Solution to homework #5

Problem 1

The formula for the pull-in voltage of the lumped model is

$$V_{PI,lumped} = \sqrt{\frac{8kg_0^3}{27\varepsilon_0 A}}$$

The spring constant k for the fixed-fixed beam is given by

$$k = \frac{32Ewt^3}{L^3} \text{ because } \delta_{\max} = \frac{qL^4}{384EI}, k = \frac{qL}{\delta_{\max}}, I = \frac{wt^3}{12}$$

which leads to

$$V_{PI,lumped} = \sqrt{\frac{8\times 32E\not{w}t^3 g_0^3}{27\varepsilon_0(\not{w}L)L^3}} = \sqrt{\frac{9.4815Et^3 g_0^3}{\varepsilon_0 L^4}}$$

If we include the fringing field effect, it becomes

$$V_{PI,lumped} = \sqrt{\frac{9.4815Et^3 g_0^3}{\varepsilon_0 L^4\left(1+0.65\dfrac{g_0}{w}\right)}}$$

For a start, let us choose the following as the pull-in voltage formula for the distributed beam model:

$$V_{PI,beam} = \sqrt{\frac{\alpha Et^3 g_0^3}{\varepsilon_0 L^4}}.$$

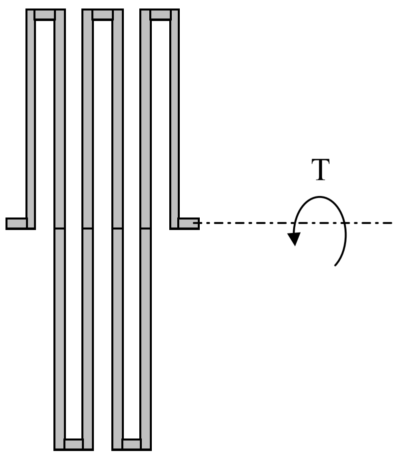
The unknown constant α can be found for a given value of (g_0/w) by determining the value of V_{PI} using `emstatic.m` for some arbitrary values of $L, g_0, w, \text{and } t$. Since the dependence on $L, g_0, \text{and } t$ is correctly captured in the $V_{PI,beam}$ formula above. This value of α will give a very good estimate for the actual V_{PI} computed with `emstatic.m`. The following three cases were considered and verified to be correct.

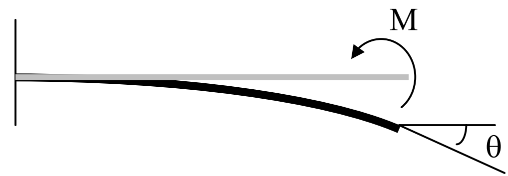
(g_0/w)	α
(1.2/35) = 0.0343	11.5003
(1.2/25) = 0.0480	11.4448
(1.2/15) = 0.0800	11.2782

Using more data points, $\alpha(g_0/w)$ function can be found by a nonlinear fit. It is worth noting that the pull-in voltage given by the lumped model will be very inaccurate whereas the above formula gives a very nice estimate, often accurate up to two decimal points.

Problem 2

The torsional stiffness of the serpentine structure is calculated on the basis of the bending of the vertical beams when a torque is applied about the axis as shown below. Since the horizontal beams in the figure are very small, their twists are neglected.





There are two vertical beams of length p and four of length $2p$. The angular rotation (twist of the serpentine spring) for a vertical beam due to torque T is given by

$$\theta = \frac{Ml}{EI} = \frac{Tp}{EI} \ or \ \frac{T(2p)}{EI}$$

where $I = \frac{bt^3}{12}$.

The total angular displacement for torque T is the summation of the rotations of all the six (four long and two short) beams. Then, we can calculate the angular stiffness κ constant as follows.

$$\theta_{total} = \frac{2Tp}{EI} + \frac{4T(2p)}{EI} = \frac{10Tp}{EI} \Rightarrow \kappa = \frac{EI}{10p}.$$

Since there are two serpentine springs, one on either side, it will be 2κ for the spring constant for the rotation about x and y axes.

The rotational inertia J of the disk is given by

$$J = J_x = J_y = \frac{1}{12}M\left(3\frac{d^2}{4} + t^2\right) = \frac{1}{12}\frac{\pi d^2 t}{4}\left(3\frac{d^2}{4} + t^2\right)$$

In this modeling, we neglect the vertical forces and vertical deflections of the disk and consider only the rotations about x and y axes. The equations of motions are given by

$$J\ddot{\phi}_x + \kappa\phi_x = T_x$$
$$J\ddot{\phi}_y + \kappa\phi_y = T_y$$

The coupling between the two axes of rotation arises due to the torque created by the electrostatic force on the disk. For this, we simply calculate the net torque by numerically integrating the electrostatic force computed using the parallel-plate approximation. For this we need to know the z -height of the point in the disk for given ϕ_x and ϕ_y.

Consider the rotation matrix approach to find the z -height of a point (x , y) of the disk.

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & \sin\phi_x \\ 0 & -\sin\phi_x & \cos\phi_x \end{bmatrix} \begin{bmatrix} \cos\phi_y & 0 & -\sin\phi_y \\ 0 & 1 & 0 \\ \sin\phi_y & 0 & \cos\phi_y \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

This gives the coordinates of all point in the disk. Note that $x = r\cos\alpha, y = r\sin\alpha, z = g_0$ for the disk in the zero position. When the disk rotates, these coordinates will change according to the rotation matrices above. The integration (summation in the discretized sense) is done over the area of the disk under which the electrode is activated. Use $\Delta\alpha = 5\pi/360$ and $\Delta r = a/20$ for the purpose of discretization. Then, the area of the parallel plate for each discretized point is $(\Delta r)(r\Delta\alpha)$. Then, the toques are given as

$$T_x = \sum \frac{\varepsilon_0 (\Delta r)(r\Delta\alpha)V^2}{2(z')^2} y'$$

$$T_y = \sum \frac{\varepsilon_0 (\Delta r)(r\Delta\alpha)V^2}{2(z')^2} x'$$