3. Rigid Body Motion and the Euclidean Group

3.1 Introduction

In the last chapter we discussed points and lines in three-dimensional space, their representations, and how they transform under rigid body. In this chapter, we will develop the fundamental concepts necessary to understand rigid body motion and to analyze instantaneous and finite screw motions. A rigid body motion is simply a rigid body displacement that is a function of time. The first derivative of the motion will give us an expression for the rigid body velocity and this will lead us to the concept of an instantaneous screw. Similarly, higher order derivatives will yield expressions for the acceleration and jerk.

3.2 The Euclidean group

In Chapter 2, we saw that the displacement of a rigid body B can be described in reference frame $\{A\}$, by establishing a reference frame $\{B\}$ on B and describing the position and orientation of $\{B\}$ in $\{A\}$ via a homogeneous transformation matrix:

$$
{}^{A}\mathbf{A}_{B} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
\tag{1}
$$

where ${}^{A}\mathbf{r}^{O'}$ is the position vector of the origin O' of $\{B\}$ in the reference frame $\{A\}$, and ${}^{A}\mathbf{R}_{B}$ is a rotation matrix that transforms the components of vectors in $\{B\}$ into components in $\{A\}$. Recall from Chapter 2, the composition of two displacements, from $\{A\}$ to $\{B\}$, and from $\{B\}$ to $\{C\}$, is achieved by the matrix multiplication of ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$:

$$
\begin{aligned}
{}^{A}\mathbf{A}_{C} &= \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{C} & {}^{A}\mathbf{r}^{O''} \\ \hdashline \mathbf{0} & 1 \end{array}\right] \\
&= \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hdashline \mathbf{0} & 1 \end{array}\right] \times \left[\begin{array}{c:c} {}^{B}\mathbf{R}_{C} & {}^{B}\mathbf{r}^{O''} \\ \hdashline \mathbf{0} & 1 \end{array}\right] \\
&= \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} & {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{r}^{O''} + {}^{A}\mathbf{r}^{O'} \\ \hdashline \mathbf{0} & 1 \end{array}\right]
\end{aligned}
$$

(2)

where the “×” refers to the standard multiplication operation between matrices (and vectors).

The set of all displacements or the set of all such matrices in Equation (1) with the composition rule above, is called $SE(3)$, the *special Euclidean group* of rigid body displacements in three-dimensions:

$$
SE(3) = \left\{ \mathbf{A} \;\middle|\; \mathbf{A} = \left[\begin{array}{c:c} \mathbf{R} & \mathbf{r} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right], \quad \mathbf{R} \in R^{3\times 3}, \mathbf{r} \in R^{3}, \mathbf{R}^{T}\mathbf{R} = \mathbf{R}\mathbf{R}^{T} = \mathbf{I} \right\}
$$

(3)

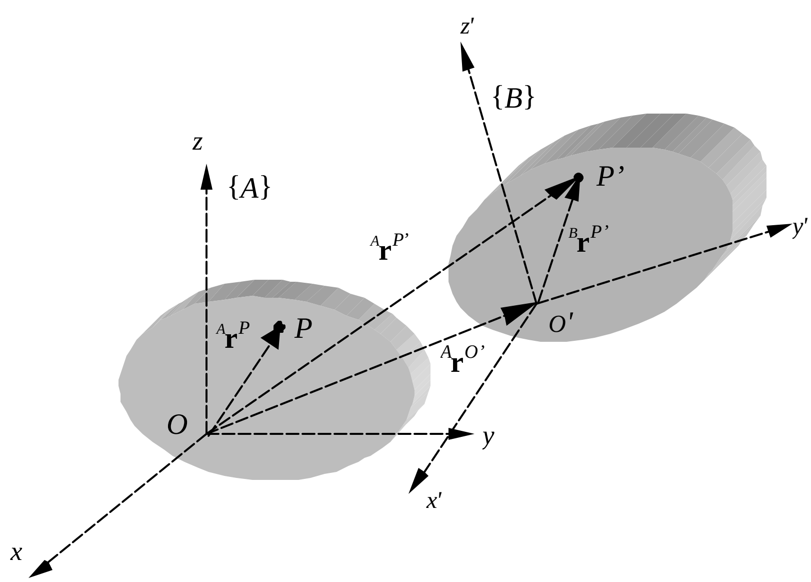

Figure 1 The rigid body displacement of a rigid body from an initial position and orientation to a final position and orientation. The body fixed reference frame is coincident with {*A*} in the initial position and orientation, and with {*B*} in its final position and orientation. The point *P* attached to the rigid body moves from *P* to *P’*.

If we consider this set of matrices with the binary operation defined by matrix multiplication, it is easy to see that $SE(3)$ satisfies the four axioms that must be satisfied by the elements of an *algebraic group*:

1. The set is closed under the binary operation. In other words, if $\mathbf{A}$ and $\mathbf{B}$ are any two matrices in $SE(3)$, $\mathbf{AB} \in SE(3)$.
2. The binary operation is associative. In other words, if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any three matrices $\in SE(3)$, then $(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$.
3. For every element $\mathbf{A} \in SE(3)$, there is an identity element given by the 4×4 identity matrix, $\mathbf{I} \in SE(3)$, such that $\mathbf{AI} = \mathbf{A}$.
4. For every element $\mathbf{A} \in SE(3)$, there is an identity inverse, $\mathbf{A}^{-1} \in SE(3)$, such that

   $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$.

It can be easily shown that (a) the binary operation in Equation (2) is a continuous operation — the product of any two elements in $SE(3)$ is a continuous function of the two elements; and (b) the inverse of any element in $SE(3)$ is a continuous function of that element. Thus $SE(3)$ is a *continuous group*. We will show that any open set of elements of $SE(3)$ has a 1-1 map onto an open set of R^6. In other words, $SE(3)$ is a *differentiable manifold*. A group that is a differentiable manifold is called a *Lie group*, after the famous mathematician Sophus Lie (1842-1899). Because $SE(3)$ is a *Lie group*, it has many interesting properties that are of interest in screw system theory.

In addition to the special Euclidean group in three dimensions, there are many other groups that are of interest in rigid body kinematics. They are all *subgroups* of $SE(3)$. A subgroup of a group consists of a collection of elements of the group which themselves form a group with the same binary operation. We list some important subgroups and their significance in kinematics in Table 1, and describe their properties below.

Table 1 The important subgroups of $SE(3)$

Subgroup	Notation	Definition	Significance
The group of rotations in three dimensions	$SO(3)$	The set of all proper orthogonal matrices. $SO(3) = \left\{ \mathbf{R} \mid \mathbf{R} \in R^{3\times3}, \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I} \right\}$	All spherical displacements. Or the set of all displacements that can be generated by a spherical joint (*S*-pair).
Special Euclidean group in two dimensions	$SE(2)$	The set of all 3×3 matrices with the structure: $\begin{bmatrix} \cos\theta & \sin\theta & r_x \\ -\sin\theta & \cos\theta & r_y \\ 0 & 0 & 1 \end{bmatrix}$ where θ, r_x, and r_y are real numbers.	All planar displacements. Or the set of displacements that can be generated by a planar pair (*E*-pair).
The group of rotations in two dimensions	$SO(2)$	The set of all 2×2 proper orthogonal matrices. They have the structure: $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where θ is a real number.	All rotations in the plane, or the set of all displacements that can be generated by a single revolute joint (*R*-pair).
The group of translations in n dimensions.	$T(n)$	The set of all $n\times1$ real vectors with vector addition as the binary operation.	All translations in n dimensions. n = 2 indicates planar, n = 3 indicates spatial displacements.
The group of translations in one dimension.	$T(1)$	The set of all real numbers with addition as the binary operation.	All translations parallel to one axis, or the set of all displacements that can be generated by a single prismatic joint (*P*-pair).
The group of cylindrical displacements	$SO(2)\times T(1)$	The Cartesian product of $SO(2)$ and $T(1)$	All rotations in the plane and translations along an axis perpendicular to the plane, or the set of all displacements that can be generated by a cylindrical joint (*C*-pair).
The group of screw displacements	$H(1)$	A one-parameter subgroup of $SE(3)$	All displacements that can be generated by a helical joint (*H*-pair).

3.2.1 The group of rotations

A rigid body B is said to rotate relative to another rigid body A, when a point of B is always fixed in $\{A\}$. Attach the frame $\{B\}$ so that its origin O' is at the fixed point. The vector ${}^{A}\mathbf{r}^{O'}$ is equal to zero in the homogeneous transformation in Equation (1).

$$
{}^{A}\mathbf{A}_{B} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 1} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
$$

The set of all such displacements, also called *spherical displacements*, can be easily seen to form a subgroup of $SE(3)$.

If we compose two rotations, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, the product is given by:

$$
\begin{aligned}
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} &= \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 1} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \times \left[\begin{array}{c:c} {}^{B}\mathbf{R}_{C} & \mathbf{0}_{3\times 1} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \\
&= \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} & \mathbf{0}_{3\times 1} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
\end{aligned}
$$

Notice that only the 3×3 submatrix of the homogeneous transformation matrix plays a role in describing rotations. Further, the binary operation of multiplying 4×4 homogeneous transformation matrices reduces to the binary operation of multiplying the corresponding 3×3 submatrices. Thus, we can simply use 3×3 rotation matrices to represent spherical displacements. This subgroup, is called the special orthogonal group in three dimensions, or simply $SO(3)$:

$$
SO(3) = \left\{ \mathbf{R} \;\middle|\; \mathbf{R} \in R^{3\times 3},\ \mathbf{R}^{T}\mathbf{R} = \mathbf{R}\mathbf{R}^{T} = \mathbf{I} \right\} \tag{4}
$$

The adjective special refers to the fact we exclude orthogonal matrices whose determinants are negative.

