Rigid Body Transformations

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Remark about Notation

Vectors

- $\mathbf{x}$, y, $\mathbf{a}$, …
- ${}^{A}\mathbf{x}$
- u, v, p, q, …

Potential for Confusion!

Matrices

- $\mathbf{A}$, B, $\mathbf{C}$, …
- ${}^{A}\mathbf{A}_{B}$
- g, h, …

The 3×1 vector **a** and its 3×3 skew symmetric matrix counterpart **a**^

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix}$$

a

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

a^ = A

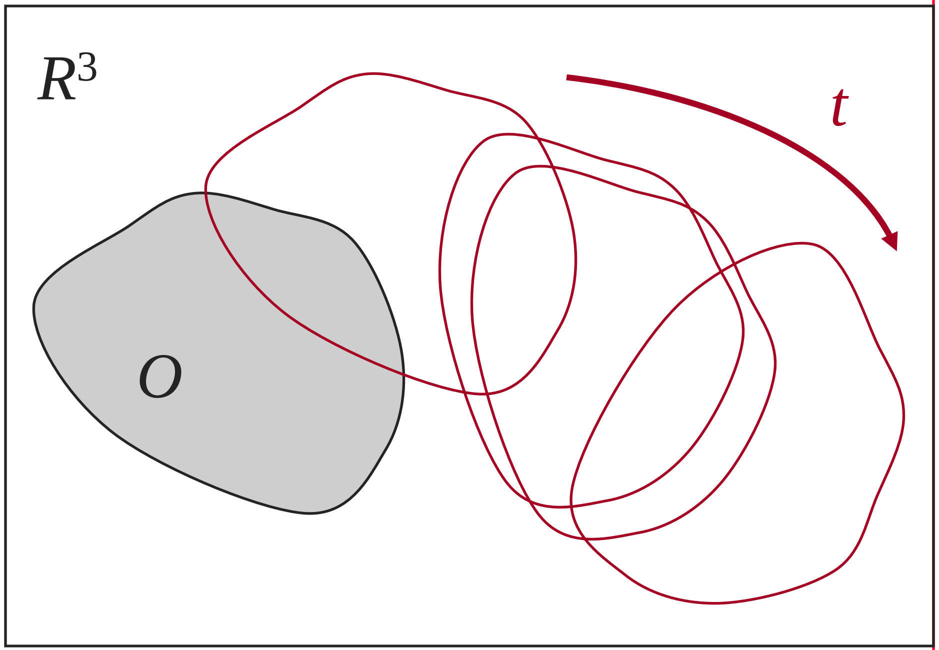
For any vector b

$$\mathbf{a} \times \mathbf{b} = \mathbf{A}\,\mathbf{b}$$

Rigid Body Transformation

Rigid Body Displacement

$$g : O \rightarrow R^3$$



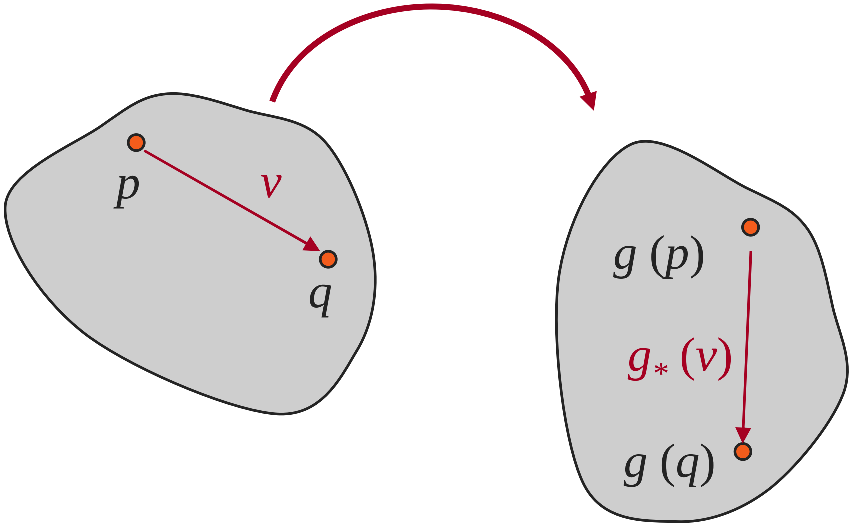
Rigid Body Motion

$$g(t) : O \rightarrow R^3$$

Coordinate Transformations and Displacements

Transformations of points

- Transformation (g) of points induces an action (g_*) on vectors

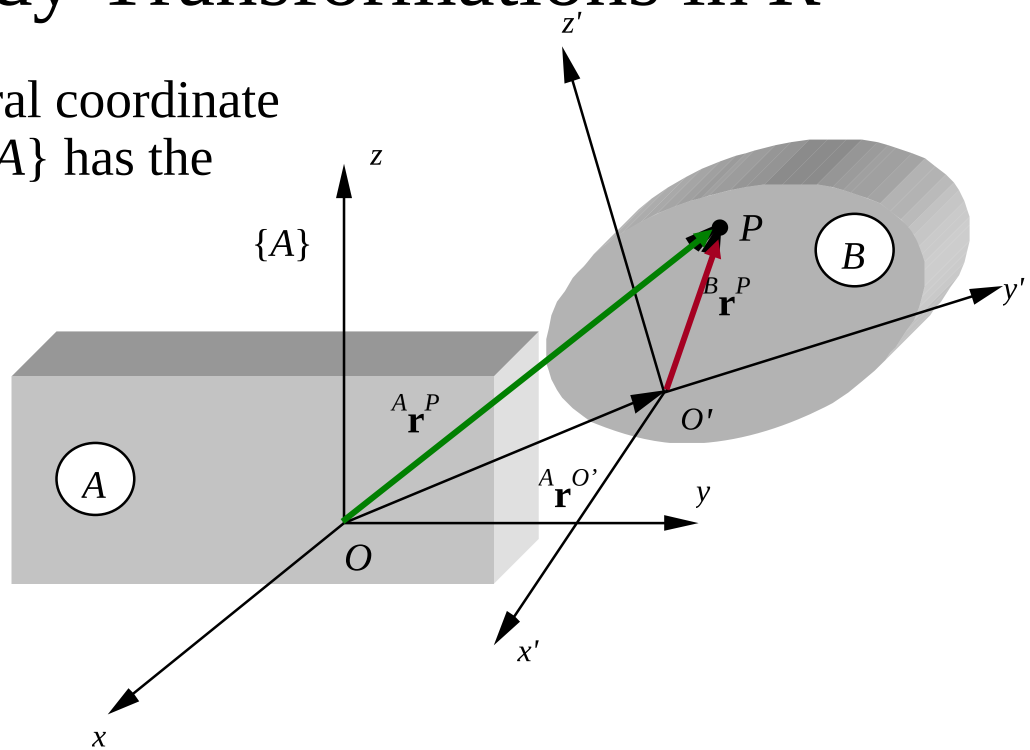


What are rigid body transformations? Displacements?

