CSE 390/MEAM 420

Kinematics of Wheeled Robots

Professor Vijay Kumar

Department of Mechanical Engineering and Applied Mechanics

University of Pennsylvania

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1 Introduction

In this chapter, we will develop the notation and a mathematical model for the kinematics of wheeled mobile robots and discuss the main issues for control. The scope of this discussion will be limited, for the most part, to differential-drive, planar robots.

2 Configuration space and constraints

Consider a system with N particles, $P_i(i = 1, \ldots, N)$, and their positions vector $\mathbf{r}_i$ in some reference frame. Each vector $\mathbf{r}_i$ can be written as a set of components $(x_i,\ y_i,\ z_i)$. The $3N$ components specify the configuration of the system. The resulting Euclidean space:

$$\mathcal{C} = \left\{X | X \in R^{3N},\ X = [x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_N, y_N, z_N]\right\}$$

is called the configuration space. The $3N$ scalar numbers are called configuration space variables or coordinates for the system. The trajectories of the system in the configuration space are always continuous. They may however have corners, double points, or points with multiple intersections. Corners are points at which the velocity is zero or discontinuous. Note that a velocity is discontinuous only when it is subject to an impulse (for example, during an impact).

If you consider a system with N rigid bodies, $B_i(i = 1, \ldots, N)$, each rigid body has three coordinates for planar systems, and six for spatial systems. Thus for planar systems, if each rigid body has coordinates $(x_i,\ y_i,\ \theta_i)$ in some reference frame, the configuration space can be written as the Cartesian product of the configuration spaces of individual rigid bodies. Note that technically θ_i belongs to a subset of R, the set of real numbers, because it is an angle and only takes values in $[0, 2\pi)$ with the values 0 and 2π being identified. This is sometimes denoted as the circle, S^1. Thus, the configuration space for a single rigid body in the plane is called the *special Euclidean group* in two-dimensions and is denoted by $SE(2)$:

$$SE(2) = R \times R \times S^1.$$

The configuration space of N planar rigid bodies is:

$$\mathcal{C} = \{X | X \in SE(2) \times SE(2) \times \ldots \times SE(2),\ X = [x_1, y_1, \theta_1, x_2, y_2, \theta_2, \ldots, x_N, y_N, \theta_N]\}$$

In a system of two or more particles (rigid bodies), unconstrained motion is simply not possible. To see this consider the simplest possible system involving two particles on a straight line or two frictionless beads sliding on a smooth, taut wire. Since the particles cannot cross each other, there is a constraint on the positions of the two particles. This constraint is an *inequality* constraint.

Similarly, consider a robot (a rigid body) moving in an environment with a fixed obstacle (another rigid body). As shown in Figure 1, the configuration space can be divided into legal configurations that avoid penetration of the robot with the obstacle and illegal configurations. The figure (bottom left) also shows the boundary of the set of configurations $\{x, y, \theta\}$ that result in collisions. Clearly, one can conceptualize the obstacle-free space or the *free space* as the set of configurations that satisfy a given set of inequalities.

