Solution to homework #6

Problem 1

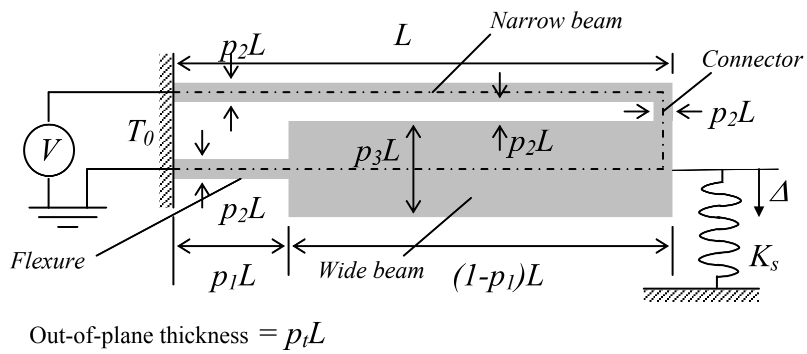


Fig. 1 Basic electro-thermal microactuator

Electrical analysis

The four segments (viz. narrow beam, connector, wide beam, and flexure) in the actuator can be treated as electrical resistors in series. The resistance of an axial conductor of this type is given by the electrical resistivity times the length divided by the cross-section area. Thus, we have

$$R_i = \frac{\rho_e L_i}{A_i} \text{ for } i = 1,2,3,4 \tag{1}$$

Noting the relative proportions shown in Fig. 1, we get the combined series-resistance as

$$R = R_1 + R_2 + R_3 + R_4 = \frac{\rho_e}{L}\left\{\frac{1}{p_t p_2} + \frac{1}{p_t} + \frac{(1-p_1)}{p_t p_3} + \frac{p_1}{p_t p_2}\right\} = \phi_e \frac{\rho_e}{L} \tag{2}$$

The current J and the electrical input power P_e are given by

$$J = \frac{V}{R} = \frac{LV}{\phi_e \rho_e} \tag{3}$$

$$P_e = J^2 R = \frac{LV^2}{\phi_e \rho_e} \tag{4}$$

The Joule heating per unit volume per unit time in each of the four segments, i.e., $\dot{Q}_{e_i}$ $(i = 1 \cdots 4)$, which acts as the heat source for the thermal analysis, can be written as follows using Eqs. (1) through (3).

$$\dot{Q}_{e_i} = \frac{J^2 R_i}{A_i L_i} = \frac{L^2 V^2}{\phi_e^2 A_i^2 \rho_e} \text{ for } i = 1,2,3,4 \tag{5}$$

i.e,

$$\dot{Q}_{e_1} = \frac{V^2}{\phi_e^2 p_t^2 p_2^2 L^2 \rho_e}$$
$$\dot{Q}_{e_2} = \frac{V^2}{\phi_e^2 p_t^2 p_2^2 L^2 \rho_e}$$
$$\dot{Q}_{e_3} = \frac{V^2}{\phi_e^2 p_t^2 p_3^2 L^2 \rho_e} \qquad (6)$$
$$\dot{Q}_{e_4} = \frac{V^2}{\phi_e^2 p_t^2 p_2^2 L^2 \rho_e}$$

Thermal analysis

One-dimensional thermal conduction modeling is appropriate for the slender segments of this microactuator. Since the connector segment is very short, the temperature distribution in it is neglected but the heat generated in it is taken into account in balancing the flux across its two interfaces. The temperature $T_i(s)$ in each of the three remaining segments is governed by the diffusion equation shown below.

$$\frac{d^2 T_i(s)}{ds^2} + \frac{\dot{Q}_{e_i}}{k_t} = 0 \text{ for } i = 1,3,4 \qquad (7)$$

where s runs from zero through the length of the respective segment. With convection and radiation neglected, the only boundary conditions at either end of each segment are either due to the temperature being equal to the ambient or the continuity of heat flux across the interface between two segments. The differential equation in Eq. (7) can be readily solved as

$$T_i(s) = -\frac{\dot{Q}_{e_i}}{2k_t} s^2 + a_i s + b_i \qquad (8)$$

where the constants a_i and b_i ($i = 1,3,4$) are solved using the following six boundary conditions:

1. Temperature at the left end is at the ambient temperature:

$$T_1(s=0) = T_0 \text{, i.e., } b_1 = T_0 \qquad (9a)$$

2. Temperature at the right end of the narrow beam is equal to that at the right end of the wide beam:

$$T_1(s = L_1) = T_3(s = 0) \text{, i.e., } -\frac{\dot{Q}_{e_1}}{2k_t} L^2 + a_1 L + T_0 = b_3 \text{ which can be simplified as}$$

$$a_1 L - b_3 = \frac{\phi_{t_1} V^2}{\phi_e^2 \rho_e k_t} - T_0 = c_1 \text{ with } \phi_{t_1} = \frac{1}{2 p_t^2 p_2^2} \qquad (9b)$$

3. Continuity of heat flux across the connector along with the heat generated in it:

$$-k_t A_1 \left.\frac{dT_1}{ds}\right|_{x=L_1} - k_t A_3 \left.\frac{dT_3}{ds}\right|_{x=0} + \dot{Q}_{e_2} A_2 L_2 = 0$$ which can be simplified as

$$-k_t p_t p_2 L^2 \left(-\frac{\dot{Q}_{e_1}}{k_t} L + a_1 \right) - k_t p_t p_3 L^2 a_3 + \dot{Q}_{e_2} p_t p_2^3 L^3 = 0$$

$$p_2 L a_1 + p_3 L a_3 = \frac{\phi_{t_2} V^2}{\phi_e^2 \rho_e k_t} = c_2 \text{ with } \phi_{t_2} = \frac{1}{p_t}\left(1 - \frac{1}{p_2}\right) \tag{9c}$$

4. Temperature at the left end of the wide beam is equal to that at the right end of the flexure:

$T_3(s = L_3) = T_4(s = 0)$, i.e., $-\frac{\dot{Q}_{e_3}}{2k_t}(1-p_1)^2 L^2 + a_3(1-p_1)L + b_3 = b_4$ which can be simplified as

$$(1-p_1)La_3 + b_3 - b_4 = \frac{\phi_{t_3} V^2}{\phi_e^2 \rho_e k_t} = c_3 \text{ with } \phi_{t_3} = \frac{(1-p_1)^2}{2 p_t^2 p_3^2} \tag{9d}$$

5. Heat flux continuity across the interface between the wide beam and the flexure:

$$k_t A_3 \left.\frac{dT_3}{ds}\right|_{x=L_3} - k_t A_4 \left.\frac{dT_4}{ds}\right|_{x=0} = 0$$ which can be simplified as

$$k_t p_t p_3 L^2 \left(-\frac{\dot{Q}_{e_3}}{k_t}(1-p_1)L + a_3 \right) - k_t p_t p_2 L^2 a_4 = 0$$

$$p_3 L a_3 - p_2 L a_4 = \frac{\phi_{t_4} V^2}{\phi_e^2 \rho_e k_t} = c_4 \text{ with } \phi_{t_4} = \frac{(1-p_1)}{p_t^2 p_3} \tag{9e}$$

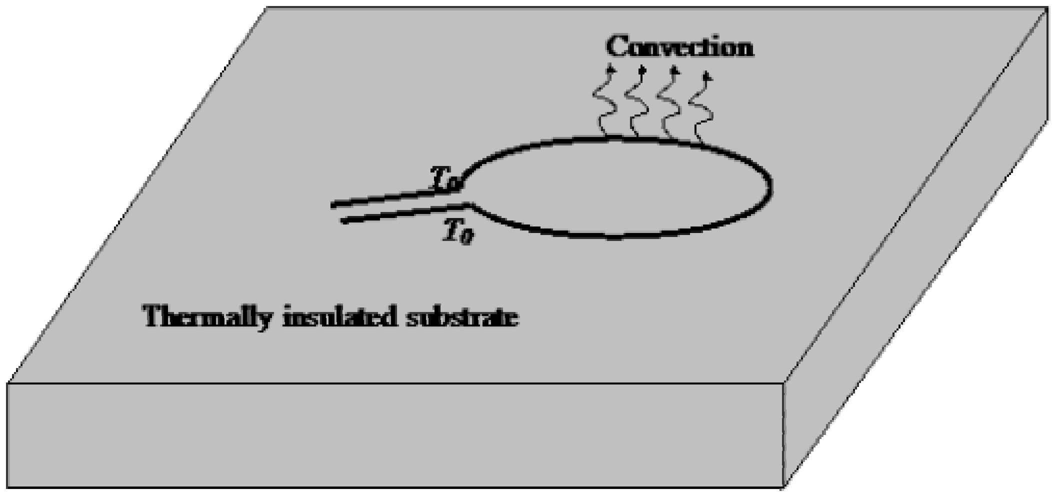
6. Temperature at the left end of the flexure is at the ambient temperature:

$T_4(s = L_4) = T_0$, i.e., $-\frac{\dot{Q}_{e_4}}{2k_t} p_1^2 L^2 + a_4 p_1 L + b_4 = T_0$ which can be simplified as

