

Principle of Virtual Work

Aristotle

Galileo (1594)

Bernoulli (1717)

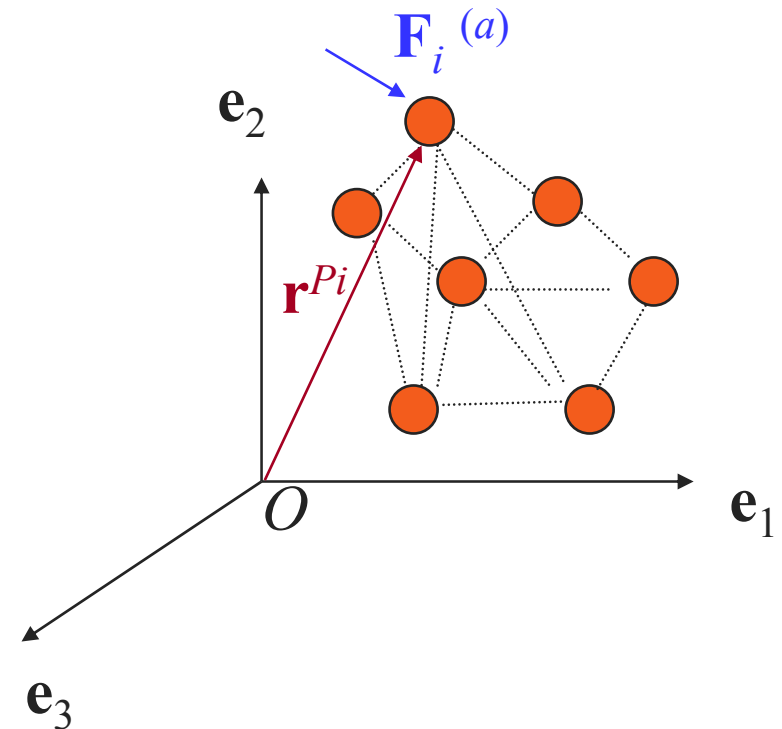
Lagrange (1788)



Virtual Work

Key Ideas

- Virtual displacement
 - ◆ Small
 - ◆ Consistent with constraints
 - ◆ Occurring without passage of time
- Applied forces (and moments)
 - ◆ Ignore constraint forces
- Static equilibrium
 - ◆ Zero acceleration, or
 - ◆ Zero mass



n generalized coordinates, q_j

$$\delta W = \sum_{i=1}^N \left[\mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}^{P_i} \right]$$

$$\delta W = \sum_{j=1}^n \left\{ \sum_{i=1}^N \left[\mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}^{P_i}}{\partial q_j} \right] \right\} \delta q_j$$

Example

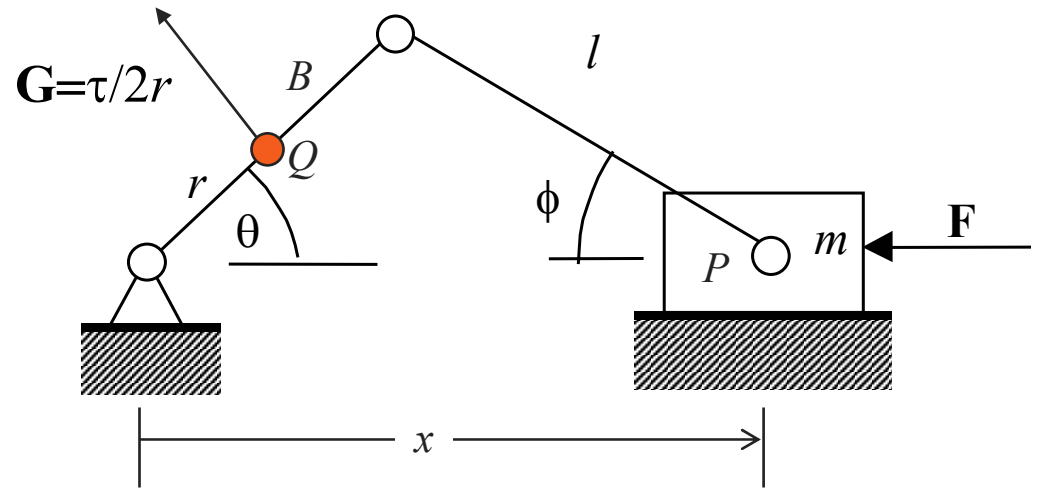
Applied forces

\mathbf{F} acting at P

\mathbf{G} acting at Q

Constraint forces

?



Single degree of freedom

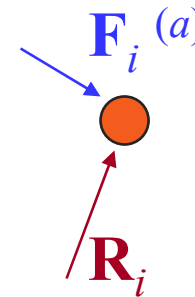
Generalized coordinate, θ

Motion of particles P and Q can be described by the generalized coordinate θ

Static Equilibrium Implies Zero Virtual Work is Done

Forces

- Forces that do work
 - ◆ Applied Forces
 - ◆ External Forces
- Forces that do no work
 - ◆ Constraint forces



Static Equilibrium

Implies sum of all forces on each particle equals zero

$$\begin{aligned}
 & \left[\mathbf{F}_i^{(a)} + \mathbf{R}_i \right] = 0 \\
 & \sum_{i=1}^N \left[\mathbf{F}_i^{(a)} + \mathbf{R}_i \right] = 0 \quad \Rightarrow \quad \sum_{i=1}^N \left[\mathbf{F}_i^{(a)} + \mathbf{R}_i \right] \delta \mathbf{r}_i = 0
 \end{aligned}$$

The Key Idea

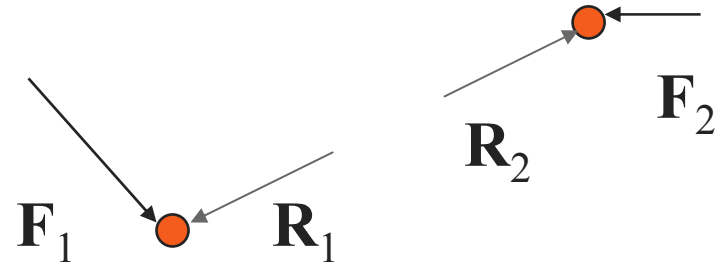
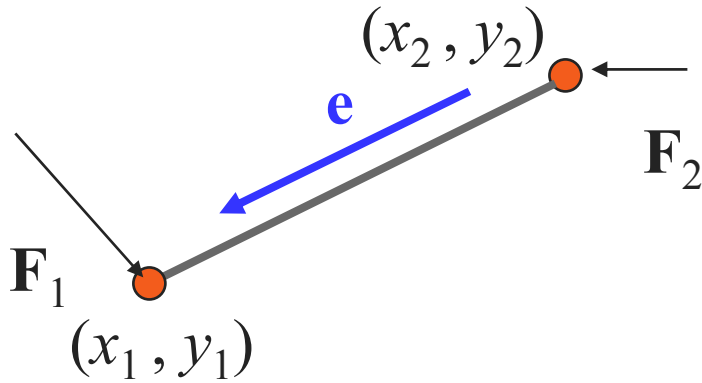
Constraint forces do zero virtual work!

$$\sum_{i=1}^N [\mathbf{F}_i^{(a)} + \mathbf{R}_i] \delta \mathbf{r}_i = 0$$

The equation above is annotated with a red arrow pointing from the bracketed term $[\mathbf{F}_i^{(a)} + \mathbf{R}_i]$ to a red '0' above the equals sign, indicating that the constraint forces \mathbf{R}_i contribute zero to the total virtual work.