It is well known that any rotation can be decomposed into three finite successive rotations, each about a different axis than the preceding rotation. The three rotation angles, called Euler angles, completely describe the given rotation. The basic idea is as follows. If we consider any two reference frames $\{A\}$ and $\{B\}$, and the rotation matrix ${}^{A}\mathbf{R}_{B}$, we can construct two intermediate reference frames $\{M\}$ and $\{N\}$, so that

$$ {}^{A}\mathbf{R}_{B} = {}^{A}\mathbf{R}_{M} \times {}^{M}\mathbf{R}_{N} \times {}^{N}\mathbf{R}_{B} $$

where

1. The rotation from $\{A\}$ to $\{M\}$ is a rotation about the x axis (of $\{A\}$) through ψ;
2. The rotation from $\{M\}$ to $\{N\}$is a rotation about the y axis (of $\{M\}$) through ϕ; and
3. The rotation from $\{N\}$ to $\{B\}$is a rotation about the z axis (of $\{N\}$) through θ.

$$ {}^{A}\mathbf{R}_{B} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \times \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5} $$

Thus any rotation can be viewed as a composition of these three elemental rotations except for rotations at which the Euler angle representation is singular[1]. This in turn means all rotations in an open neighborhood in $SO(3)$ can be described by three real numbers (coordinates). With a little work it can be shown that there is a 1-1, continuous map from $SO(3)$ onto an open set in R^3. This gives $SO(3)$ the structure of a three-dimensional differentiable manifold, and therefore a Lie group.

The rotations in the plane, or more precisely rotations about axes that are perpendicular to a plane, form a subgroup of $SO(3)$, and therefore of $SE(3)$. To see this, consider the *canonical form* of this set of rotations, the rotations about the z axis. In other words, connect the rigid bodies A and B with a revolute joint whose axis is along the z axis in Figure 1. The homogeneous transformation matrix has the form:

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[1] These singularities are easily found by writing out the right hand side of Equation (5) explicitly and identifying points at which the Euler angles are not unique. Note that we have chosen the so-called *x-y-z* representation for Euler angles, in which the first rotation is about the *x*-axis, the second about the *y*-axis and third about the *z*-axis. There are eleven other choices of Euler angle representations which can be derived by choosing different axes for the three elemental rotations. For any rotation, it is always possible to find a suitable non singular Euler angle representation.

$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where θ is the angle of rotation. If we compose two such rotations, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, through θ_1 and θ_2 respectively, the product is given by:

$$
\begin{aligned}
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} &= \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & 0 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

All matrices in this subgroup are the same periodic function of one real variable, θ, given by:

$$
\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

This subgroup is called $SO(2)$. Further, since $\mathbf{R}(\theta_1) \times \mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2)$, we can think of the subgroup as being locally isomorphic[2] to R^1 with the binary operation being addition.

3.2.2 The group of translations

A rigid body B is said to translate relative to another rigid body A, if we can attach reference frames to A and to B that are always parallel. The rotation matrix ${}^{A}\mathbf{R}_{B}$ equals the identity in the homogeneous transformation in Equation (1).

[2] The isomorphism is only local because the map from R^1 to $SO(2)$ is many to one. Strictly speaking, the subgroup is isomorphic to the unit circle in the complex plane with multiplication as the group operation.

$$
{}^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
$$

The set of all such homogeneous transformation matrices is the group of translations in three dimensions and is denoted by $T(3)$.

If we compose two translations, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, the product is given by:

$$
\begin{aligned}
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} &= \left[\begin{array}{c|c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \times \left[\begin{array}{c|c} \mathbf{I}_{3\times 3} & {}^{B}\mathbf{r}^{O''} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \\
&= \left[\begin{array}{c|c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} + {}^{B}\mathbf{r}^{O''} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
\end{aligned}
$$

Notice that only the 3×1 vector part of the homogeneous transformation matrix plays a role in describing translations. Thus we can think of a element of $T(3)$ as simply a 3×1 vector, ${}^{A}\mathbf{r}^{O'}$. Since the composition of two translations is captured by simply adding the two corresponding 3×1 vectors, ${}^{A}\mathbf{r}^{O'}$ and ${}^{B}\mathbf{r}^{O''}$, we can define the subgroup, $T(3)$, as the real vector space R^3 with the binary operation being vector addition.

Similarly, we can describe the two subgroups of $T(3)$, $T(1)$ and $T(2)$, the group of translations in one and two dimensions respectively. Because they are subgroups of $T(3)$, they are also subgroups of $T(3)$. It is worth noting that $T(1)$ consists of all translations along an axis and this is exactly the set of displacements that can be generated by connecting A and B with a single prismatic joint.

3.2.3 The special Euclidean group in two dimensions

If we consider all rotations and translations in the plane, we get the set of all displacements that are studied in *planar kinematics*. These are also the displacements generated by the *Ebene*-pair, the planar E-pair. If we let the rigid body B translate along the x and y axis and rotate about the z axis relative to the frame $\{A\}$, we get the canonical set of homogeneous transformation matrices of the form:

$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & {}^{A}r_x^{O'} \\ -\sin\theta & \cos\theta & 0 & {}^{A}r_y^{O'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where θ is the angle of rotation, and ${}^{A}r_x^{O'}$ and ${}^{A}r_y^{O'}$ are the two components of translation of the origin *O'*. If we compose two such displacements, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, the product is given by:

$$
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & {}^{A}r_x^{O'} \\ -\sin\theta_1 & \cos\theta_1 & 0 & {}^{A}r_y^{O'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & {}^{B}r_x^{O''} \\ -\sin\theta_2 & \cos\theta_2 & 0 & {}^{B}r_y^{O''} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & \left({}^{A}r_x^{O'} + {}^{B}r_x^{O''}\cos\theta_1 + {}^{B}r_y^{O''}\sin\theta_1\right) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & \left({}^{A}r_y^{O'} - {}^{B}r_x^{O''}\sin\theta_1 + {}^{B}r_y^{O''}\cos\theta_1\right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Because the set of matrices can be continuously parameterized by three variables, θ, ${}^{A}r_x^{O'}$, and ${}^{A}r_y^{O'}$, *SE*(2) is a differentiable, three-dimensional manifold.

3.2.4 The one-parameter subgroup in *SE*(3)

The group of cylindrical motions is the group of motions admitted by a *cylindrical pair*, or a *C*-pair. If we let the rigid body *B* translate along and rotate about the *z* axis relative to the frame $\{A\}$, we get the canonical set of homogeneous transformation matrices of the form:

$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where θ is the angle of rotation and k is the translation. The set of such matrices is continuously parameterized by these two variables. Thus, this subgroup is a two-dimensional Lie group. In fact, it is nothing but the Cartesian product $SO(2) \times T(1)$. Physically this means we can realize the cylindrical pair by arranging a revolute joint and a prismatic joint in series (in any order) along the same axis.

A one-dimensional subgroup is obtained by coupling the translation and the rotation so that they are proportional. The canonical homogeneous transformation matrix is of the form:

$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & h\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where h is a scalar constant called the pitch. Because the displacement involves a rotation and a co-axial translation that is linearly coupled to the rotation, this displacement is called a *screw displacement*. It is exactly the displacement generated by a *helical pair*, or the *H*-pair.

$$
\begin{aligned}
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} &= \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & h\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & h\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & 0 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & 0 \\ 0 & 0 & 1 & h(\theta_1+\theta_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

The set of all screw displacements about the *z*-axis can be described by a matrix function $\mathbf{A}(\theta)$, with the property $\mathbf{A}(\theta_1) \times \mathbf{A}(\theta_2) = \mathbf{A}(\theta_1 + \theta_2)$. Thus this one-dimensional subgroup is isomorphic to the set R^1 with the binary operation of addition. Such one-dimensional subgroups are called *one-parameter subgroups* and, as we will see later, they have an important geometric significance.

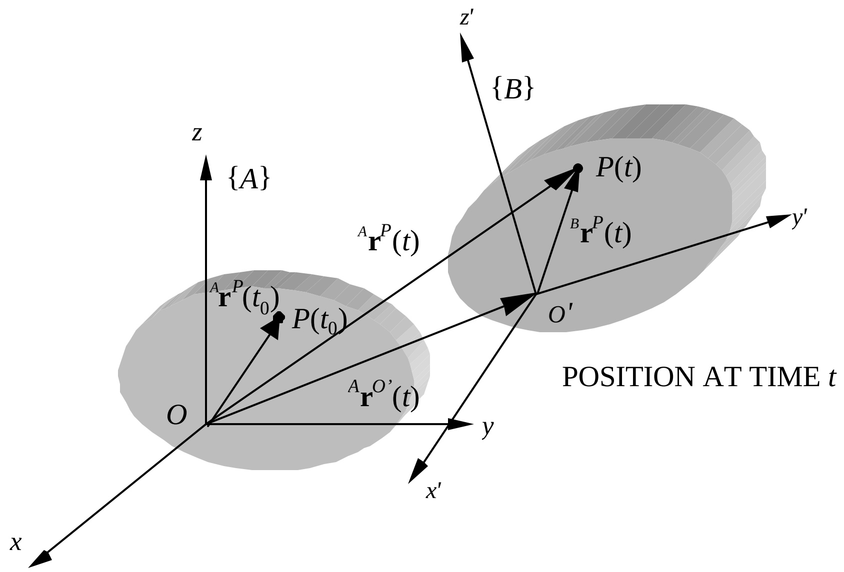

Figure 2 The motion of the rigid body B, as seen from $\{A\}$. The body fixed reference frame is coincident with $\{A\}$ in the initial position and orientation at time t_0, and with $\{B\}$ in the current position and orientation at time t. The point P is attached to the rigid body.