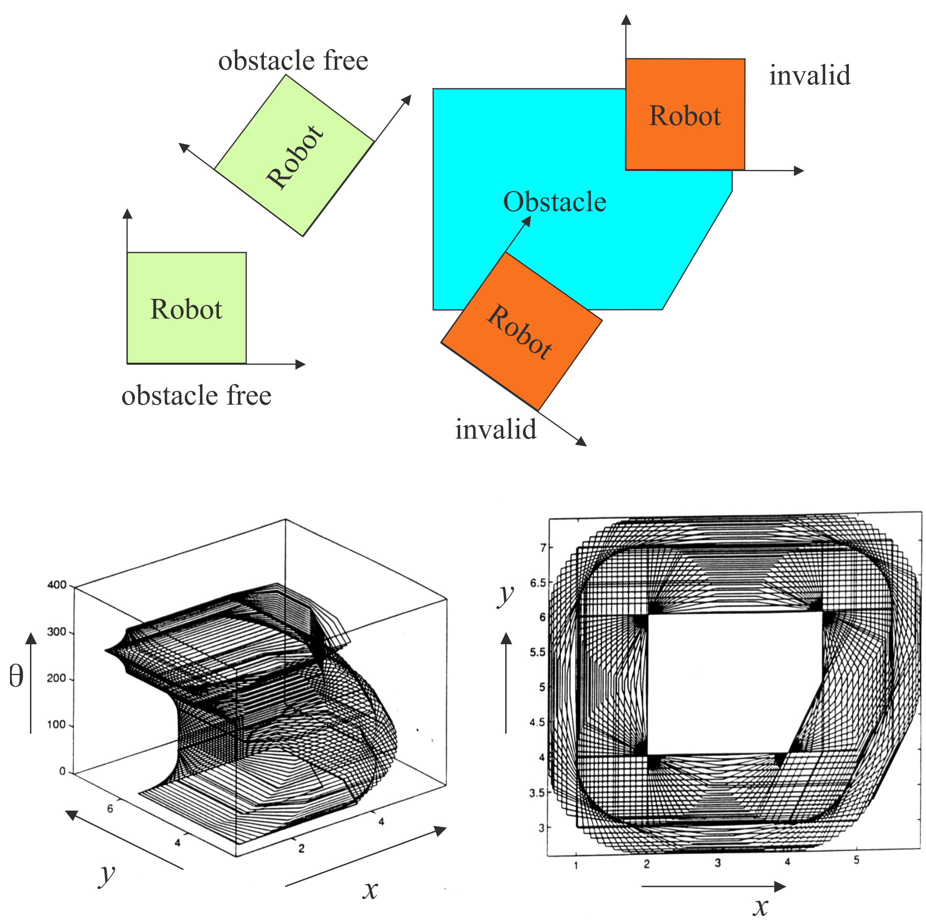

Figure 1: The presence of an obstacle divides the configuration space into free space and the set of invalid configurations (top). The set of invalid configurations in three dimensional space is shown on the bottom left — the projection of this solid on the $x - y$ plane is shown on the bottom right.

We will now turn our attention from inequalities to constraint equalities. Consider a system of N planar rigid bodies. We have already seen that there is a $3N$-dimensional configuration space associated with the system. However, when there are one or more configuration constraints (as in the case of planar kinematic chains) not all of the $3N$ variables describing the system configuration are independent. We have seen that the minimum number of variables (also called coordinates) to completely specify the configuration (position of every particle) of a system is called the number of degrees of freedom for that system. Thus if there are m independent configuration constraints, the number of degrees of freedom n is given by:

$$n = 3N - m \tag{1}$$

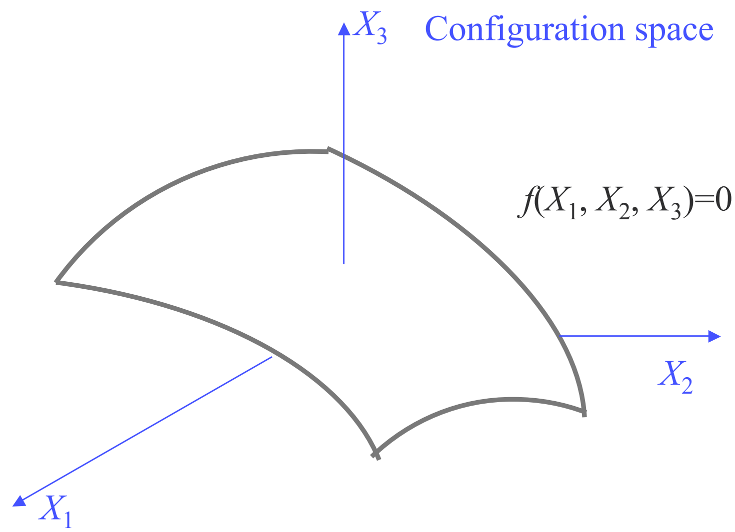

Figure 2: A single constraint in a three-dimensional configuration space forces the resulting two degree-of-freedom system to evolve on a two-dimensional surfaces in the configuration space

Constraints on the position of a system of particles (as in the examples with kinematic chains) are called *holonomic constraints*. The positions of the particles are constrained by holonomic equations. The system is constrained to move in a subset of the $3N$-dimensional configuration space. For example, if you take a single rigid body that has a three dimensional configuration space and force it to pivot about a fixed point, you introduce two constraints and force it to remain in a one-dimensional subset of the configuration space.

In contrast to holonomic constraints in which the positions of the particles are constrained, we may have constraints in which the velocities of the particles are constrained but the positions are not. As an example, consider the constraint called the *knife-edge constraint* illustrated by the skate edge in Figure 3.