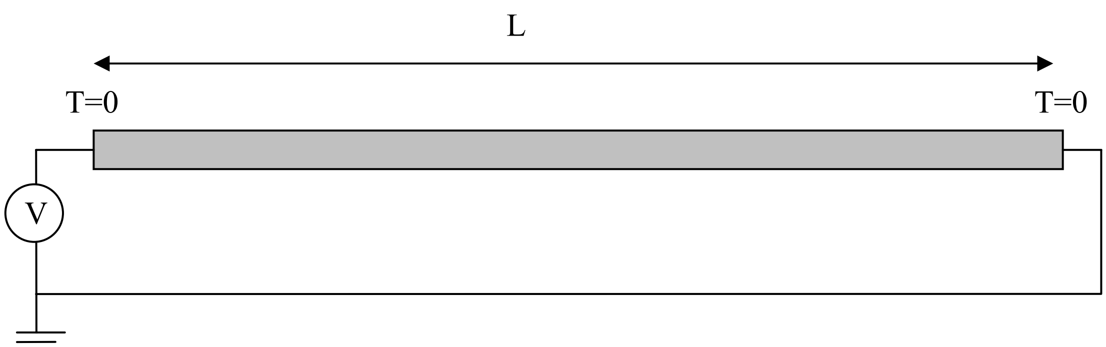
$$p_1 L a_4 + b_4 = \frac{\phi_{t_5} V^2}{\phi_e^2 \rho_e k_t} + T_0 = c_5 \text{ with } \phi_{t_5} = \frac{p_1^2}{2 p_t^2 p_2^2} \tag{9f}$$

The linear equations in Eqs. (9b) through (9f) can be solved for $\{a_1, a_3, b_3, a_4, b_4\}$. Notice that, for consistency, $b_i s$ should have units of temperature, and $a_i s$ temperature per unit length. Thus, in view of Eq. (8), the temperature profile depends on the material properties and relative proportions but not on the size-factor, L.

Problem 2



The circular conductor can be modeled as a 1-D conductor for electrical and thermal analysis.



The electrical resistance at the ambient temperature is given by

$$R_0 = \frac{\rho L}{A} = \frac{\rho(2\pi r)}{wt} \tag{1}$$

Due to TCR effect, the resistance at any other temperature (denoted as temperature raise from the ambient, which is taken as zero) is

$$R = R_0(1+\alpha T) \tag{2}$$

The current can then be calculated as

$$I = \frac{V}{R} = \frac{Vwt}{2\pi\rho r(1+\alpha T)} \tag{3}$$

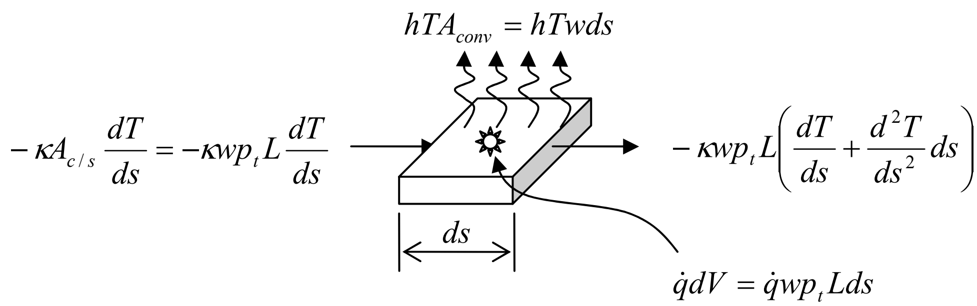
The Joule heating induced power is equal to

$$\dot{Q} = I^2 R = \frac{V^2}{R} = \frac{V^2 wt}{2\pi\rho r\left(1+\alpha T\right)} \tag{4}$$

The above expression gives the total power in the loop. The power generated per unit volume is given by

$$\dot{q} = \frac{V^2}{R(volume)} = \frac{V^2 wt}{2\pi\rho r\left(1+\alpha T\right)} \cdot \frac{1}{2\pi rwt} = \frac{V^2}{4\pi^2 r^2 \rho(1+\alpha T)}$$

For 1-D thermal analysis, let us consider a differential element and sum up the heat entering, leaving it, and being generated within it.