- g preserves lengths
- g_* preserves cross products

Rigid Body Transformations in R^3

Can show that the most general coordinate transformation from {B} to {A} has the following form



- position vector of P in {B} is transformed to position vector of P in {A}
- description of {B} as seen from an observer in {A}

$$\boxed{{}^{A}\mathbf{r}^{P}} = {}^{A}\mathbf{R}_{B}\,\boxed{{}^{B}\mathbf{r}^{P}} + {}^{A}\mathbf{r}^{O'}$$

Rotation of {B} with respect to {A}

Translation of the origin of {B} with respect to origin of {A}

Rotational transformations in R^3

Properties of rotation matrices

- Transpose is the inverse
- Determinant is +1

- Rotations preserve cross products

$$\mathbf{R}\,\mathbf{u} \times \mathbf{R}\,\mathbf{v} = \mathbf{R}\,(\mathbf{u} \times \mathbf{v})$$

- Rotation of skew symmetric matrices

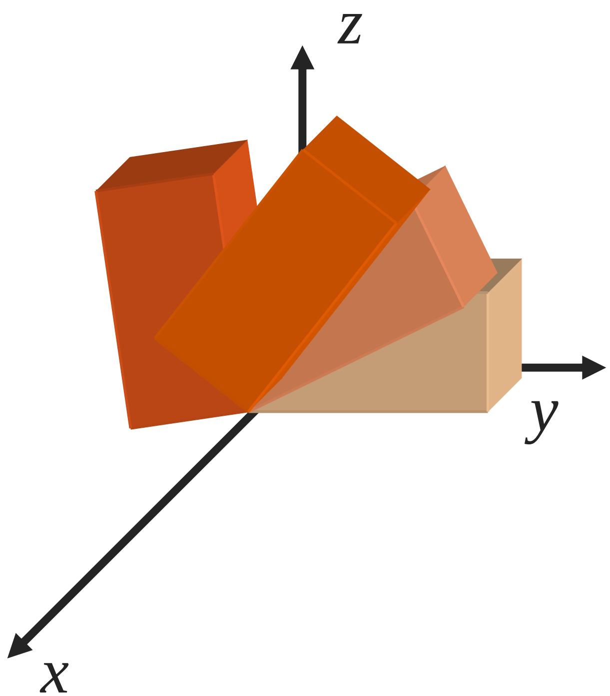
For any rotation matrix **R**:

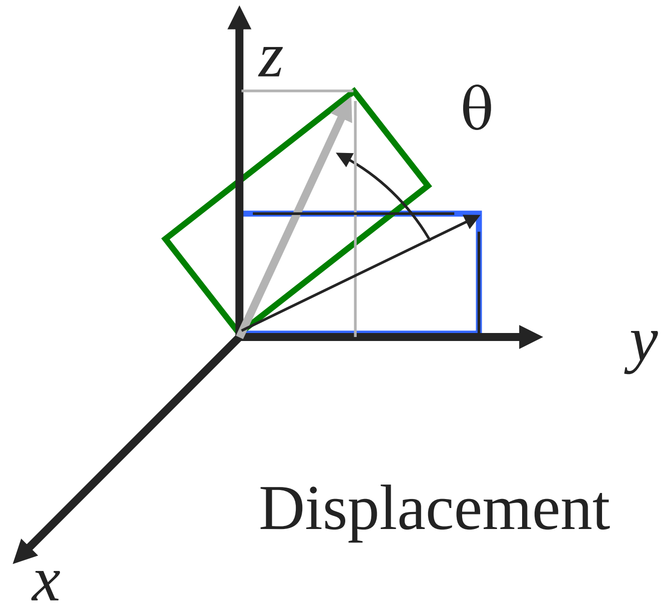
$$\mathbf{R}\;\mathbf{w}^\wedge\;\;\mathbf{R}^T \;=\; (\mathbf{R}\;\;\mathbf{w})^\wedge$$