The rigid body can be described by the coordinates of a reference point C that is the (single) point of contact on the plane (x, y) and the angle (θ) between the longitudinal axes and the x-axis. Since the skate cannot slide in a lateral direction, the velocity of the point C must be along the longitudinal axis (shown by the vector $\mathbf{e}_f$). In other words, the velocity of the point C in the lateral direction $(\mathbf{e}_l)$ must be zero. Let us formalize this with equations.

The position and orientation or the *pose* of the robot is given by a 3×1 vector:

$$q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \tag{2}$$

Differentiating this equation gives us the velocity:

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \tag{3}$$

The velocity of the point C is the 2×1 vector:

$$\mathbf{v}_C = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \tag{4}$$

and the vectors $\mathbf{e}_f$ and $\mathbf{e}_l$ are:

$$\mathbf{e}_f = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}, \qquad \mathbf{e}_l = \begin{bmatrix} sin\theta \\ -cos\theta \end{bmatrix}.$$

The longitudinal component of velocity or the component of $\mathbf{v}_C$ along $\mathbf{u}$, denoted by $\mathbf{v}_f$:

$$v_f = \mathbf{v}_C^T \, \mathbf{e}_f = \dot{x}cos\theta + \dot{y}sin\theta \tag{5}$$

while the lateral component of velocity is:

$$v_l = \mathbf{v}_C^T \, \mathbf{e}_l = \dot{x}sin\theta - \dot{y}cos\theta \tag{6}$$

If there is no slip in the lateral direction, we get the constraint equation $v_l = 0$ or:

$$\dot{x} \; sin\theta - \dot{y} \; cos\theta = 0 \tag{7}$$

Or in differential form after rearranging terms:

$$dy = tan\theta \; dx \tag{8}$$

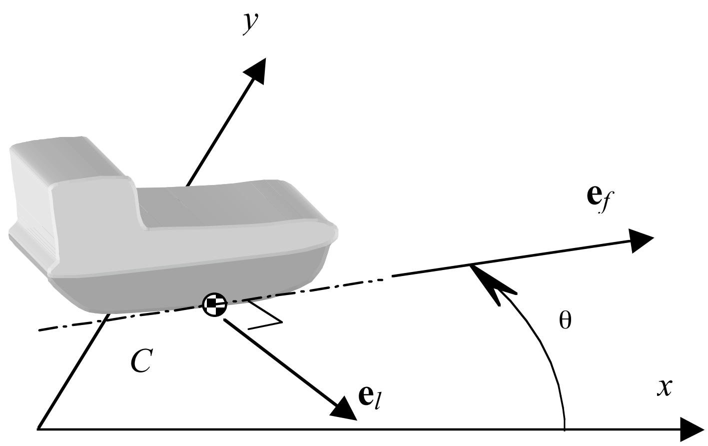

Figure 3: The position of the skate in the plane is describe by three coordinates. Thus its configuration space is R^3 or $R^2 \times S^1$. But its velocity can only be along the vector $\mathbf{u}$. Thus the rate of change of the three position coordinates is constrained.

You should try to verify that Equation (8) cannot be integrated to obtain a constraint on x, y, and θ. Rather this equation simply constrains the velocities and not the positions.

You cannot assume all velocity constraints are non integrable. Indeed, it is possible to differentiate any position constraint to obtain a velocity constraint that can be integrated back to the position constraint. Figure 4 provides a pertinent example of a non trivial velocity constraint that can be integrated. Consider the rolling wheel with a point of contact P with the (stationary) ground. Because it is rolling, the velocity of the point P is zero. Suppose the wheel rolls along the y axis on the $y - z$ plane given by $x = 0$. The velocity of the point C can be obtained from the vector equation [1]:

$$\begin{aligned} \mathbf{v}_C &= \mathbf{v}_P + \omega \times \overrightarrow{PC} \\ &= \dot{\phi}\hat{\mathbf{i}} \times r\hat{\mathbf{k}} \\ &= -r\dot{\phi}\hat{\mathbf{j}} \end{aligned} \tag{9}$$

If we were to denote the coordinates of the center of the wheel by (x, y, z), we can write the equations:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ -r\dot{\phi} \\ 0 \end{bmatrix} \tag{10}$$