By balancing heat we get

$$\kappa wp_t L\frac{d^2T}{ds^2}ds - hTwds + \dot{q}wp_t Lds = 0$$

$$\Rightarrow \kappa\frac{d^2T}{ds^2} - \frac{hT}{p_t L} + \dot{q} = 0 \tag{5}$$

In view of Eq. (4), Eq. (5) becomes

$$\kappa\frac{d^2T}{ds^2} - \frac{hT}{p_t L} + \frac{V^2}{4\pi^2 r^2 \rho(1+\alpha T)} = 0 \tag{6}$$

which is an inhomogeneous second order differential equation that is difficult to solve analytically (at least based on inspection). The boundary conditions are: $T_{s=0} = T_{s=2\pi r} = 0$. So, it is a two-point boundary value problem (BVP).

We will use bvp4c function in Matlab to solve Eq. (6). For this purpose, we define two functions as follows.

$$\begin{aligned} y_1(s) &= T(s) \\ y_2(s) &= \frac{dT(s)}{ds} \end{aligned} \tag{7}$$

From Eqs. (7) and (6), we can write

$$\begin{Bmatrix} \dfrac{dy_1}{ds} \\ \\ \dfrac{dy_2}{ds} \end{Bmatrix} = \begin{Bmatrix} y_2 \\ \\ \dfrac{1}{\kappa}\left(\dfrac{hy_1}{w} - \dfrac{V^2}{4\pi^2 r^2 \rho(1+\alpha y_1)}\right) \end{Bmatrix} = 0 \tag{8}$$

The above notation helps understand the Matlab script below.

Save as hw6p2.m

```
% Solution to problem 2 in homework #6 in MEAM 550 in Spring 2004
% The script is below is adapted from Constantin Hatzis's code
% submitted as a solution.
%
% Solution of the two-point boundary value problem (bvp) concerning
% a Joule-heated circular loop with convection from top considering
% TCR effect and zero temperature at the two ends.
clear all
clc
% Global variables
global t kappa alpha V h rho r
% Data
r = 150E-6;
w = 3e-6;
t = 1e-6;
kappa = 146.4;
alpha = 2000E-6;
rho = 4.2E-4;
V = 10;
h = 30;
R0 = rho*2*pi*r/w/t;
% Prepare data for the bvp4c function
L = 2*pi*r;
s = 0:L/100:L; % Discretize the circular loop
% Initial guess for the solution of the bvp
solinit = bvpinit(s,[0 0]);
% Call to bvp4c function to get the solution
soln = bvp4c(@deriv, @bcs, solinit);
y1 = deval(soln,s);
figure(1)
clf
plot(s*1E6,y1(1,:),'LineWidth',2);
xlabel('s (microns)','FontSize',16);
ylabel('T (K)','FontSize',16);
```

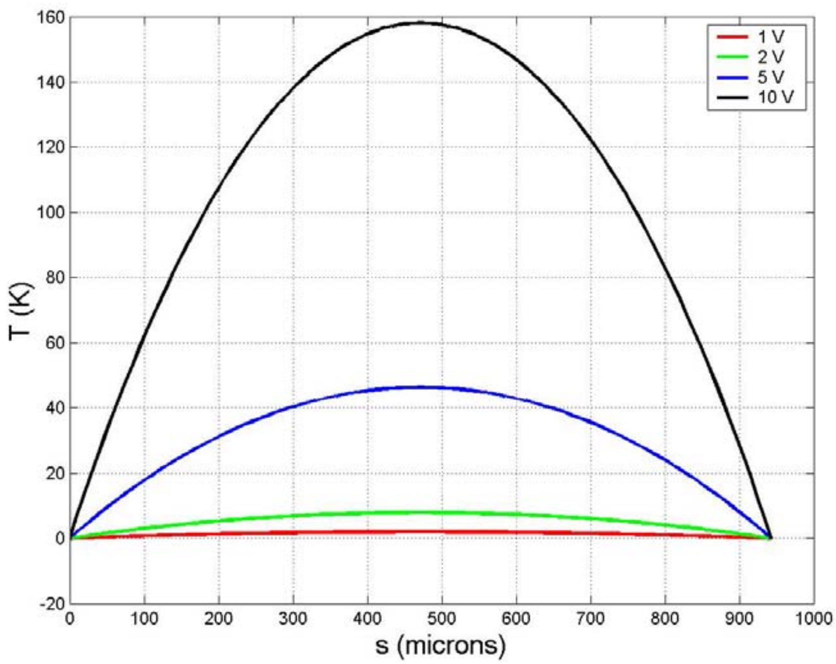
Save as derive.m

```
function dYds = deriv(x,y)
global t kappa alpha V h rho r
d2Tds2 = ( h*y(1)/t - V^2/(4*pi^2*r^2*rho)/(1+alpha*y(1)) ) / kappa;
dYds = [y(2); d2Tds2];
```