Why?

Constraints: Two Particles Connected by Rigid Massless Rod



$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = r^2$$

$$(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) = 0$$

$$\begin{bmatrix} (x_1 - x_2) & (y_1 - y_2) \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta y_1 \end{bmatrix} - \begin{bmatrix} (x_1 - x_2) & (y_1 - y_2) \end{bmatrix} \begin{bmatrix} \delta x_2 \\ \delta y_2 \end{bmatrix} = 0$$

$$\mathbf{e} \cdot (\delta \mathbf{r}_1 - \delta \mathbf{r}_2) = 0 \quad \longrightarrow$$

$$\mathbf{R}_1 = -\mathbf{R}_2 = \alpha \mathbf{e}$$



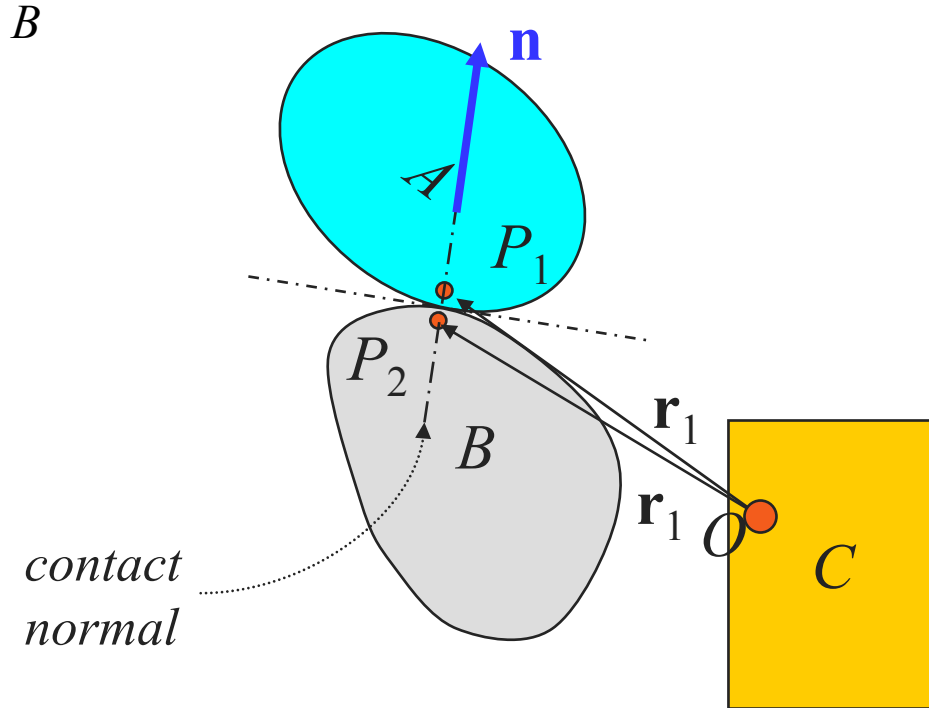
$$\begin{aligned} \mathbf{R}_1 \cdot \delta \mathbf{r}_1 + \mathbf{R}_2 \cdot \delta \mathbf{r}_2 &= \alpha \mathbf{e} \cdot \delta \mathbf{r}_1 - \alpha \mathbf{e} \cdot \delta \mathbf{r}_2 \\ &= \alpha \mathbf{e} \cdot (\delta \mathbf{r}_1 - \delta \mathbf{r}_2) \\ &= 0 \end{aligned}$$

Rigid Body: A System of Particles

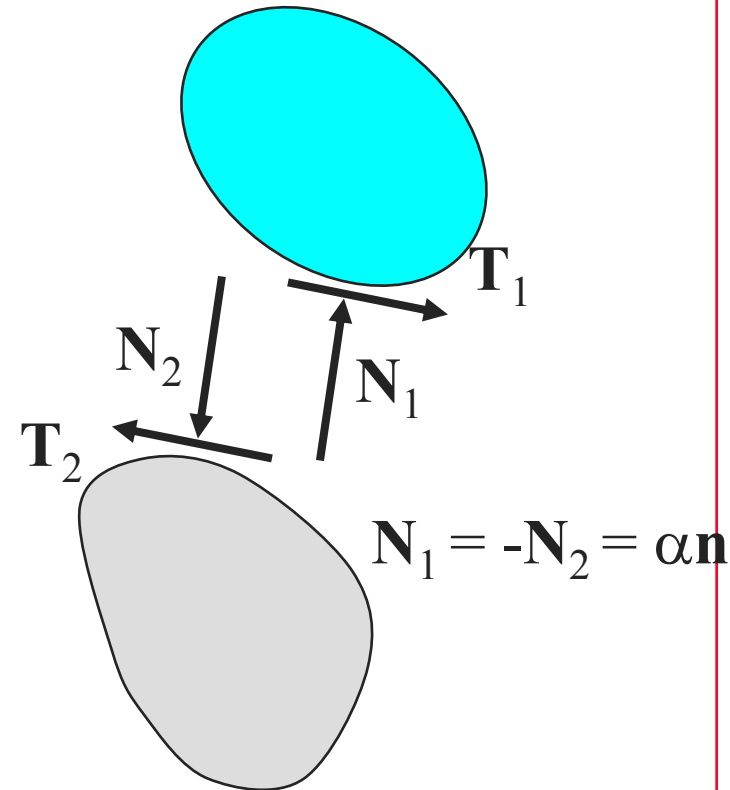
- A rigid body is a system of infinite particles.
- The distance between any pair of particles stays constant through its motion.
- Each pair of particles can be considered as connected by a massless, rigid rod.
- The internal forces associated with this distance constraint are constraint forces.
- The internal forces do no virtual work!