3.3 Velocity analysis

3.3.1 Twist

We study the motion of a rigid body B in the reference frame $\{A\}$ attached to the rigid body A. For all practical purposes, A can be considered to be a fixed rigid body, so that $\{A\}$ can be considered an inertial or an absolute frame. We choose $\{A\}$ to be the frame with which the body fixed reference frame is coincident at some initial time t_0. We consider the body fixed reference frame in its current position and orientation, $\{B\}$, at time t. The homogeneous transformation matrix ${}^{A}\mathbf{A}_{B}(t)$ is a function of time, as is the rotation matrix ${}^{A}\mathbf{R}_{B}(t)$ and the translation vector ${}^{A}\mathbf{r}^{O'}(t)$.

We consider, as before, a generic point P that is attached to the rigid body. In other words, ${}^{A}\mathbf{r}^{P}(t)$ is a function of time, but ${}^{B}\mathbf{r}^{P}(t)$ is a constant which is equal to ${}^{A}\mathbf{r}^{P}(t_0)$. The velocity of the point P on the rigid body B as seen in the reference frame $\{A\}$ is obtained by differentiating the position vector ${}^A\mathbf{r}^P(t_0)$ in the reference frame $\{A\}$:

$$ {}^A\mathbf{v}^P(t) = \frac{d}{dt}\left({}^A\mathbf{r}^P(t) \right) = {}^A\dot{\mathbf{r}}^P(t) $$

where $\dot{\mathbf{a}}$ denotes the differentiation of the quantity $\mathbf{a}$ in the reference frame $\{A\}$.

The velocity ${}^A\mathbf{v}^P(t)$ is found by is given by writing the equation for the position vector,

$$ \begin{bmatrix} {}^A\mathbf{r}^P(t) \\ \hline 1 \end{bmatrix} = {}^A\mathbf{A}_B \begin{bmatrix} {}^A\mathbf{r}^P(t_0) \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^A\mathbf{R}_B(t) & {}^A\mathbf{r}^{O'}(t) \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array} \right] \begin{bmatrix} {}^A\mathbf{r}^P(t_0) \\ \hline 1 \end{bmatrix}, \tag{6} $$

and differentiating it with respect to time:

$$ \begin{bmatrix} {}^A\mathbf{v}^P(t) \\ \hline 1 \end{bmatrix} = {}^A\dot{\mathbf{A}}_B(t) \begin{bmatrix} {}^A\mathbf{r}^P(t_0) \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} \frac{d}{dt}\left({}^A\mathbf{R}_B(t) \right) & \frac{d}{dt}\left({}^A\mathbf{r}^{O'}(t) \right) \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array} \right] \begin{bmatrix} {}^A\mathbf{r}^P(t_0) \\ \hline 1 \end{bmatrix}, $$

Substituting for ${}^A\mathbf{r}^P(t_0)$ from Equation (6), we get:

$$ \begin{aligned} \begin{bmatrix} {}^A\mathbf{v}^P(t) \\ \hline 1 \end{bmatrix} &= {}^A\dot{\mathbf{A}}_B \left[{}^A\mathbf{A}_B \right]^{-1} \begin{bmatrix} {}^A\mathbf{r}^P(t) \\ \hline 1 \end{bmatrix} \\ &= \left[\begin{array}{c|c} {}^A\dot{\mathbf{R}}_B(t) & {}^A\dot{\mathbf{r}}^{O'}(t) \\ \hline \mathbf{0}_{1\times 3} & 0 \end{array} \right] \left[\begin{array}{c|c} \left[{}^A\mathbf{R}_B(t) \right]^T & -\left[{}^A\mathbf{R}_B(t) \right]^T \, {}^A\mathbf{r}^{O'}(t) \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array} \right] \begin{bmatrix} {}^A\mathbf{r}^P(t) \\ \hline 1 \end{bmatrix} \\ &= \left[\begin{array}{c|c} {}^A\dot{\mathbf{R}}_B(t) \left[{}^A\mathbf{R}_B(t) \right]^T & {}^A\dot{\mathbf{r}}^{O'}(t) - {}^A\dot{\mathbf{R}}_B(t) \left[{}^A\mathbf{R}_B(t) \right]^T \, {}^A\mathbf{r}^{O'}(t) \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array} \right] \begin{bmatrix} {}^A\mathbf{r}^P(t) \\ \hline 1 \end{bmatrix} \end{aligned} \tag{7} $$

Thus the velocity of any point P on the rigid body B in the reference frame $\{A\}$ can be obtained by premultiplying the position vector of the point P in $\{A\}$ with the matrix, ${}^A\mathbf{T}_B$,

$$\left[\begin{array}{c} {}^{A}\mathbf{v}^{P}(t) \\ \hdashline 1 \end{array}\right] = \left[{}^{A}\mathbf{T}_{B}\right]\left[\begin{array}{c} {}^{A}\mathbf{r}^{P}(t_0) \\ \hdashline 1 \end{array}\right], \tag{8}$$

where

$${}^{A}\mathbf{T}_{B} = {}^{A}\dot{\mathbf{A}}_{B}\left[{}^{A}\mathbf{A}_{B}\right]^{-1} = \left[\begin{array}{c:c} {}^{A}\mathrm{W}_{B} & {}^{A}\mathbf{v}^{\hat{O}}(t) \\ \hdashline \mathbf{0}_{3\times 1} & 0 \end{array}\right] \tag{9}$$

and

$$\begin{aligned} {}^{A}\mathrm{W}_{B} &= {}^{A}\dot{\mathbf{R}}_{B}\left[{}^{A}\mathbf{R}_{B}\right]^{T} \\ {}^{A}\mathbf{v}^{\hat{O}} &= {}^{A}\dot{\mathbf{r}}^{O'} - {}^{A}\mathrm{W}_{B}{}^{A}\mathbf{r}^{O'} \end{aligned}.$$

$\hat{O}$ is a point on the rigid body B that we will shortly establish as the point that is instantaneously coincident with the origin of $\{A\}$.

The 3×3 matrix ${}^{A}\mathrm{W}_{B}$ is easily seen to be skew symmetric. Because ${}^{A}\mathbf{R}_{B}$ $[{}^{A}\mathbf{R}_{B}]^{T}$ is the identity matrix, its time derivative is the zero matrix which means

$${}^{A}\dot{\mathbf{R}}_{B}\left[{}^{A}\mathbf{R}_{B}\right]^{T} + {}^{A}\mathbf{R}_{B}\left[{}^{A}\dot{\mathbf{R}}_{B}\right]^{T} = \mathbf{0}$$

or,

$${}^{A}\mathrm{W}_{B} + \left[{}^{A}\mathrm{W}_{B}\right]^{T} = 0$$

Thus ${}^{A}\mathrm{W}_{B}(t)$ is a skew-symmetric matrix operator and has the form:

$${}^{A}\mathrm{W}_{B} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

where ${}^{A}\mathrm{w}_{B}(t) = [\omega_1, \omega_2, \omega_3]^{T}$ is the 3×1 vector associated with the matrix operator.

The physical significance of this operator is immediately seen if we take the special case of a *spherical motion* of B relative to A, in which we can choose the origin of $\{B\}$ to be coincident with the origin of $\{A\}$. In this special case,

$$ {}^{A}\mathbf{r}^{O'} = 0, \quad {}^{A}\dot{\mathbf{r}}^{O'} = 0, $$

and Equation (7) gives us the result,

$$ \begin{aligned} {}^{A}\mathbf{v}^{P}(t) &= {}^{A}W_{B}\ {}^{A}\mathbf{r}^{P} \\ &= {}^{A}w_{B} \times {}^{A}\mathbf{r}^{P}, \end{aligned} $$

which means the vector ${}^{A}w_{B}$ must be the angular velocity vector of the rigid body B as seen in reference frame $\{A\}$. The matrix ${}^{A}W_{B}$ is called the *angular velocity matrix* of the rigid body B as seen in reference frame $\{A\}$[3].

Once we see that ${}^{A}w_{B}$ is the angular velocity vector of the rigid body B in $\{A\}$, we see that

$$ {}^{A}\mathbf{v}^{\hat{O}} = {}^{A}\mathbf{v}^{O'} + {}^{A}w_{B} \times \left(-\,{}^{A}\mathbf{r}^{O'}\right) $$

is the velocity of the point $\hat{O}$ on B whose position is the same as that of the point O on A.

Thus, ${}^{A}\mathbf{T}_{\boldsymbol{B}}$ is essentially a matrix operator that yields the velocity of any point attached to B in frame $\{A\}$. It consists of the angular velocity matrix of $\{B\}$ and the velocity of the point $\hat{O}$, both as seen in frame $\{A\}$. Because ${}^{A}\mathbf{T}_{\boldsymbol{B}}$ depends on only six parameters — the three components of the vector ${}^{A}w_{B}$ and the three components of the linear velocity ${}^{A}\mathbf{v}^{\hat{O}}$ — the six components may be assembled into a 6×1 vector[4] called the *twist vector*:

$$ {}^{A}\mathbf{t}_{B} = \begin{bmatrix} {}^{A}w_{B} \\ \hdashline {}^{A}\mathbf{v}^{\hat{O}} \end{bmatrix} \qquad (10) $$

---

[3] It is worth emphasizing that ${}^{A}\mathbf{r}^{P}$, ${}^{A}w_{B}$, ${}^{A}\mathbf{v}^{P}$, and ${}^{A}W_{B}$ are components of physical quantities in the reference frame $\{A\}$. The choice of $\{A\}$ is somewhat arbitrary, as is the choice of the time t_0. The components themselves will depend on the exact choice of the coordinate system O-x-y-z in Figure 2.

[4] Some authors prefer an ordering with the linear velocity in the first three slots and the angular velocity at the bottom of the 6×1 vector.

In analogy to the two representations of the angular velocity, the *twist* of body B in reference frame $\{A\}$ can be represented either as the *twist matrix* ${}^{A}\mathbf{T}_{B}$ in (9) or as the twist vector ${}^{A}\mathbf{t}_{B}$ in (10). We will pursue the geometric significance of the *twist* in the next subsection.