Save as bcs.m

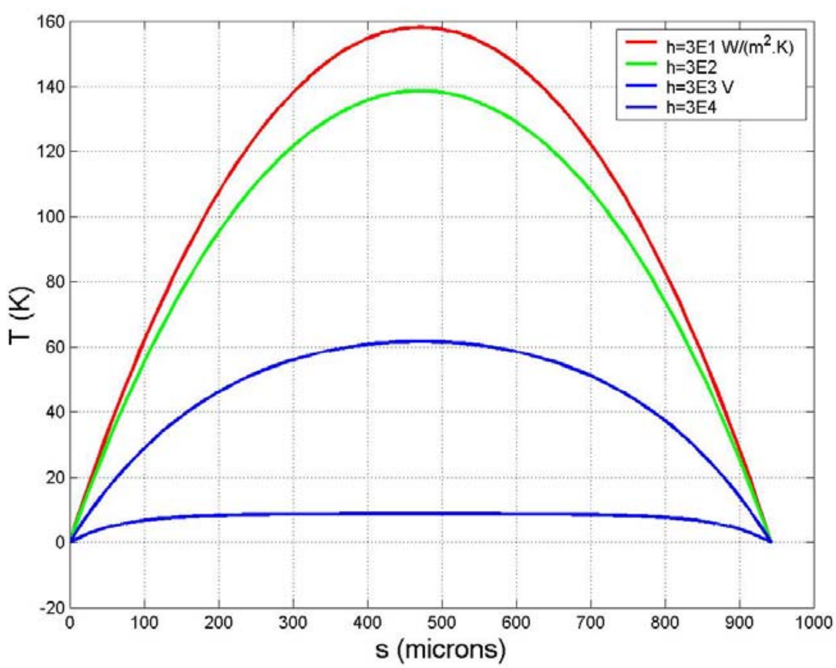
```
function bc = bcs(ya,yb)
bc = [ya(1); yb(1)];
```

The temperature plot for four different voltages is shown below.

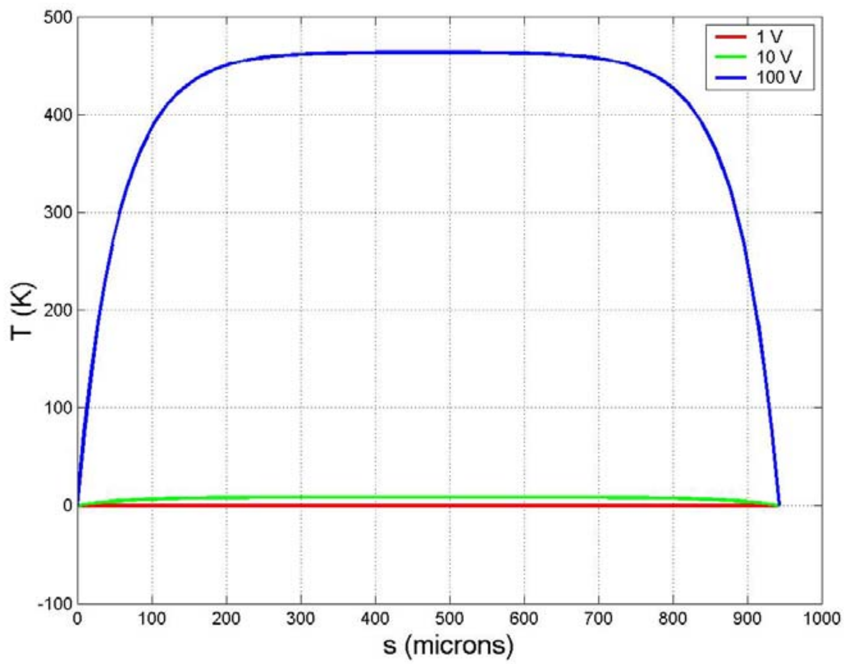


Notice that the maximum voltage does not scale as the square of the voltages. This is because of the TCR effect. You may want to turn off the TCR effect in the Matlab code and see if this is true.

Now, let us see the effect of h on the solution. The following plot shows this for $V = 10V$. As can be seen in the plot, the temperature profile begins to change as the convection effects gets bigger. Furthermore, the maximum temperature also decreases significantly.



Next, for $h = 3E4W/(m^2 K)$, the temperature plot is shown for three different voltages.



Go on play with the Matlab script to see what happens as you change other parameters. These simulations help us get some insight into the design. If you wanted the temperature profile to be constant throughout most of the circular loop, what would you do? That is, how would you choose the parameters?