Contact Constraints and Normal Contact Forces

Rigid body A rolls and slides on rigid body B



Contact Forces



Contact Kinematics

$$A\mathbf{v}^{P_2} \cdot \mathbf{n} = B\mathbf{v}^{P_1} \cdot \mathbf{n}$$

$$C\mathbf{v}^{P_2} \cdot \mathbf{n} = C\mathbf{v}^{P_1} \cdot \mathbf{n}$$

$$\delta \mathbf{r}_2 \cdot \mathbf{n} = \delta \mathbf{r}_1 \cdot \mathbf{n}$$

$$\begin{aligned} \mathbf{N}_1 \cdot \delta \mathbf{r}_1 + \mathbf{N}_2 \cdot \delta \mathbf{r}_2 &= \alpha \mathbf{n} \cdot \delta \mathbf{r}_1 - \alpha \mathbf{n} \cdot \delta \mathbf{r}_2 \\ &= \alpha \mathbf{n} \cdot (\delta \mathbf{r}_1 - \delta \mathbf{r}_2) \\ &= 0 \end{aligned}$$

Normal and Tangential Contact Forces

1. Normal contact forces

- Normal contact forces are constraint forces
- Equivalently, normal forces do no virtual work

2. Tangential contact forces

- If A rolls on B (equivalently B rolls on A)

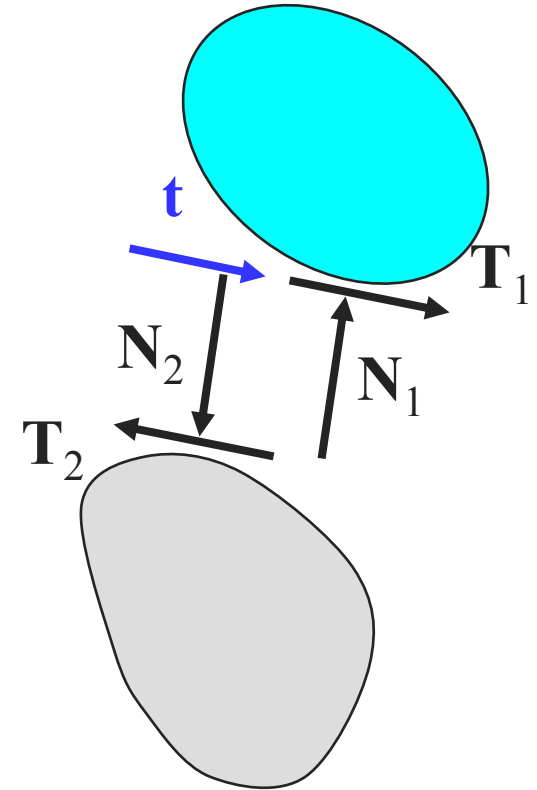
$$A_{\mathbf{V}}P_2 = B_{\mathbf{V}}P_1$$

then, tangential contact forces are constraint forces

- In general (sliding with friction), tangential forces will contribute to virtual work

$$A_{\mathbf{V}}P_2 \neq B_{\mathbf{V}}P_1$$

$$\begin{aligned} \mathbf{T}_1 \cdot \delta \mathbf{r}_1 + \mathbf{T}_2 \cdot \delta \mathbf{r}_2 &= \beta \mathbf{t} \cdot \delta \mathbf{r}_1 - \beta \mathbf{t} \cdot \delta \mathbf{r}_2 \\ &= \beta \mathbf{t} \cdot (\delta \mathbf{r}_1 - \delta \mathbf{r}_2) \\ &\neq 0 \end{aligned}$$



Statement

A system of N particles (P_1, P_2, \dots, P_N) is in static equilibrium if and only if the virtual work done by all the applied (active) forces through any (arbitrary) virtual displacement is zero.


$$\delta W = \sum_{j=1}^n \left\{ \sum_{i=1}^N \left[\mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}^{P_i}}{\partial q_j} \right] \right\} \delta q_j = 0$$

A holonomic system of N particles is in static equilibrium if and only if all the **generalized (active) forces** are zero.

- Only “applied” or “active” forces contribute to the generalized force
- The j th generalized force is given by

$$Q_j = \sum_{i=1}^N \left[\mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}^{P_i}}{\partial q_j} \right]$$

$$= \sum_{i=1}^N \left[\mathbf{F}_i^{(a)} \cdot \frac{\partial \dot{\mathbf{r}}^{P_i}}{\partial \dot{q}_j} \right]$$


Why?

Velocity Partialals

In any frame A

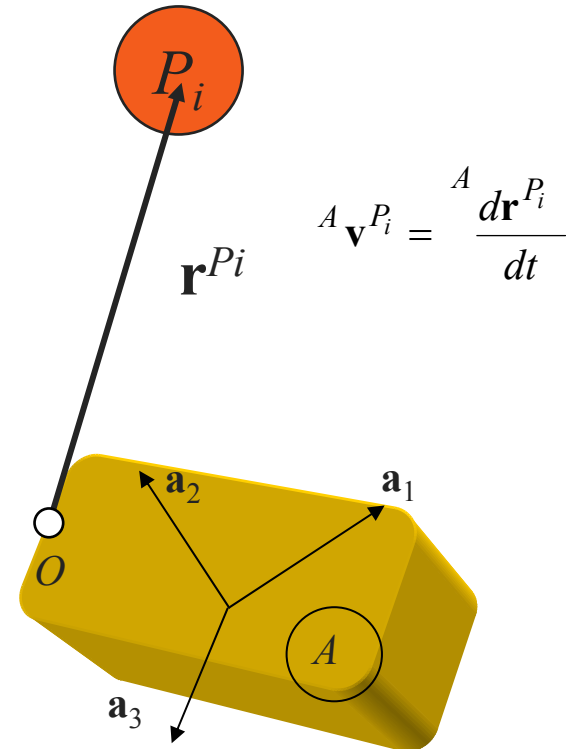
$$\mathbf{r}^{P_i} = \mathbf{r}^{P_i}(q_1, q_2, \dots, q_n, t)$$

n speeds

$${}^A \mathbf{v}^{P_i} = \frac{d\mathbf{r}^{P_i}}{dt}$$

$$= \frac{\partial \mathbf{r}^{P_i}}{\partial t} + \frac{\partial \mathbf{r}^{P_i}}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial \mathbf{r}^{P_i}}{\partial q_2} \frac{dq_2}{dt} + \dots + \frac{\partial \mathbf{r}^{P_i}}{\partial q_n} \frac{dq_n}{dt}$$

$${}^A \mathbf{v}^{P_i} = \frac{\partial \mathbf{r}^{P_i}}{\partial t} + \mathbf{v}_1^{P_i} \dot{q}_1 + \mathbf{v}_2^{P_i} \dot{q}_2 + \dots + \mathbf{v}_n^{P_i} \dot{q}_n$$