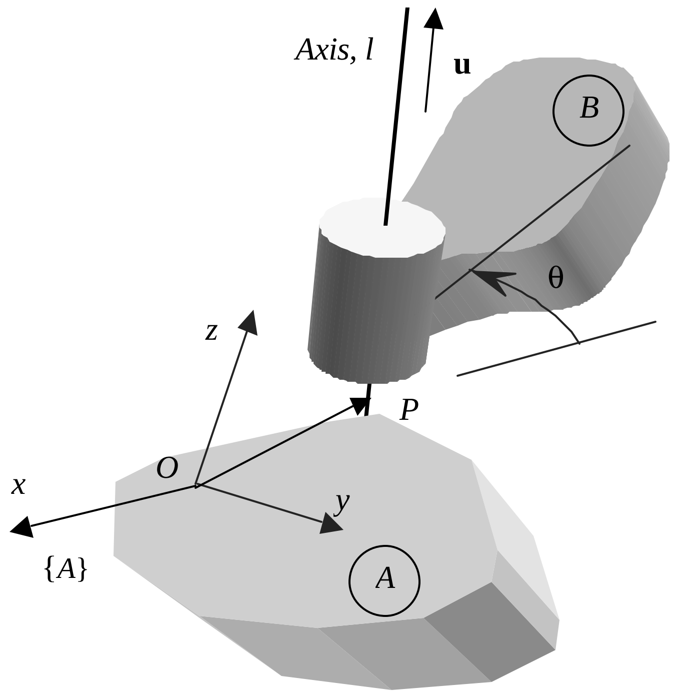

Figure 3 The rigid bodies A and B are connected by a revolute joint with the axis l. $\mathbf{u}$ is a unit vector along the axis and P is a point on the axis. O-x-y-z is the reference frame $\{A\}$.

3.3.2 Instantaneous screw axis

In order to obtain a better understanding of the geometric significance of a twist vector (or matrix), it is productive to first study the two special cases of rigid body rotation and rigid body translation.

Consider the two rigid bodies A and B, connected by a revolute joint with the axis l as shown in Figure 3. $\mathbf{u}$ is a unit vector along the axis and P is a point on the axis. The twist, ${}^{A}\mathbf{t}_{B}$, can be found by inspection to be:

$$
{}^{A}\mathbf{t}_{B} = \begin{bmatrix} {}^{A}\mathrm{w}_{B} \\ \hdashline {}^{A}\mathbf{v}^{\hat{O}} \end{bmatrix} = \begin{bmatrix} {}^{A}\mathbf{u}\dot{\theta} \\ \hdashline {}^{A}\mathbf{r}^{P} \times \mathbf{u}\dot{\theta} \end{bmatrix} = \dot{\theta} \begin{bmatrix} {}^{A}\mathbf{u} \\ \hdashline {}^{A}\mathbf{r}^{P} \times {}^{A}\mathbf{u} \end{bmatrix}
\tag{11}
$$

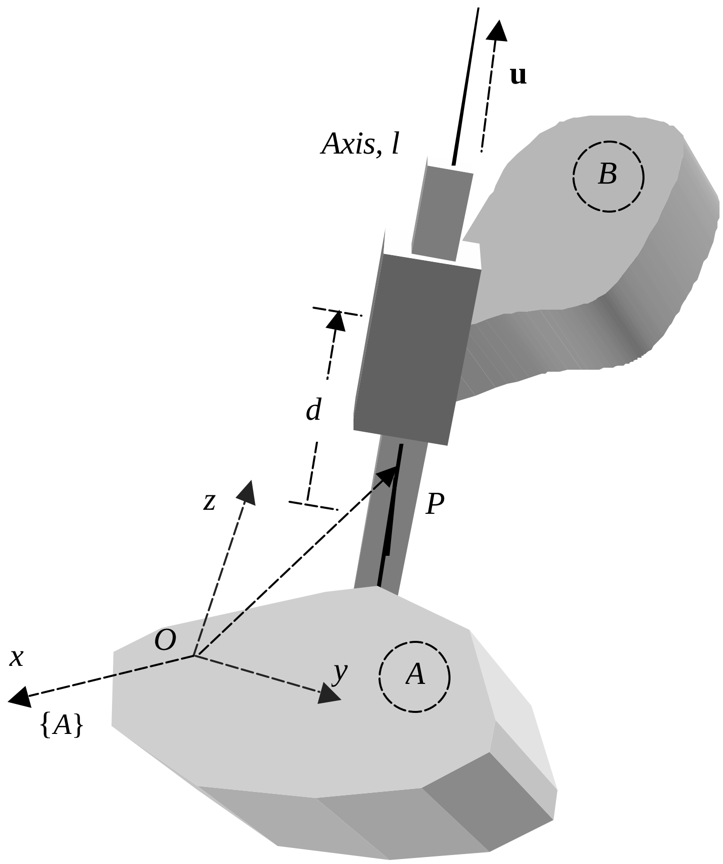

Figure 4 The rigid bodies A and B are connected by a prismatic joint with the axis l. $\mathbf{u}$ is a unit vector along the axis and P is a point on the axis. O-x-y-z is the reference frame $\{A\}$.

Similarly, in Figure 4, the two rigid bodies A and B, are connected by a prismatic joint with an axis[5] parallel to the line l. $\mathbf{u}$ is a unit vector along the axis and P is a point on the axis. The twist, ${}^{A}\mathbf{t}_{B}$, can be found by inspection to be:

[5] The axis for a prismatic joint is not uniquely defined. The direction of translation determines the direction of the axis, but the axis can be any line along this direction.

$$
{}^{A}\mathbf{t}_B = \begin{bmatrix} {}^{A}\mathrm{w}_B \\ \hdashline {}^{A}\mathbf{v}\hat{O} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hdashline {}^{A}\mathbf{u}\dot{d} \end{bmatrix} = \dot{d}\begin{bmatrix} \mathbf{0} \\ \hdashline {}^{A}\mathbf{u} \end{bmatrix}
$$

(12)

In both these cases, the twist vector can be associated with an axis or a line whose Plucker coordinates are easily identified. In Equation (11), the line associated with the twist is the axis given by the unit line vector,

$$
\begin{bmatrix} {}^{A}\mathbf{u} \\ \hdashline {}^{A}\mathbf{r}^{P} \times {}^{A}\mathbf{u} \end{bmatrix},
$$

while in Equation (12), the unit line vector is a line at infinity given by:

$$
\begin{bmatrix} \mathbf{0} \\ \hdashline {}^{A}\mathbf{u} \end{bmatrix}.
$$

In both cases, the twist vector is simply unit line vector multiplied by a scalar quantity which is the rate at which the joint is displaced.

This natural association of a line with a twist vector extends to the most general type of motion. Given any twist ${}^{A}\mathbf{T}_B$, we can always find an axis such that ${}^{A}\mathrm{w}_B$ is parallel to the axis, and points on the boxy B that lie on the axis translate along the axis. In other words, there is an axis such that if the origin is chosen to be at any point on the axis, ${}^{A}\mathrm{w}_B$ and ${}^{A}\mathbf{v}^{P}$ are parallel. This is the "infinitesimal version" of Chasles' theorem, and the axis obtained in this way is called *the instantaneous screw axis*. A proof for this follows.

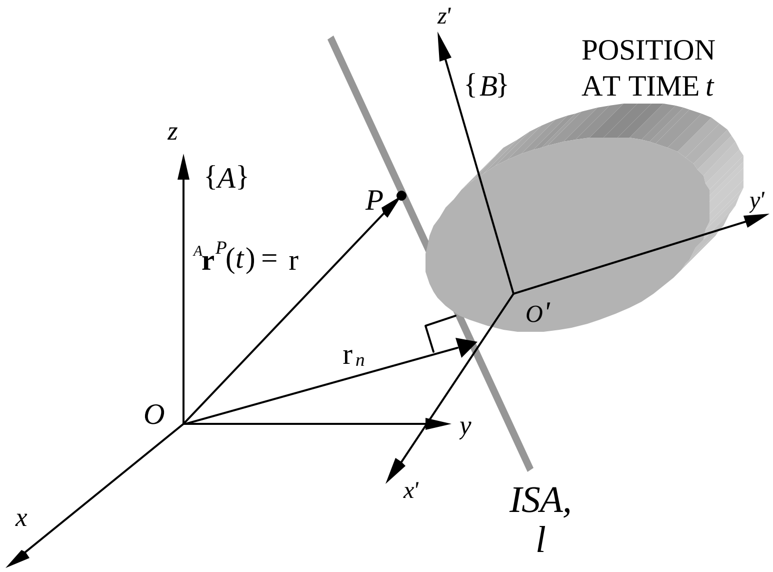

Figure 5 The instantaneous screw axis (*ISA*) with the axis *l*.