[1]Recall from physics that the velocities of two points A and B fixed to a rigid body with angular velocity ω are related by the equation, $\mathbf{v}_B = \mathbf{v}_A + \omega \times \overrightarrow{AB}$

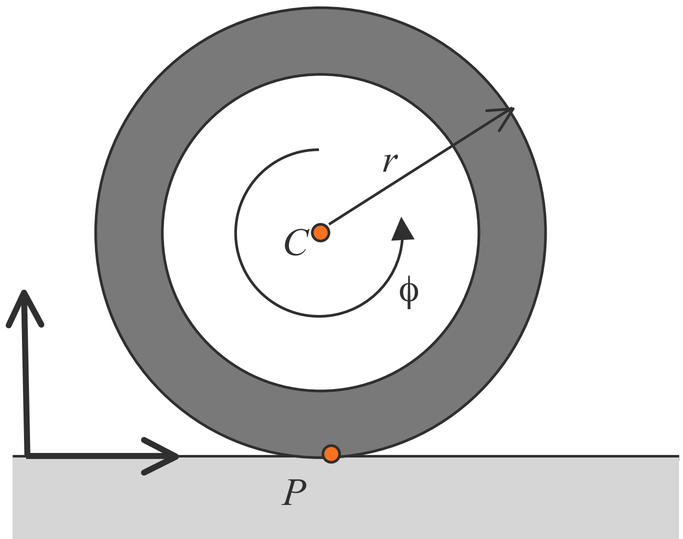

Figure 4: The velocity of the point of contact (P) on a wheel rolling on a stationary surface is zero.

which integrates to:

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -r\int_0^t \dot{\phi} d\tau \\ r \end{bmatrix} \tag{11}$$

Given the history of forward wheel rotation, we can integrate to get the position (x, y, z). Thus, the velocity constraints in Equation (10) are holonomic.

3 Kinematics of a Differential Drive Wheeled Robot

Mobile robots for operation on flat terrain have several simplifying features that make them easier to model than real-world trucks or passenger cars. In particular, many robots have two independently-driven, coaxial wheels. The speed difference between both wheels results in a rotation of the vehicle about the center of the axle while the wheels act in concert to produce motion in the forward or reverse direction. Second, these robots are two-dimensional and lack suspensions. In cars, the suspension compensates for vertical motion caused by the cars dynamics at high speeds. Mobile robots operate at relatively low speeds and we assume vertical motion is absent.

Figure 5 shows two possible schematics for designs of differential drive robots. In both cases the kinematics is determined by the axle and the wheel radii. Denote the centers of the wheels by C_1 and C_2 respectively, and let their radius be r. Let the axle width or the length of the vector $\overrightarrow{C_1C_2}$ be l. Let (x_i, y_i, z_i) denote the position of center C_i and let ϕ_i denote the wheel speed of the ith wheel. And let the

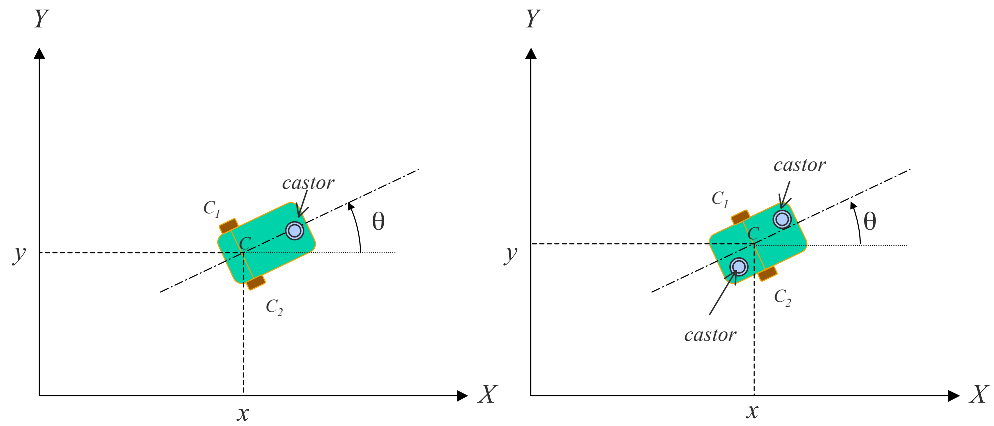

Figure 5: Two possible configurations for differential drive robots. The castors serve to support the weight of the vehicle and under ideal conditions do not affect the kinematics of the robot.