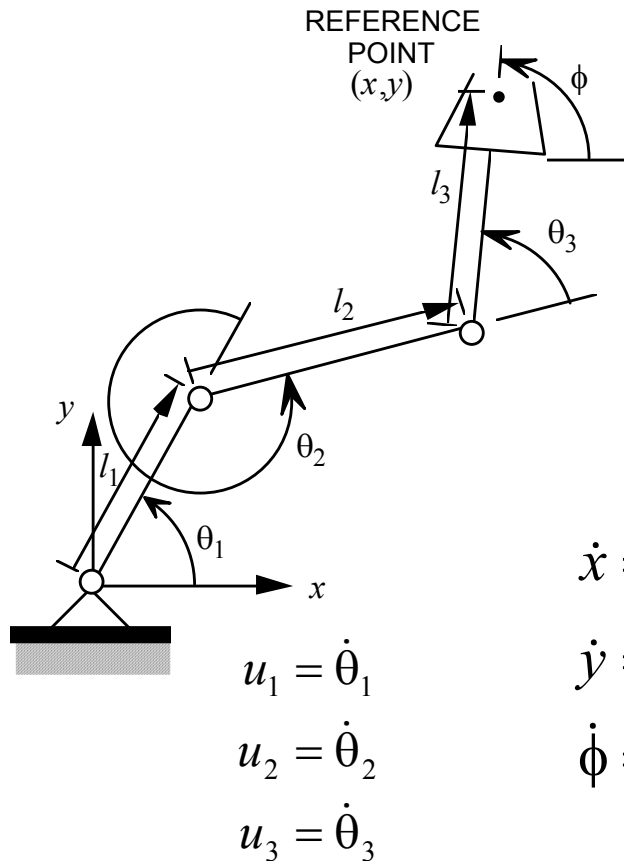
Define the j th velocity partial

$$\mathbf{v}_j^{P_i} = \frac{\partial \mathbf{r}^{P_i}}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}^{P_i}}{\partial \dot{q}_j}$$

The j th generalized force is $Q_j = \sum_{i=1}^N [\mathbf{F}_i^{(a)} \cdot \mathbf{v}_j^{P_i}]$

Example Illustrating Partial Velocities

Three Degree-of-Freedom Robot Arm



$$x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

$$\phi = (\theta_1 + \theta_2 + \theta_3)$$

differentiating

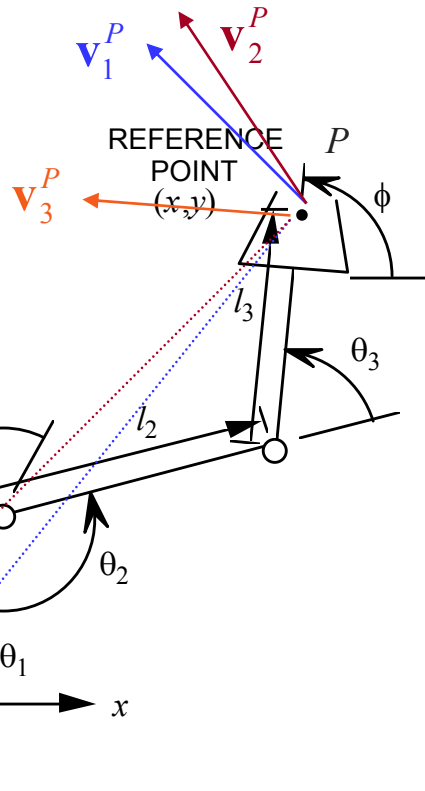
$$\dot{x} = -l_1 \dot{\theta}_1 s_1 - l_2 (\dot{\theta}_1 + \dot{\theta}_2) s_{12} - l_3 (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) s_{123}$$

$$\dot{y} = l_1 \dot{\theta}_1 c_1 + l_2 (\dot{\theta}_1 + \dot{\theta}_2) c_{12} + l_3 (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) c_{123}$$

$$\dot{\phi} = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

Example (continued)

Equations relating the joint velocities and the end effector velocities



$$\dot{x} = -l_1\dot{\theta}_1 s_1 - l_2(\dot{\theta}_1 + \dot{\theta}_2) s_{12} - l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) s_{123}$$

$$\dot{y} = l_1\dot{\theta}_1 c_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) c_{12} + l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) c_{123}$$

in matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12} + l_3 s_{123}) \\ (l_1 c_1 + l_2 c_{12} + l_3 c_{123}) \end{bmatrix} \mathbf{v}_1^P + \begin{bmatrix} -(l_2 s_{12} + l_3 s_{123}) \\ (l_2 c_{12} + l_3 c_{123}) \end{bmatrix} \mathbf{v}_2^P + \begin{bmatrix} -l_3 s_{123} \\ l_3 c_{123} \end{bmatrix} \mathbf{v}_3^P = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

The three partial velocities of the point P (omitting leading superscript A) are columns of the “Jacobian” matrix

Example 1

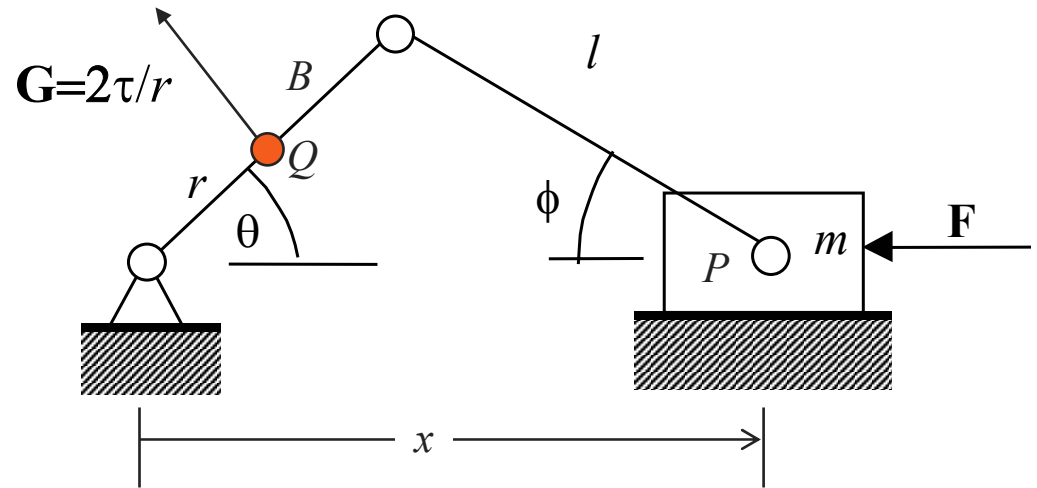
Generalized speed:

- $u = d\theta/dt$

Velocities

- ${}^A \mathbf{v}_1^P = -\frac{R \sin(\theta + \phi)}{\cos \phi} \mathbf{a}_1$

- ${}^A \mathbf{v}_1^Q = -\frac{r}{2} (-\sin \theta \mathbf{a}_1 + \cos \theta \mathbf{a}_2)$



Generalized Active Forces

- $\mathbf{F} = -F \mathbf{a}_1$

- $\mathbf{G} = \frac{2\tau}{r} (-\sin \theta \mathbf{a}_1 + \cos \theta \mathbf{a}_2)$

- No friction, gravity

$$x = r \cos \theta + l \cos \phi; \quad r \sin \theta = l \sin \phi$$

$$\dot{x} = -r \sin \theta \dot{\theta} - l \sin \phi \dot{\phi}; \quad r \cos \theta \dot{\theta} = l \cos \phi \dot{\phi}$$