Consider a general twist vector of the form in pair of vectors:

$$^A\mathbf{t}_B = \begin{bmatrix} ^A\mathrm{w}_B \\ \text{----} \\ ^A\mathbf{v}^{\hat{O}} \end{bmatrix}$$

Define $\mathbf{u}$ as the unit vector along $^A\mathrm{w}_B$, and let us write:

$$^A\mathrm{w}_B = \omega\, \mathbf{u} \tag{13}$$

Decompose the linear velocity into two components, $\mathbf{v}_{par}$ and $\mathbf{v}_{perp}$, where $\mathbf{v}_{par}$ is parallel to $\mathbf{u}$ and $\mathbf{v}_{perp}$ is perpendicular to $\mathbf{u}$. Because it is perpendicular to $^A\mathrm{w}_B$, $\mathbf{v}_{perp}$ can be written in the form, $\mathrm{r} \times {}^A\mathrm{w}_B$, for some position vector r . $\mathbf{v}_{par}$ can be written as the product of a scalar, h, with $^A\mathrm{w}_B$. Thus we can write:

$$\begin{aligned} ^A\mathbf{v}^{\hat{O}} &= \mathbf{v}_{par} + \mathbf{v}_{perp} \\ &= h\, {}^A\mathrm{w}_B \; + \mathrm{r} \times {}^A\mathrm{w}_B \end{aligned}$$

Now let P be a point whose position vector in $\{A\}$ is r . In other words, $\mathrm{r} = {}^A\mathbf{r}^P$, and:

$$^A\mathbf{v}^{\hat{O}} = h\, {}^A\mathrm{w}_B \; + \; {}^A\mathbf{r}^P \times {}^A\mathrm{w}_B . \tag{14}$$

Since $\hat{O}$ and P are both points on the rigid body B, we can write:

$$ {}^{A}\mathbf{v}^{\hat{O}} = {}^{A}\mathbf{v}^{P} + {}^{A}\mathbf{r}^{P} \times {}^{A}\mathrm{w}_{B} $$

(15)

Comparing these two equations, it is clear that both are satisfied only if ${}^{A}\mathbf{v}^{P} = h\,{}^{A}\mathrm{w}_{B}$. In other words, if P is a point whose position vector from O satisfies Equation (14), its velocity must be parallel to ${}^{A}\mathrm{w}_{B}$. In fact, there is a whole set of points that satisfy Equation (14). Any point P' with the position vector,

$$ {}^{A}\mathbf{r}^{P'} = {}^{A}\mathbf{r}^{P} + k\mathbf{u} $$

has the property,

$$ {}^{A}\mathbf{r}^{P'} \times {}^{A}\mathrm{w}_{B} = {}^{A}\mathbf{r}^{P} \times {}^{A}\mathrm{w}_{B}, $$

and will also satisfy Equation (14). The locus of such points[6] is a line, l, shown in Figure 5, that is parallel to $\mathbf{u}$. Thus, for a general twist of the rigid body B relative to A, there is a line that is parallel to the angular velocity of B in A, consisting of points attached to B, such that their velocities are parallel to the line. The velocity of any other point Q on the rigid body B, is given by:

$$ {}^{A}\mathbf{v}^{Q} = {}^{A}\mathbf{v}^{P} + {}^{A}\mathrm{w}^{B} \times \overrightarrow{PQ} $$

(16)

The first term on the right hand side is a component that is parallel to the axis, which is the same regardless of where Q lies, and the second term is a component that is perpendicular to the axis whose magnitude is proportional to the distance from the axis. The velocity ${}^{A}\mathbf{v}^{Q}$, and therefore the velocity of any point on the body, is tangential to a right circular helix with the axis l passing through that point, whose pitch is given by h, the ratio of the magnitudes of ${}^{A}\mathbf{v}^{P}$ and ${}^{A}\mathrm{w}_{B}$. Because of this geometry, l is called the *instantaneous screw axis* (ISA) of motion, and h the *pitch* of the screw axis. The magnitude of the angular velocity, ω, is called the *amplitude*. The

[6] If the origin, O, had been chosen at at any of these points, say P, ${}^{A}\mathbf{r}^{P}=0$, and the velocity vector and the angular velocity vectors would be parallel.

body B is said to undergo a twist of amplitude ω about the instantaneous screw axis l relative to body A.

A compact description of the twist and the instantaneous screw axis is obtained if we define r_n be the position vector of a point on the axis such that r_n is perpendicular to ${}^A\mathrm{w}_B$, as shown in Figure 5. The parameters that describe the twist and the instantaneous screw axis are given by:

$$\begin{aligned}
\omega &= \left|{}^A\mathrm{w}_B\right| = \sqrt{{}^A\mathrm{w}_B \cdot {}^A\mathrm{w}_B} \\
\mathbf{u} &= \frac{{}^A\mathrm{w}_B}{\omega} \\
\mathrm{r}_n &= \frac{{}^A\mathrm{w}_B \times {}^A\mathbf{v}_{\hat{O}}}{\omega^2} \\
h &= \frac{{}^A\mathrm{w}_B \cdot {}^A\mathbf{v}_{\hat{O}}}{\omega^2}
\end{aligned}$$

(17)

The *ISA* associated with a twist can be made explicit using the equation:

$$ {}^A\mathbf{t}_B = \left[\begin{array}{c} {}^A\mathrm{w}_B \\ \hline {}^A\mathbf{v}_{\hat{O}} \end{array}\right] = \omega \left[\begin{array}{c} \mathbf{u} \\ \hline h\mathbf{u} + \mathrm{r} \times \mathbf{u} \end{array}\right] $$

(18)

Thus, if the Plucker coordinates of the line vector (without normalization) are given by the vector $[L, M, N, P, Q, R]^T$, the twist vector is given by:

$$ {}^A\mathbf{t}_B = \begin{bmatrix} L \\ M \\ N \\ P^* \\ Q^* \\ R^* \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \\ P + hL \\ Q + hM \\ R + hN \end{bmatrix} $$

(19)

The components of the twist vector are called *screw coordinates* in analogy to the Plucker *line coordinates* for lines.

3.4 Force analysis and the wrench about an axis

In the previous section we showed that the instantaneous kinematics of any rigid body motion can be described by the instantaneous screw axis, and we derived a set of formulae that allow us to compute the location and the pitch of the screw axis. In this section we argue that the idea of a screw axis is also central to the description of a general system of forces and couples acting on a rigid body, and we show how to compute the pitch and location of this axis.

A general system of forces and couples acting on a rigid body cannot be reduced to a pure resultant force or a pure resultant couple. Instead, if we add all the forces to get a resultant force, **F**, and add all the moments about the origin O to get a resultant couple, **C**, we have a pure force *and* a pure couple. Such a force-couple combination was called a *dyname* by Plücker (1862) and later by Routh (1892). It is shown next that any such force-couple dyname can be described by an equivalent combination of a force and a couple such that the vectors representing the pure force and the pure couple are parallel.

In Figure 6, a rigid body is acted upon by n forces, $\mathbf{F}_1,...,\mathbf{F}_n$, and m pure couples, $\mathbf{C}_1, ..., \mathbf{C}_m$. The resultant can be described by a force-couple combination:

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i, \quad \mathbf{M}^O = \sum_{i=1}^{m} \mathbf{C}_i + \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i \tag{20}$$

We say that the system of **F** and $\mathbf{M}^O$ is equipollent to the system consisting of $\mathbf{F}_1,...,\mathbf{F}_n$, and $\mathbf{C}_1,..., \mathbf{C}_m$. In order to develop another equipollent system, we decompose $\mathbf{M}^O$ into two vector components, **C** and **C'**, such that **C** is parallel to **F** and **C'** is perpendicular to **F**. Now we find a vector r_n that is perpendicular to **F** such that $\mathbf{C'} = \mathrm{r}_n \times \mathbf{F}$.

In other words,

$$\mathrm{r}_n = \frac{\mathbf{F} \times \mathbf{C}}{\mathbf{F} \cdot \mathbf{F}} \tag{21}$$

By translating the line of action of the force through $\mathbf{r}_n$ (as shown in the figure), we generate a moment about O given by $\mathbf{r}_n \times \mathbf{F}$, which is equal to **C'**. The force along the new line of action, l, with the couple **C** is a system that is equipollent to the n forces, $\mathbf{F}_1, \ldots, \mathbf{F}_n$, and m pure couples, $\mathbf{C}_1, \ldots, \mathbf{C}_m$.

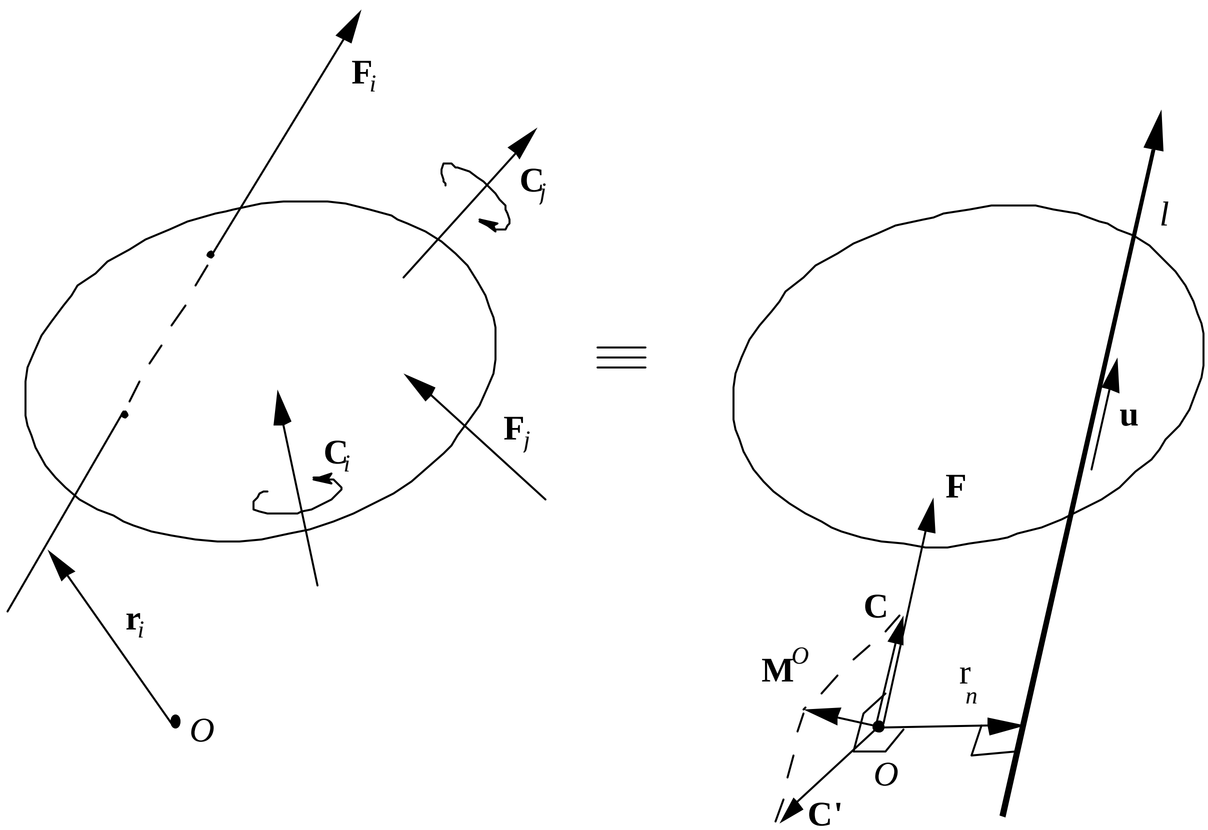

Figure 6 A system of forces and couples acting on a rigid body can be reduced to a *wrench*, a combination of a force and a couple such that the two vectors are parallel.

