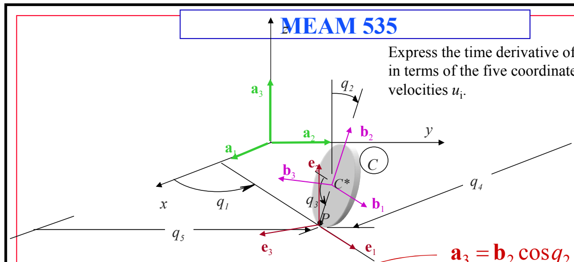
Express the time derivative of each coordinate in terms of the five coordinates and the velocities u_i.



$$\mathbf{a}_3 = \mathbf{b}_2 \cos q_2 + \mathbf{b}_3 \sin q_2$$

$${}^{A}\omega^{C} = {}^{A}\omega^{E} + {}^{E}\omega^{B} + {}^{B}\omega^{C} = \dot{q}_1\mathbf{a}_3 - \dot{q}_2\mathbf{b}_1 + \dot{q}_3\mathbf{b}_3$$

$${}^{A}\omega^{C} = u_1\mathbf{b}_1 + u_2\mathbf{b}_2 + u_3\mathbf{b}_3 = -\dot{q}_2\mathbf{b}_1 + \dot{q}_1\cos q_2\mathbf{b}_2 + (\dot{q}_1\sin q_2 + \dot{q}_3)\mathbf{b}_3$$

$$u_1 = -\dot{q}_2 \quad u_2 = \dot{q}_1\cos q_2 \quad u_3 = \dot{q}_1\sin q_2 + \dot{q}_3 \quad u_4 = \dot{q}_4 \quad u_5 = \dot{q}_5$$

$$\dot{q}_1 = \frac{u_2}{\cos q_2} \quad \dot{q}_2 = -u_1 \quad \dot{q}_3 = u_3 - \frac{u_2 \sin q_2}{\cos q_2}$$

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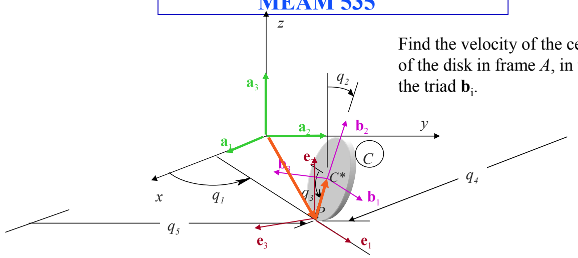
Find the velocity of the center (C^*) of the disk in frame A, in terms of the triad $\mathbf{b}_i$.



1. Need position vector of C^* in A

$$\overrightarrow{OC^*} = \overrightarrow{OQ} + \overrightarrow{QC^*} \qquad q_4\mathbf{a}_1 + q_5\mathbf{a}_2$$

2. Recognize Q is fixed in B

$${}^{A}\boldsymbol{\omega}^{B} \times R\mathbf{b}_2$$

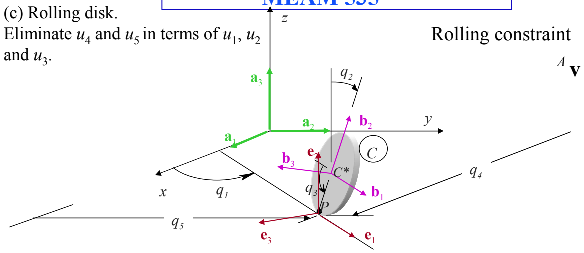
$${}^{A}\mathbf{v}^{C^*} = \frac{{}^{A}d}{dt}\overrightarrow{OC^*} = \frac{{}^{A}d}{dt}\overrightarrow{OQ} + \frac{{}^{A}d}{dt}\overrightarrow{QC^*}$$

3. Substitute

$${}^{A}\mathbf{v}^{C^*} = \frac{{}^{A}d}{dt}(q_4\mathbf{a}_1 + q_5\mathbf{a}_2) + \left({}^{A}\boldsymbol{\omega}^{E} + {}^{E}\boldsymbol{\omega}^{B}\right) \times (R\mathbf{b}_2)$$

$$\begin{aligned} {}^{A}\mathbf{v}^{C^*} = &(u_4\cos q_1 + u_5\sin q_1 - R\cdot u_2\tan q_2)\mathbf{b}_1 \\ &+(-u_4\sin q_1\sin q_2 + u_5\cos q_1\sin q_2)\mathbf{b}_2 \\ &+(u_4\sin q_1\cos q_2 - u_5\cos q_1\cos q_2 + R\cdot u_1)\mathbf{b}_3 \end{aligned}$$

(c) Rolling disk.
Eliminate u_4 and u_5 in terms of u_1, u_2 and u_3.



Rolling constraint

$$^A\mathbf{v}^P = 0$$

$$^A\mathbf{v}^P = {}^A\mathbf{v}^{C^*} + {}^A\omega^C \times \overrightarrow{C^*P}$$

$$= {}^A\mathbf{v}^{C^*} + (u_1\mathbf{b}_1 + u_2\mathbf{b}_2 + u_3\mathbf{b}_3) \times (-R\mathbf{b}_2)$$

$$= 0$$

$$u_4 \cos q_1 + u_5 \sin q_1 - Ru_2 \tan q_2 + Ru_3 = 0$$

$$-u_4 \sin q_1 \sin q_2 + u_5 \cos q_1 \sin q_2 = 0$$

Two constraints which reduce the disk's degrees of freedom from 5 to 3… More about this later!