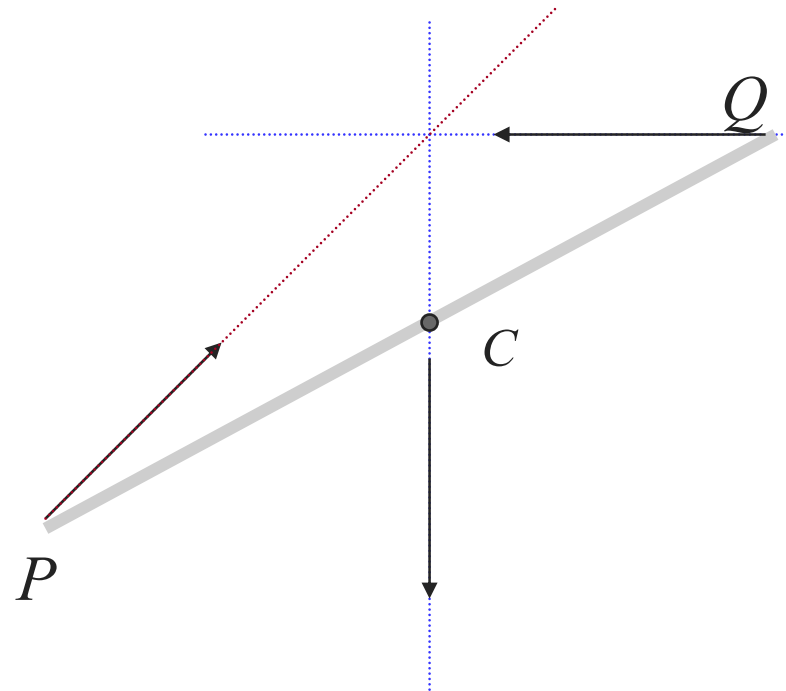
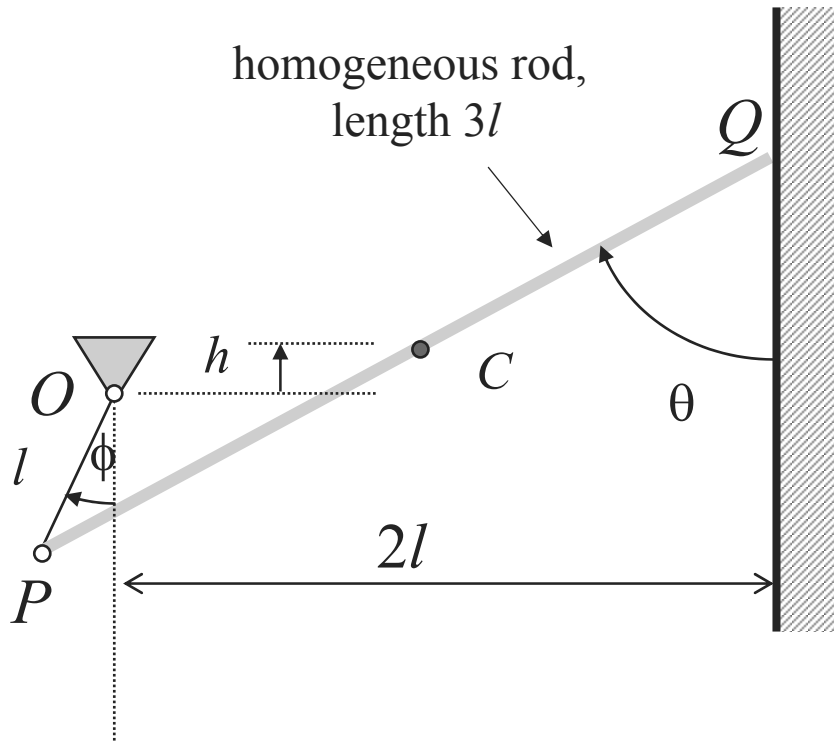
$$\dot{\phi} = \frac{r \cos \theta}{l \cos \phi} \dot{\theta}$$

$$Q_1 = \mathbf{F} \cdot \mathbf{v}_1^P + \mathbf{G} \cdot \mathbf{v}_1^Q \quad \longrightarrow \quad Q_1 = \tau + \frac{FR \sin(\theta + \phi)}{\cos \phi}$$

Example 2

Assumptions

- No friction at the wall
- Gravity (center of mass is at midpoint, C)
- Massless string, OP



Equivalent System of Forces

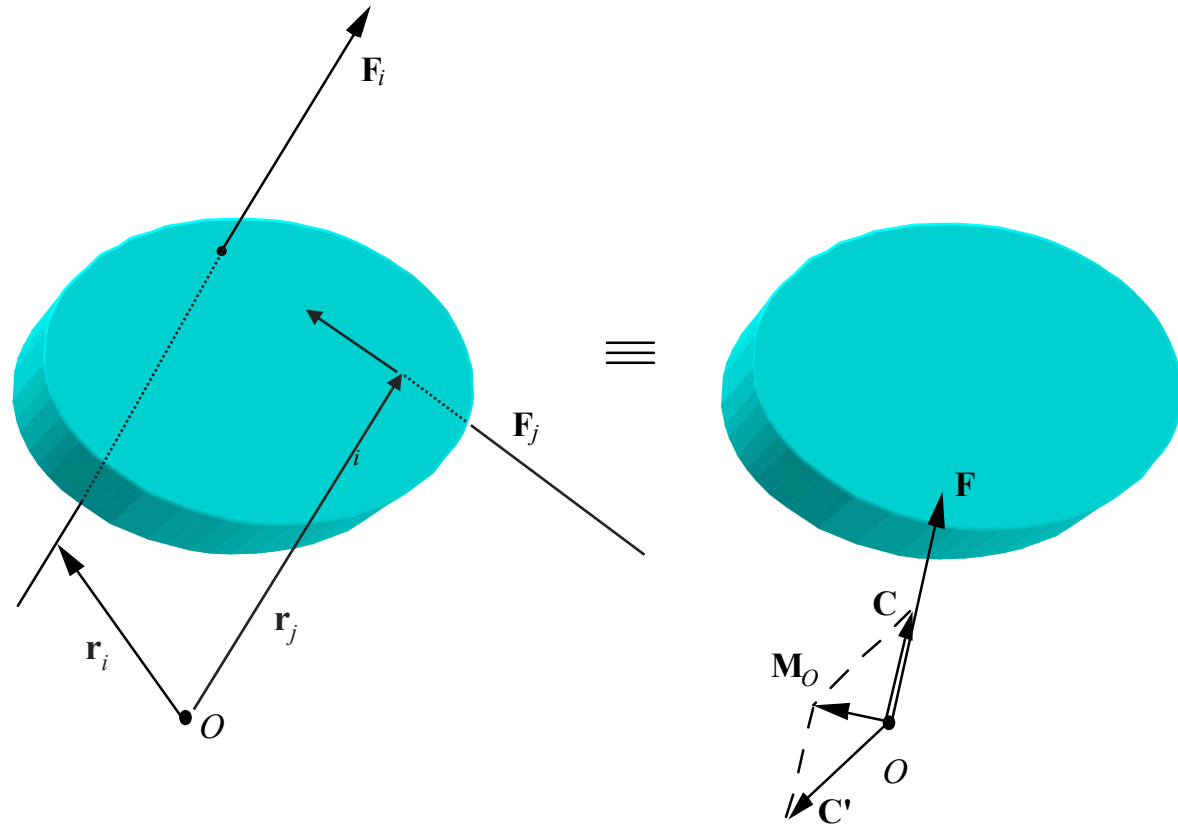
A system of forces acting on a rigid body can be replaced by