Thus any system of forces and couples can be described by an equivalent combination of a force and a couple such that the vectors representing the pure force and the pure couple are parallel. Such a combination is called a *wrench*. In vector notation, a wrench is described by a 6×1 vector, ${}^{A}\mathbf{w}$:

$$
{}^{A}\mathbf{w} = \begin{bmatrix} \mathbf{F} \\ \mathbf{M}^{O} \end{bmatrix} \tag{22}
$$

where the leading superscript A denotes the fact that the vectors are written with respect to basis vectors in reference frame $\{A\}$ and the moment is the moment about the origin of $\{A\}$.

The wrench acts along a line which is the line of action of the force (l in the figure). This line is called the wrench axis. The wrench has a *pitch*, λ, which is the ratio of the magnitude of the couple and the force.

$$\lambda = \frac{|\mathbf{C}|}{|\mathbf{F}|}$$

λ is positive when the couple and the force have the same direction and is negative when the directions are contrary. The magnitude of the force, F, is the *intensity* of the wrench. Finally, note that pure forces and pure couples are special cases of a wrench — a pure force is a wrench of zero pitch and a pure couple is a wrench of infinite pitch.

The concept of the wrench and the derivation above are very similar to the presentation of the twist in kinematic analysis in Section 3.3.2. The geometric concept of a screw[7] is central to both a twist and a wrench. If we ignore the amplitude of a twist (or the intensity of a wrench), what remains is the axis of the twist (or the axis of the wrench) and the pitch associated with it. We define a screw as a line to which is attached a scalar parameter, a pitch. We speak of a wrench about a screw of a certain intensity or a twist about a screw with a certain amplitude.

3.5 Transformation laws for twists and wrenches

In the previous section, we developed expressions for the twist of a rigid body by attaching a frame $\{B\}$ to the rigid body and describing the motion of the frame $\{B\}$ in a frame $\{A\}$ attached to the rigid body $\{A\}$. It is worth recalling that we started with the homogeneous transformation matrix ${}^{A}\mathbf{A}_{B}$, the representation of the position and orientation of $\{B\}$ in the frame $\{A\}$, and we derived expressions for ${}^{A}\mathbf{t}_{B}$, the 6×1 twist vector describing the instantaneous motion of $\{B\}$ as seen by an observer attached to $\{A\}$. Note the components of the twist

[7]The instantaneous screw axis was first used by Mozzi (1763) although Chasles (1830) is credited with this discovery. The basic idea of a wrench can be traced back to Poinsot's work in 1806, but the the concept of the wrench and the twist were formalized by Plücker (1865) and later by Ball in his treatise *The theory of screws* in 1900.

vector only make sense in the reference frame $\{A\}$. In this subsection, we examine how we can find components of quantities like the twist, ${}^{A}\mathbf{t}_{B}$, in a frame other than $\{A\}$.

We have seen that the displacement of a frame attached to a rigid body from $\{A\}$ to $\{B\}$ can be represented in a frame $\{F\}$ that is different from the first frame ($\{A\}$) via a similarity transform. In the frame $\{F\}$, the displacement is represented by the homogeneous transform:

$$
{}^{F}\mathbf{A}_{G} = {}^{F}\mathbf{A}_{A}\,{}^{A}\mathbf{A}_{B}\,({}^{F}\mathbf{A}_{A})^{-1} \tag{23}
$$

Such similarity transforms can be used to transform any matrix quantity in one frame to another frame.

In order to see this, consider the matrix representation of the twist ${}^{A}\mathbf{T}_{B}$ obtained in frame $\{A\}$ by differentiating the matrix ${}^{A}\mathbf{A}_{B}(t)$:

$$
{}^{A}\mathbf{T}_{B} = {}^{A}\dot{\mathbf{A}}_{B}\left[{}^{A}\mathbf{A}_{B}\right]^{-1} = \begin{bmatrix} {}^{A}\mathrm{W}_{B} & {}^{A}\mathbf{v}^{\hat{O}} \\ 0 & 0 \end{bmatrix} \tag{24}
$$

The same instantaneous motion can be described in $\{F\}$, as shown in Figure 7, by differentiating the matrix ${}^{F}\mathbf{A}_{G}(t)$:

$$
\begin{aligned}
{}^{F}\dot{\mathbf{A}}_{G}\left[{}^{F}\mathbf{A}_{G}\right]^{-1} &= \frac{d}{dt}\left[{}^{F}\mathbf{A}_{A}\;{}^{A}\mathbf{A}_{B}\left({}^{F}\mathbf{A}_{A}\right)^{-1}\right]\left[{}^{F}\mathbf{A}_{A}\;{}^{A}\mathbf{A}_{B}\left({}^{F}\mathbf{A}_{A}\right)^{-1}\right]^{-1} \\
&= \left[{}^{F}\mathbf{A}_{A}\,\frac{d}{dt}\left({}^{A}\mathbf{A}_{B}\right)\left({}^{F}\mathbf{A}_{A}\right)^{-1}\right]\left[\left({}^{F}\mathbf{A}_{A}\right)\left({}^{A}\mathbf{A}_{B}\right)^{-1}\left({}^{F}\mathbf{A}_{A}\right)^{-1}\right] \\
&= {}^{F}\mathbf{A}_{A}\;{}^{\mathbf{A}}\mathbf{T}_{B}\left({}^{F}\mathbf{A}_{A}\right)^{-1}
\end{aligned} \tag{25}
$$

In the above differentiation, notice that that we are interested in the motion of $\{B\}$ in $\{A\}$ but in a frame $\{F\}$ that is rigidly attached to $\{A\}$. Therefore ${}^{F}\mathbf{A}_{A}$ is a constant.

An interesting result is obtained if we choose a frame $\{F\}$ that is coincident with the reference frame $\{B\}$. This gives us the twist matrix in a frame that is attached to the moving rigid body B.

$$ {}^{B}\mathbf{A}_{A}\ {}^{\mathbf{A}}\mathbf{T}_{B}\left({}^{B}\mathbf{A}_{A}\right)^{-1} = {}^{B}\mathbf{A}_{A}\left[{}^{\mathbf{A}}\dot{\mathbf{A}}_{B}\left({}^{A}\mathbf{A}_{B}\right)^{-1}\right]\left({}^{B}\mathbf{A}_{A}\right)^{-1} $$
$$ = \left({}^{A}\mathbf{A}_{B}\right)^{-1}\ {}^{\mathbf{A}}\dot{\mathbf{A}}_{B} $$

Note that this new twist matrix represents the components of the same instantaneous motion (B relative to A}, but in a coordinate system attached to {B}.

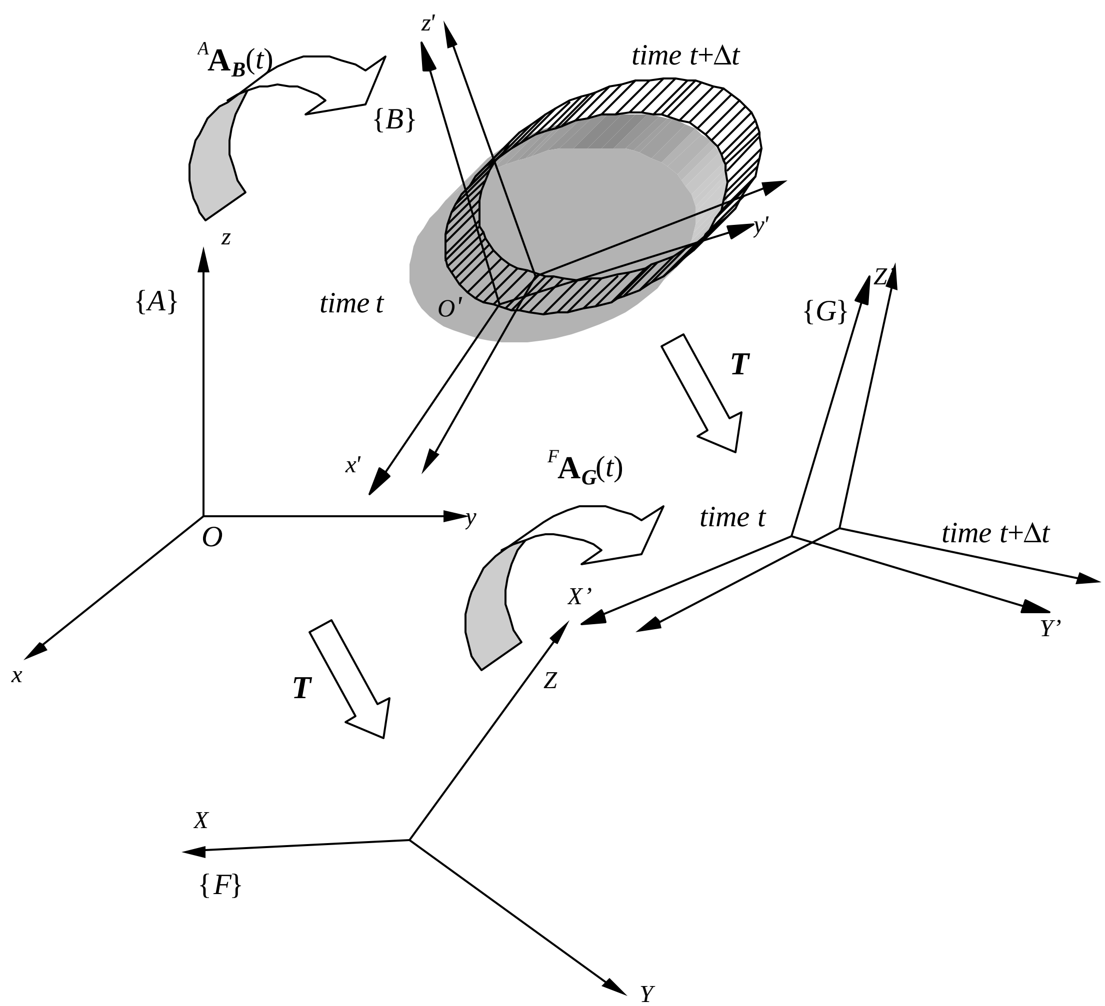

Figure 7 The movement of a frame attached to the moving rigid body B can be studied from frame {F} or from frame {A}. The instantaneous motion can be described in reference frame {A} by the twist matrix ${}^{A}\mathbf{T}_{B}$. The same motion can be described in reference frame {F} by ${}^{F}\mathbf{T}_{G} = {}^{F}\mathbf{A}_{A}\,{}^{A}\mathbf{T}_{B}\,({}^{F}\mathbf{A}_{A})^{-1}$.