component of speed in the longitudinal direction (see Figure 3) be given by $v_{f,i}$. According to Equation (10):

$$\begin{aligned} v_{f,i} &= -r\dot{\phi}_i \\ v_{l,i} &= 0 \end{aligned} \tag{12}$$

From Equation (5) we get

$$v_{f,i} = \dot{x}_i cos\theta + \dot{y}_i sin\theta \tag{13}$$

and from Equation (6), we have:

$$0 = \dot{x}_i sin\theta - \dot{y}_i cos\theta \tag{14}$$

Note that we are measuring ϕ_i in a counterclockwise direction from one (arbitrarily chosen) side of the mobile robot as shown in Figure 6.

Now consider the coordinates of the center of the axle (x, y) which is clearly half way between C_1 and C_2:

$$x = \frac{x_1 + x_2}{2} \quad y = \frac{y_1 + y_2}{2}.$$

The velocity of the point C is given by:

$$\mathbf{v}_C = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{\dot{x}_1 + \dot{x}_2}{2} \\ \frac{\dot{y}_1 + \dot{y}_2}{2} \end{bmatrix} \tag{15}$$

The forward speed or the velocity component in the longitudinal direction can be obtained by projecting

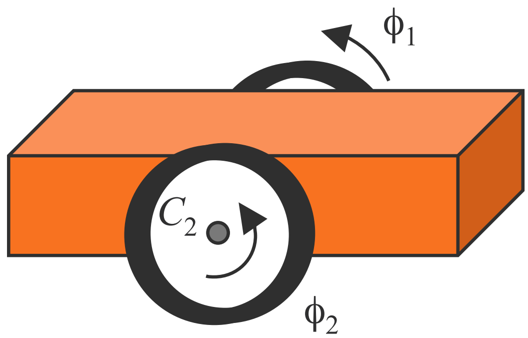

Figure 6: The wheel angles ϕ_i are taken to be positive in a counterclockwise direction from one (arbitrarily chosen) side of the mobile robot.

along $\mathbf{e}_l$:

$$\begin{aligned} v_f &= \mathbf{e}_f^T \mathbf{v}_C = \dot{x}cos\theta + \dot{y}sin\theta \\ &= \frac{\dot{x}_1 cos\theta + \dot{y}_1 sin\theta}{2} + \frac{\dot{x}_2 cos\theta + \dot{y}_2 sin\theta}{2} \\ &= \frac{v_{f,1} + v_{f,2}}{2} \\ &= -\frac{r\dot{\phi}_1 + r\dot{\phi}_2}{2} \end{aligned} \tag{16}$$

Now if we consider the two points C_1 and C_2 which are rigidly attached to the axle and the mobile robot, the velocities of these two points are related by the equation:

$$\mathbf{v}_{C_2} = \mathbf{v}_{C_1} + \dot{\theta}\hat{\mathbf{k}} \times \overrightarrow{C_1 C_2} \tag{17}$$

where

$$\mathbf{v}_{C_1} = v_{f,1}\mathbf{e}_f, \quad \mathbf{v}_{C_2} = v_{f,2}\mathbf{e}_f.$$

Let the wheel axle width $|\overrightarrow{C_1 C_2}|$ be given by l. From the above equation we can write:

$$-r\dot{\phi}_2 \mathbf{e}_f = -r\dot{\phi}_1 \mathbf{e}_f + \dot{\theta}\hat{\mathbf{k}} \times l\mathbf{e}_l.$$

If we write the components along $\mathbf{e}$, we get:

$$-r\dot{\phi}_2 = -r\dot{\phi}_1 + l\dot{\theta} \tag{18}$$

Thus, we have the two equations that relate the velocity of the mobile robot to the velocities of the wheels:

$$v_f = -\frac{r\dot{\phi}_1 + r\dot{\phi}_2}{2} \tag{19}$$

$$\dot{\theta} = \frac{r\dot{\phi}_1 - r\dot{\phi}_2}{l} \tag{20}$$

References

[1] Tongue, Benson H. and Sheri D. Sheppard, *Dynamics: Analysis and Design of Systems in Motion,* John Wiley and Sons, Inc., 2005.