- A resultant force \mathbf{F}

$$\mathbf{F} = \sum_{i=1}^r \mathbf{F}_i$$

- A moment about a convenient reference point O

$$\mathbf{M}_O = \sum_{i=1}^r \mathbf{r}_i \times \mathbf{F}_i$$



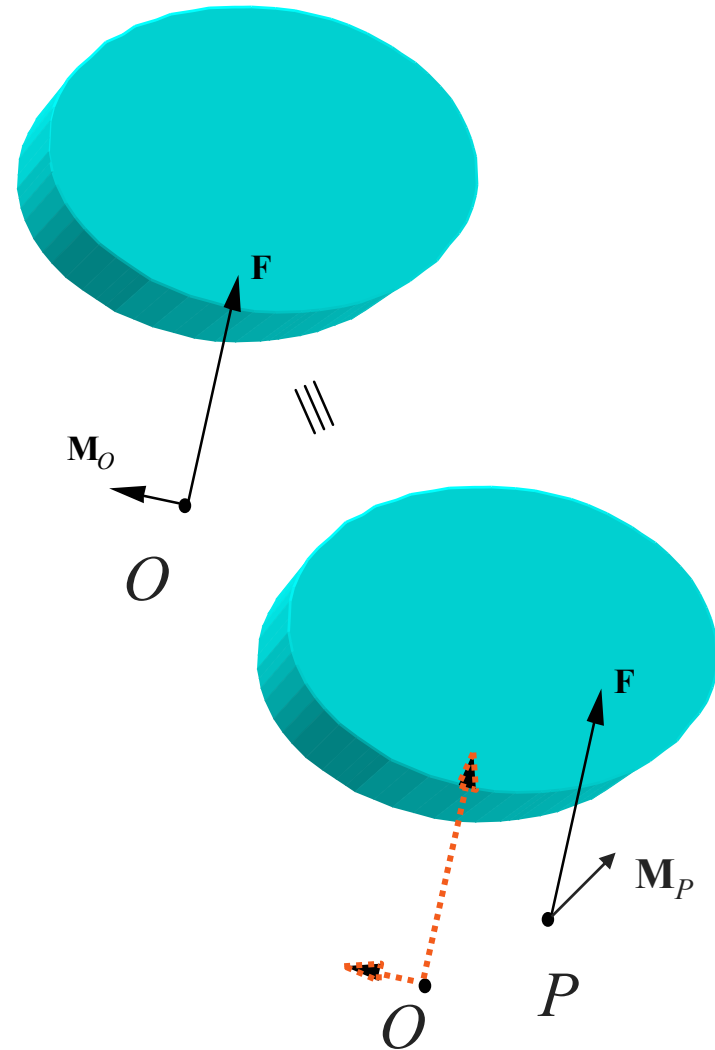
A couple is a set of forces whose resultant force is zero, but the resultant moment is non zero.

Resultant Moment Depends on Reference Point

Resultant force is independent of origin
(reference point)

Resultant moment is dependent on the
origin

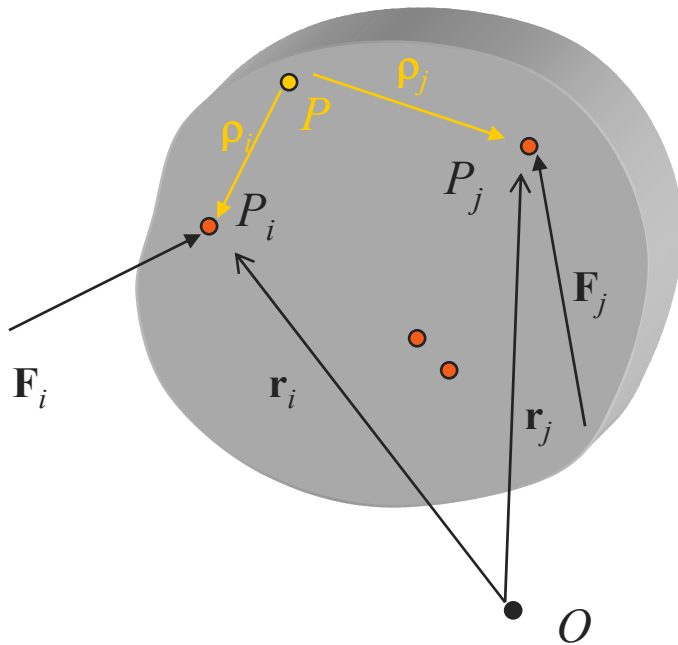
$$\mathbf{M}_P = \mathbf{M}_O + \sum_{i=1}^r \mathbf{r}^{\overline{PO}} \times \mathbf{F}$$



Generalized Forces for Rigid Bodies

Generalized force

$$Q_j = \sum_{i=1}^N [\mathbf{F}_i^{(a)} \cdot \mathbf{v}_j^{P_i}]$$



Velocity partials

$${}^A \mathbf{v}^{P_i} = \frac{\partial \mathbf{r}^{P_i}}{\partial t} + \mathbf{v}_1^{P_i} \dot{q}_1 + \mathbf{v}_2^{P_i} \dot{q}_2 + \dots + \mathbf{v}_n^{P_i} \dot{q}_n$$

But,

$${}^A \mathbf{v}^{P_i} = {}^A \mathbf{v}^P + {}^A \boldsymbol{\omega}^B \times \rho_i$$

Velocity partials can be rewritten

$$\begin{aligned} {}^A \mathbf{v}^{P_i} &= \frac{\partial \mathbf{r}^{P_i}}{\partial t} + \sum_{j=1}^n \frac{\partial [{}^A \mathbf{v}^{P_i}]}{\partial \dot{q}_j} \dot{q}_j \\ &= \frac{\partial \mathbf{r}^{\overline{OP}}}{\partial t} + \frac{\partial ({}^A \boldsymbol{\omega}^B)}{\partial t} \times \mathbf{r}^{\overline{OP}} \\ &\quad + \sum_{j=1}^n \frac{\partial [{}^A \mathbf{v}^P + {}^A \boldsymbol{\omega}^B \times \rho_i]}{\partial \dot{q}_j} \dot{q}_j \end{aligned}$$

Generalized Forces for Rigid Bodies

Generalized force

$$Q_j = \sum_{i=1}^N [\mathbf{F}_i^{(a)} \cdot \mathbf{v}_j^P]$$

Velocity partials

$$\begin{aligned} {}^A \mathbf{v}_j^P &= \frac{\partial {}^A \mathbf{v}^P}{\partial \dot{q}_j} + \frac{\partial ({}^A \boldsymbol{\omega}^B \times \boldsymbol{\rho}_i)}{\partial \dot{q}_j} \\ &= {}^A \mathbf{v}_j^P + {}^A \boldsymbol{\omega}_j^B \times \boldsymbol{\rho}_i \end{aligned}$$