The ability to express the instantaneous motion as a twist matrix in a frame other than the frame in which the motion is described, necessitates some new notation. The instantaneous motion of B relative to A can be described by a twist matrix ${}^{A}\mathbf{T}_{B}$ in frame $\{A\}$. However, if we use a different frame, say $\{F\}$, to describe the same instantaneous motion, we will want to explicitly denote the fact that the twist is obtained by considering frames $\{A\}$ and $\{B\}$, but expressed in $\{F\}$, using the notation, ${}^{F}[{}^{A}\mathbf{T}_{B}]$. When the first leading superscript F is absent, it should be clear that the twist matrix consists of components in $\{A\}$. Thus, the instantaneous motion of the body B relative to A, in the frame $\{B\}$ is given by:

$$ {}^{B}\left[{}^{\mathbf{A}}\mathbf{T}_{B}\right] = \left({}^{A}\mathbf{A}_{B}\right)^{-1} {}^{\mathbf{A}}\dot{\mathbf{A}}_{B} \tag{26} $$

The term *spatial velocity* is sometimes used to refer to ${}^{A}[{}^{A}\mathbf{T}_{B}]$, while *body velocity* is used to denote ${}^{B}[{}^{A}\mathbf{T}_{B}]$. See [MLS 94].

A similar notation works for angular velocity. The instantaneous rotational motion of B relative to A can be described by an angular velocity matrix ${}^{A}\mathrm{W}_{B}$ in frame $\{A\}$. This motion in any other reference frame is given by:

$$ {}^{F}\left[{}^{\mathbf{A}}\mathrm{W}_{B}\right] = {}^{F}\mathbf{R}_{A}\ {}^{\mathbf{A}}\mathrm{W}_{B}\left({}^{F}\mathbf{R}_{A}\right)^{-1} = {}^{F}\mathbf{R}_{A}\left[{}^{\mathbf{A}}\dot{\mathbf{R}}_{B}\left({}^{A}\mathbf{R}_{B}\right)^{T}\right]\left({}^{F}\mathbf{R}_{A}\right)^{T} $$

A straightforward application of this result gives the expression for the instantaneous rotational motion of the body B relative to A, in the frame $\{B\}$:

$$ {}^{B}\left[{}^{\mathbf{A}}\mathrm{W}_{B}\right] = \left[\left({}^{A}\mathbf{R}_{B}\right)^{T}\ {}^{\mathbf{A}}\dot{\mathbf{R}}_{B}\right] \tag{27} $$

For vectors, the transformation is much more straightforward. For example, the angular velocity vector, ${}^{A}\mathrm{w}_{B}$ (components expressed in A) can be expressed in any other frame by merely premultiplying by the appropriate rotation matrix:

$$
\begin{aligned}
{}^{F}\left[{}^{A}\mathrm{w}_{B}\right] &= {}^{F}\mathbf{R}_{A} \quad {}^{A}\left[{}^{A}\mathrm{w}_{B}\right] \\
{}^{B}\left[{}^{A}\mathrm{w}_{B}\right] &= {}^{B}\mathbf{R}_{A} \quad {}^{A}\left[{}^{A}\mathrm{w}_{B}\right]
\end{aligned}
\tag{28}
$$

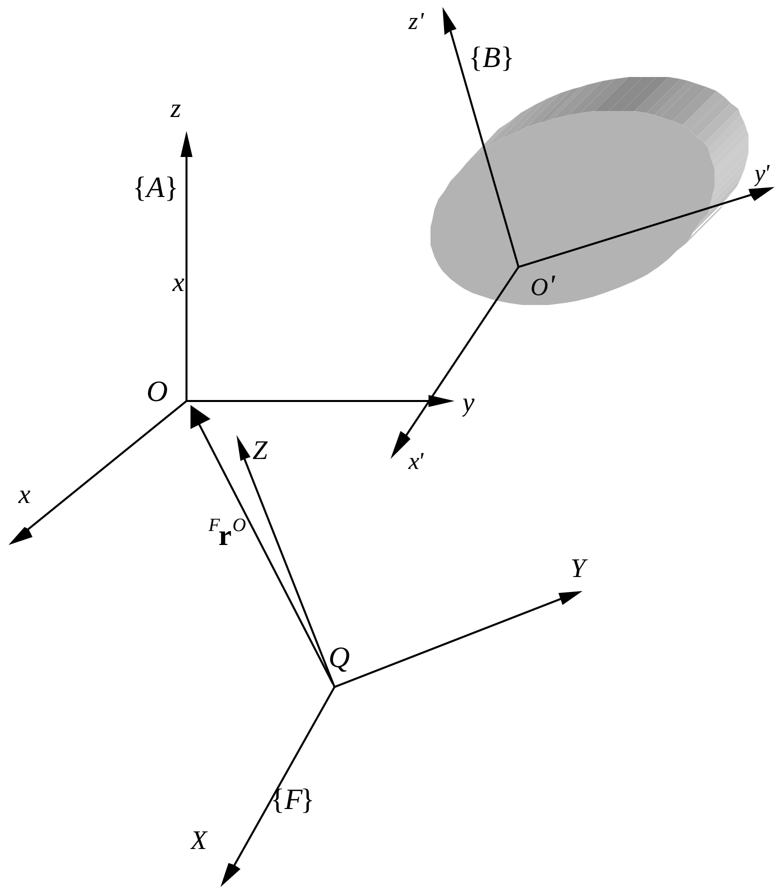

Figure 8 The motion of the rigid body B relative to A is described in terms of the motion of the frame $\{B\}$ relative to $\{A\}$. The instantaneous motion is represented by the twist vector ${}^{A}\mathbf{t}_{B}$ in $\{A\}$. The same motion is also given by the twist vector ${}^{F}[{}^{A}\mathbf{t}_{B}]$ in $\{F\}$.

A similar approach works for twist vectors. Consider an instantaneous motion with the twist vector ${}^{A}\mathbf{t}_{B}$ in $\{A\}$ and ${}^{F}[{}^{A}\mathbf{t}_{B}]$ in $\{F\}$.

$$
\begin{aligned}
{}^{A}\mathbf{t}_{B} &= {}^{A}\left[{}^{A}\mathbf{t}_{B}\right] = \begin{bmatrix} {}^{A}\mathrm{w}_{B} \\ {}^{A}\mathbf{v}^{\hat{O}} \end{bmatrix} \\
{}^{F}\left[{}^{A}\mathbf{t}_{B}\right] &= \begin{bmatrix} {}^{F}\left[{}^{A}\mathrm{w}_{B}\right] \\ {}^{F}\mathbf{v}^{\hat{Q}} \end{bmatrix}
\end{aligned}
$$

The angular velocity vector in both twists refers to the same quantity except with components in different frames. However, the linear velocity vectors in the two twist vectors are different. ${}^{A}\mathbf{v}^{\hat{O}}$ is the velocity of a point on the body B that is instantaneously coincident with O, while ${}^{F}\mathbf{v}^{\hat{Q}}$ is the velocity of a point on the body B that is instantaneously coincident with Q. In addition to the fact that the velocity vectors refer to components in different frames, the two velocities are different quantities. Since $\hat{O}$ and $\hat{Q}$ refer to two different points on the rigid body $\{B\}$, it is clear that their velocities are related by:

$$
{}^{A}\mathbf{v}^{\hat{Q}} = {}^{A}\mathbf{v}^{\hat{O}} + {}^{A}\mathrm{w}_{B} \times {}^{A}\mathbf{r}^{Q}
$$

or,

$$
{}^{A}\mathbf{v}^{\hat{Q}} = {}^{A}\mathbf{v}^{\hat{O}} + \left(-\,{}^{A}\mathbf{r}^{Q}\right) \times {}^{A}\mathrm{w}_{B} \tag{29}
$$

Given that vectors are transformed using rotation matrices, we can write:

$$
\begin{aligned}
{}^{F}\left[{}^{A}\mathrm{w}_{B}\right] &= {}^{F}\mathbf{R}_{A}\ {}^{A}\mathrm{w}_{B} \\
{}^{F}\mathbf{v}^{\hat{Q}} &= {}^{F}\mathbf{R}_{A}\ {}^{A}\mathbf{v}^{\hat{Q}} \\
&= {}^{F}\mathbf{R}_{A}\ {}^{A}\mathbf{v}^{\hat{O}} + {}^{F}\mathbf{R}_{A}\left[-\,{}^{A}\mathbf{r}^{Q} \times {}^{A}\mathrm{w}_{B}\right] \\
&= {}^{F}\mathbf{R}_{A}\ {}^{A}\mathbf{v}^{\hat{O}} + {}^{F}\mathbf{R}_{A}\left(-\,{}^{A}\mathbf{r}^{Q}\right) \times \left({}^{F}\mathbf{R}_{A}\ {}^{A}\mathrm{w}_{B}\right) \\
&= {}^{F}\mathbf{R}_{A}\ {}^{A}\mathbf{v}^{\hat{O}} + {}^{F}\mathbf{r}^{O} \times \left({}^{F}\mathbf{R}_{A}\ {}^{A}\mathrm{w}_{B}\right)
\end{aligned}
$$