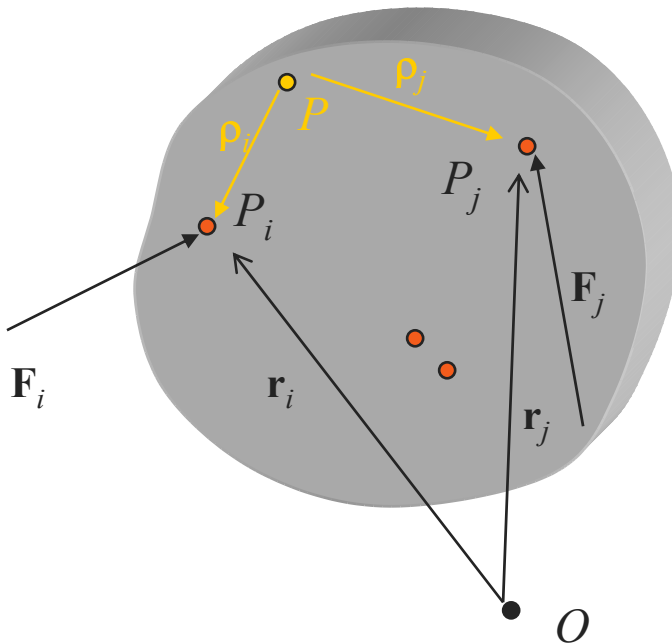
Angular velocity partial

$${}^A \boldsymbol{\omega}_j^B = \frac{\partial ({}^A \boldsymbol{\omega}^B)}{\partial \dot{q}_j}$$

Generalized force can be rewritten

$$Q_j = \sum_{i=1}^N [\mathbf{F}_i \cdot \mathbf{v}_j^P] + \sum_{i=1}^N [\mathbf{F}_i \cdot ({}^A \boldsymbol{\omega}_j^B \times \boldsymbol{\rho}_i)]$$

$$\sum_{i=1}^N [(\boldsymbol{\rho}_i \times \mathbf{F}_i) \cdot {}^A \boldsymbol{\omega}_j^B]$$



$$Q_j = \mathbf{F} \cdot \mathbf{v}_j^P + \mathbf{M}_P \cdot {}^A \boldsymbol{\omega}_j^B$$

Example

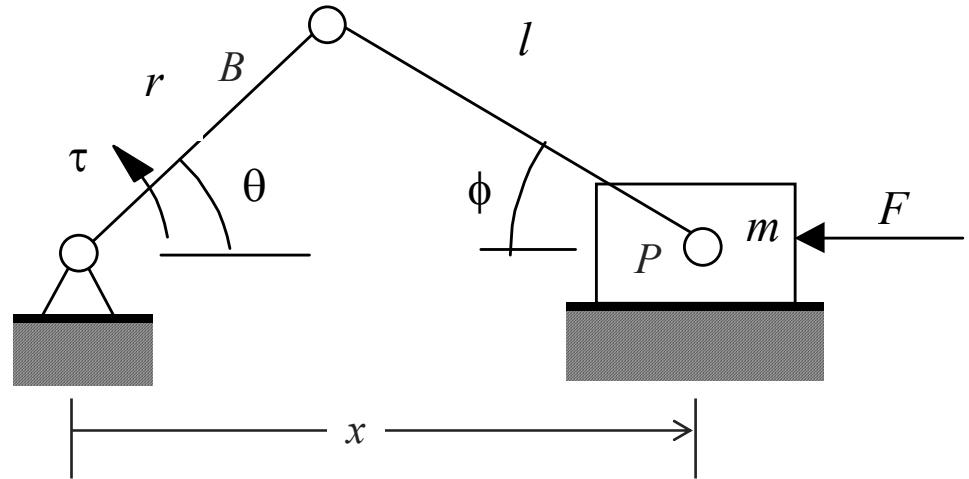
Generalized speed:

- $u = d\theta/dt$

Velocities

- ${}^A\omega_1^B = \mathbf{a}_3$

$${}^A\mathbf{v}_1^P = -\frac{R \sin(\theta + \phi)}{\cos \phi} \mathbf{a}_1$$



Generalized Active Forces

- $-F\mathbf{a}_1$

- $\tau\mathbf{a}_3$

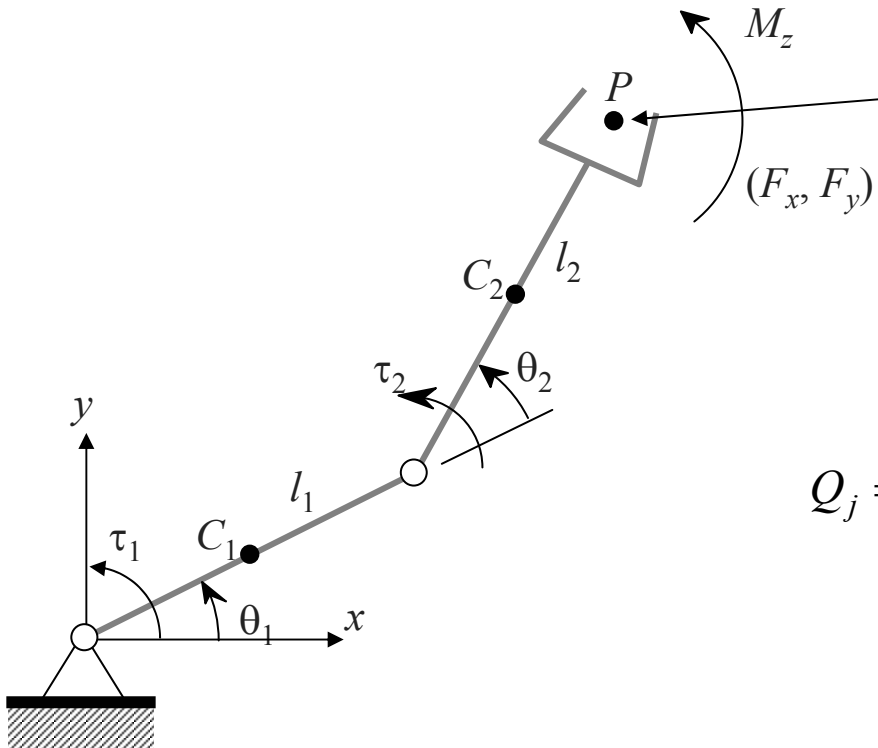
$$Q_1 = \tau + \frac{FR \sin(\theta + \phi)}{\cos \phi}$$

$$x = r \cos \theta + l \cos \phi; \quad r \sin \theta = l \sin \phi$$

$$\dot{x} = -r \sin \theta \dot{\theta} - l \sin \phi \dot{\phi}; \quad r \cos \theta \dot{\theta} = l \cos \phi \dot{\phi}$$

$$\dot{\phi} = \frac{r \cos \theta}{l \cos \phi} \dot{\theta}$$

Example



Generalized coordinates

- (θ_1, θ_2)

Generalized speeds

- (u_1, u_2)

$$u_j = \frac{d\theta_j}{dt}$$

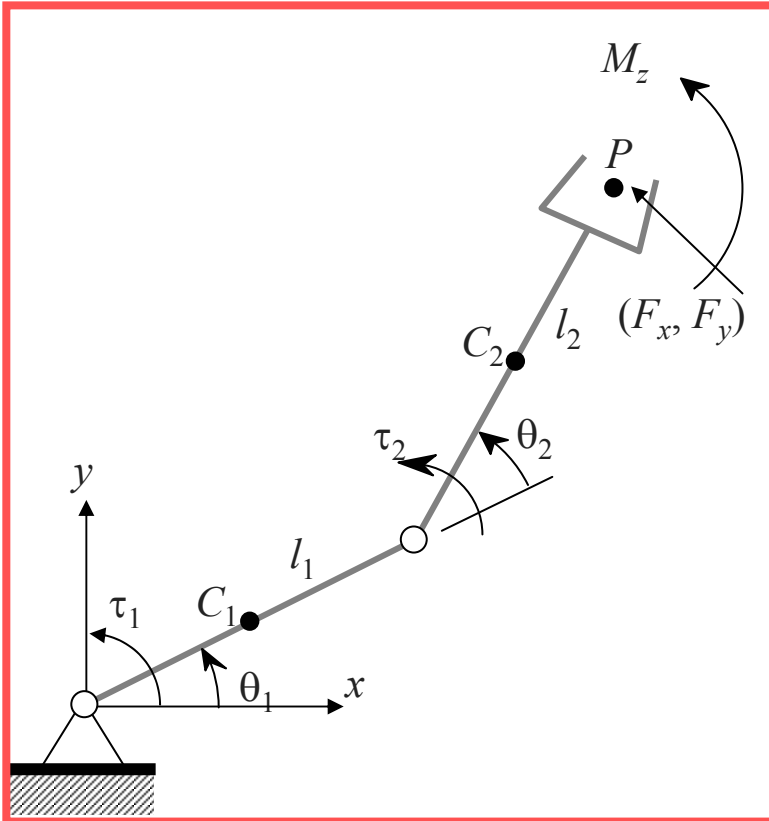
Velocity Partialals

$${}^A\omega_j^{B_1}, \quad {}^A\omega_j^{B_2}, \quad {}^A\mathbf{v}_j^{C_1}, \quad {}^A\mathbf{v}_j^{C_2}, \quad {}^A\mathbf{v}_j^P$$

Generalized forces

$$Q_j = (\tau_1 \mathbf{a}_3 - \tau_2 \mathbf{a}_3) \cdot {}^A\omega_j^{B_1} + \tau_2 \mathbf{a}_3 \cdot {}^A\omega_j^{B_2} - m_1 \mathbf{g} \mathbf{a}_2 \cdot {}^A\mathbf{v}_j^{C_1} \\ - m_2 \mathbf{g} \mathbf{a}_2 \cdot {}^A\mathbf{v}_j^{C_2} + (F_x \mathbf{a}_1 + F_y \mathbf{a}_2) \cdot {}^A\mathbf{v}_j^P + M_z \mathbf{a}_3 \cdot {}^A\omega_j^{B_2}$$

Example (continued)



Generalized forces

$$Q_j = (\tau_1 \mathbf{a}_3 - \tau_2 \mathbf{a}_3) \cdot {}^A \boldsymbol{\omega}_j^{B_1} + \tau_2 \mathbf{a}_3 \cdot {}^A \boldsymbol{\omega}_j^{B_2} - m_1 g \mathbf{a}_2 \cdot {}^A \mathbf{v}_j^{C_1} \\ - m_2 g \mathbf{a}_2 \cdot {}^A \mathbf{v}_j^{C_2} + (F_x \mathbf{a}_1 + F_y \mathbf{a}_2) \cdot {}^A \mathbf{v}_j^P + M_z \mathbf{a}_3 \cdot {}^A \boldsymbol{\omega}_j^{B_2}$$

Velocities

$${}^A \boldsymbol{\omega}^{B_1} = u_1 \mathbf{a}_3, \quad {}^A \boldsymbol{\omega}^{B_2} = (u_1 + u_2) \mathbf{a}_3,$$

$${}^A \mathbf{v}^{C_1} = \frac{1}{2} l_1 (-s_1 \mathbf{a}_1 + c_1 \mathbf{a}_2) u_1$$

$${}^A \mathbf{v}^{C_2} = l_1 (-s_1 \mathbf{a}_1 + c_1 \mathbf{a}_2) u_1 + \frac{1}{2} l_2 (-s_{12} \mathbf{a}_1 + c_{12} \mathbf{a}_2) (u_1 + u_2)$$

$${}^A \mathbf{v}^P = l_1 (-s_1 \mathbf{a}_1 + c_1 \mathbf{a}_2) u_1 + l_2 (-s_{12} \mathbf{a}_1 + c_{12} \mathbf{a}_2) (u_1 + u_2)$$

Generalized Forces

$$Q_1 = \tau_1 - F_x (l_1 s_1 + l_2 s_{12}) + F_y (l_1 c_1 + l_2 c_{12}) \\ - \frac{1}{2} m_1 g l_1 c_1 - m_2 g (l_1 c_1 + \frac{1}{2} l_2 c_{12}) + M_z$$

$$Q_2 = \tau_2 - F_x (l_2 s_{12}) + F_y (l_2 c_{12}) - m_2 g (\frac{1}{2} l_2 c_{12}) + M_z$$

Conservative Holonomic Systems

All applied forces are conservative

Or

There exists a scalar potential function such that all applied forces are given by:

$$\mathbf{F}_i^{(a)} = -\nabla\phi(q_1, q_2, \dots, q_n, t)$$

$$\mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}^{P_i}}{\partial q_j} = - \begin{bmatrix} \frac{\partial \phi}{\partial x_i} & \frac{\partial \phi}{\partial y_i} & \frac{\partial \phi}{\partial z_i} \end{bmatrix} \begin{bmatrix} \frac{\partial x_i}{\partial q_j} \\ \frac{\partial y_i}{\partial q_j} \\ \frac{\partial z_i}{\partial q_j} \end{bmatrix}$$

The virtual work done by the applied forces is:

$$\delta W = \sum_{j=1}^n \left\{ \sum_{i=1}^N \left[\mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}^{P_i}}{\partial q_j} \right] \right\} \delta q_j$$

$$Q_j = -\frac{\partial \phi}{\partial q_j}$$

$$\delta W = -\sum_{j=1}^n \frac{\partial \phi}{\partial q_j} \delta q_j = -\delta \phi$$



Statement

A conservative, holonomic system of N particles (P_1, P_2, \dots, P_N) is in static equilibrium if and only if the change in potential energy through any (arbitrary) virtual displacement is zero.

$$\delta\phi = 0$$