The twist vectors ${}^{A}\mathbf{t}_{B}$ and ${}^{F}[{}^{A}\mathbf{t}_{B}]$ can be related by:

$$\begin{bmatrix} {}^F\left[{}^A\mathrm{w}_B\right] \\ {}^F\mathbf{v}^{\hat{Q}} \end{bmatrix} = \begin{bmatrix} {}^F\mathbf{R}_A & \mathbf{0} \\ \left[{}^A\hat{\mathbf{r}}^O\right]{}^F\mathbf{R}_A & {}^F\mathbf{R}_A \end{bmatrix} \begin{bmatrix} {}^A\left[{}^A\mathrm{w}_B\right] \\ {}^A\mathbf{v}^{\hat{O}} \end{bmatrix}$$

where the "hat" over the vector $\mathbf{a}$ denotes the 3×3 skew symmetric matrix operator $[\hat{\mathbf{a}}]$ corresponding to the 3×1 vector $\mathbf{a}$. Thus the two 6×1 twist vectors are related by the 6×6 transformation matrix, ${}^F\mathrm{G}_A$, given by:

$$^F\left[{}^A\mathbf{t}_B\right] = {}^F\mathrm{G}_A\ {}^A\left[{}^A\mathbf{t}_B\right]$$

$${}^F\mathrm{G}_A = \begin{bmatrix} {}^F\mathbf{R}_A & \mathbf{0} \\ \left[{}^A\hat{\mathbf{r}}^O\right]{}^F\mathbf{R}_A & {}^F\mathbf{R}_A \end{bmatrix} \tag{30}$$

where $[{}^F\hat{\mathbf{r}}^O]$ and ${}^A\mathbf{R}_B$ are 3×3 matrices and $\mathbf{0}$ is a 3×3 zero matrix.

Note that this is the same 6×6 transformation matrix used to transform line vectors from one reference frame to another. It is left as an exercise to verify that the same transformation matrix allows us to transform wrenches from one frame to another.

3.6 Reciprocity

When the line of action of a force acting on a particle is perpendicular to the direction of the velocity vector associated with the motion of the particle, we know that the force cannot do work on the particle. Mathematically, the power, P, given by the scalar product of the force and the velocity, equals zero. Sometimes we say that the force is orthogonal to the velocity. In mechanics, we are always interested in situations where the acting force(s) are "orthogonal" to the allowable direction(s) of motion. In fact, we call such forces constraint forces.

When we consider forces and moments, or angular and linear velocities, we need a new terminology. Twists are the natural generalizations of velocity vectors. Similarly, force vectors are now wrenches. Reciprocity is the natural generalization of this intuitive notion of orthogonality[8].

Formally, two screws are said to be *reciprocal* to each other if a wrench applied about one does no work on a twist about the other. Since twists represent instantaneous motion, it is more appropriate to consider the power associated with the action of a wrench on a body undergoing a twist. Omitting the leading and trailing subscripts and superscripts for the time being, The rate of work done by a wrench $\mathbf{w} = [\mathbf{F}^T,\ \mathbf{M}^T]^T$ on a twist $\mathbf{t} = [\mathrm{w}^T,\ \mathbf{v}^T]^T$ is given by

$$P = \mathbf{F} \cdot \mathbf{v} + \mathbf{M} \cdot \mathrm{w}$$

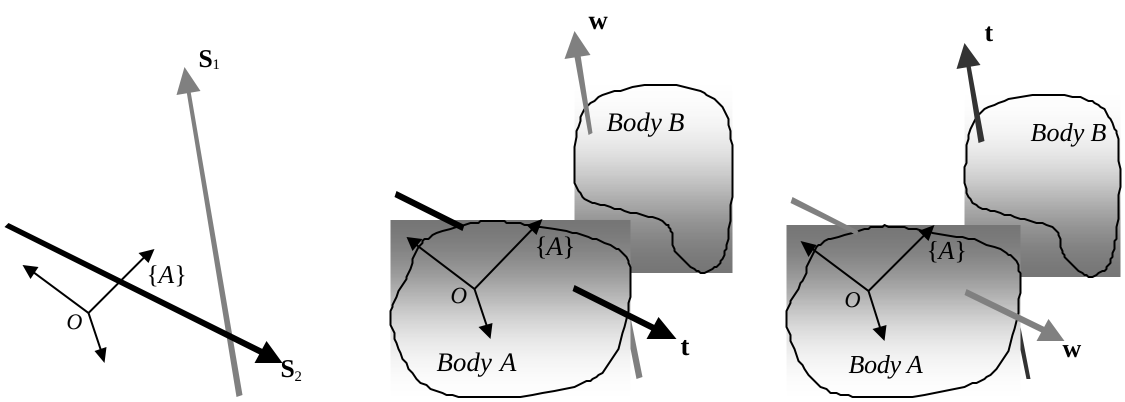

Figure 9 The instantaneous motion of body B relative to body A is described by the twist $\mathbf{t}$ while $\mathbf{w}$ is the wrench exerted by A on B. The two screws $\mathbf{S}_1$ and $\mathbf{S}_2$ are reciprocal (left) and only if a wrench about $\mathbf{S}_1$ does no work on a twist about $\mathbf{S}_2$ (center) ch about $\mathbf{S}_2$ does no work on a twist about $\mathbf{S}_1$ (right).

The above equation can be written in more formal notation. Writing the twists and wrenches in frame $\{A\}$, we get:

[8] It is incorrect to say that a force vector is orthogonal to a velocity vector. Strictly speaking, a velocity vector can be orthogonal to another velocity vector and a force vector can be orthogonal to a force vector. But since forces and velocities "live" in different vector spaces, a force cannot be said to be orthogonal to a velocity.

$$P = {}^{A}\mathbf{t}_{B}^{T}\ \mathrm{D}\ {}^{A}\mathbf{w} = \begin{bmatrix} {}^{?}\mathrm{w}_{B} & {}^{A}\mathbf{v}^{O} \end{bmatrix} \begin{bmatrix} {}^{A}\mathbf{M}^{O} \\ {}^{A}\mathbf{F} \end{bmatrix}$$

$$= {}^{A}\mathbf{w}^{T}\ \mathrm{D}\ {}^{A}\mathbf{t}_{B} = \begin{bmatrix} {}^{A}\mathbf{F} & {}^{A}\mathbf{M}^{O} \end{bmatrix} \begin{bmatrix} {}^{A}\mathbf{v}^{O} \\ {}^{?}\mathrm{w}_{B} \end{bmatrix}$$

(31)

where Δ is the 6×6 matrix:

$$\mathrm{D} = \left[\begin{array}{ccc:ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

which reorders the components of 6×1 twist or wrench vectors.

If we consider two arbitrarily oriented lines in space and associate screws with different pitches (see figure), we get the following necessary and sufficient condition for reciprocity. Two screws $\mathbf{S}_1$ (pitch h_1) $\mathbf{S}_2$ (pitch h_2) are reciprocal if and only if

$$(h_1 + h_2)\cos\phi - d\sin\phi = 0$$

(32)

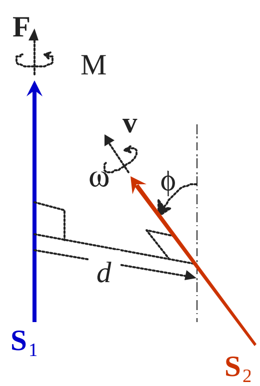

Figure 10 The reciprocity condition, $(h_1 + h_2)\cos\phi - d\sin\phi = 0$, is a geometric condition that relates the pitches of the two screws, the distance between the axes and the relative angle between the axes.

To show this, consider a coordinate system whose x-axis is aligned with $\mathbf{S}_1$, and the z-axis is aligned with the mutual perpendicular going from $\mathbf{S}_1$ to $\mathbf{S}_2$. The screw coordinates for the two screws:

$$\mathbf{S}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ h_1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \\ h_2\cos\phi - d\sin\phi \\ h_2\sin\phi + d\cos\phi \\ 0 \end{bmatrix}$$

and the wrench $\mathbf{w}$ and the twist $\mathbf{t}$ given by:

$$\mathbf{w} = \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix} = f\begin{bmatrix} 1 \\ 0 \\ 0 \\ h_1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{t} = \begin{bmatrix} \mathrm{w} \\ \mathbf{v} \end{bmatrix} = \omega\begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \\ h_2\cos\phi - d\sin\phi \\ h_2\sin\phi + d\cos\phi \\ 0 \end{bmatrix}$$

Since (31) must hold for the wrench and twist above, for any amplitude ω and any intensity f, the result (32) directly follows.

The reader is invited to prove the following facts are true.

1. A wrench acting on a rigid body free to rotate about a revolute joint does no work on the rigid body if one of the following is true
   - The wrench is of zero pitch and the axis intersects the axis of rotation; or
   - The pitch is non zero but equal to $d\tan\phi$.
2. The contact wrench at a frictionless point contact does no work on the rigid body if one of the following is true
   - The twist is of zero pitch and the axis intersects the contact normal; or
   - The pitch of the twist is non zero but equal to $d\tan\phi$.

3. A wrench acting on a rigid body free to translate along a prismatic joint does no work on the rigid body if
   - The wrench is of infinite pitch; or
   - The pitch is zero or finite, but the axis is perpendicular to the axis of the prismatic joint.

3.7 References

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