Example: Rotation

- Rotation about the *x*-axis through θ

$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$$



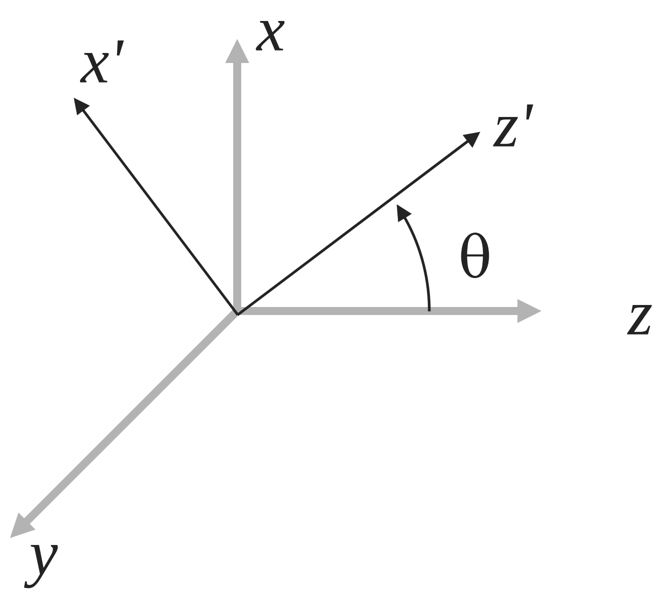


Displacement

Example: Rotation

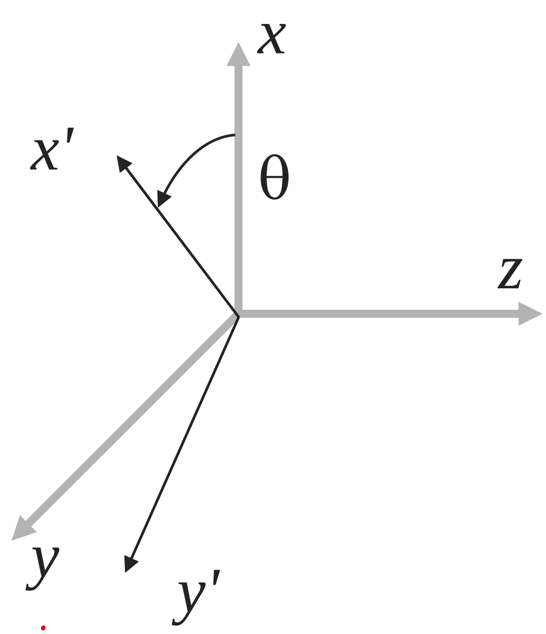
Rotation about the *y*-axis through θ

$$Rot(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

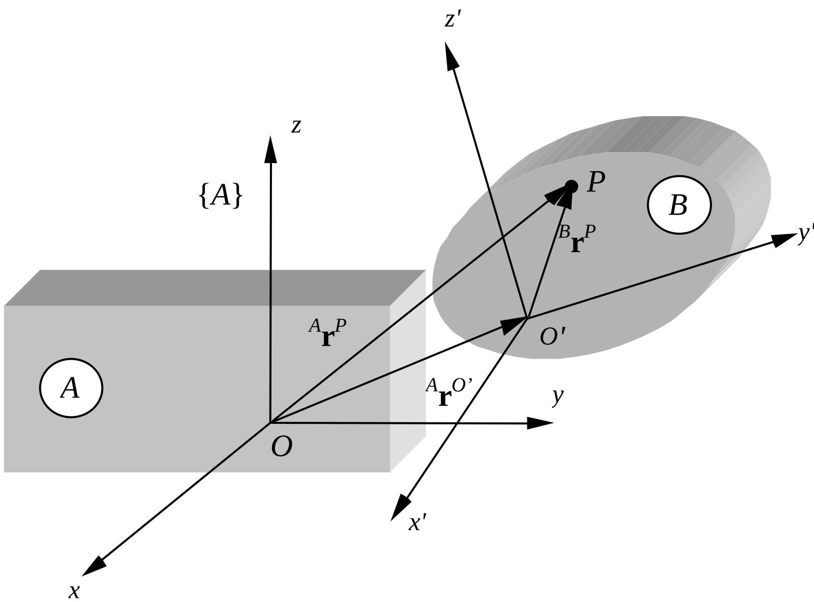


Rotation about the *z*-axis through θ

$$Rot(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

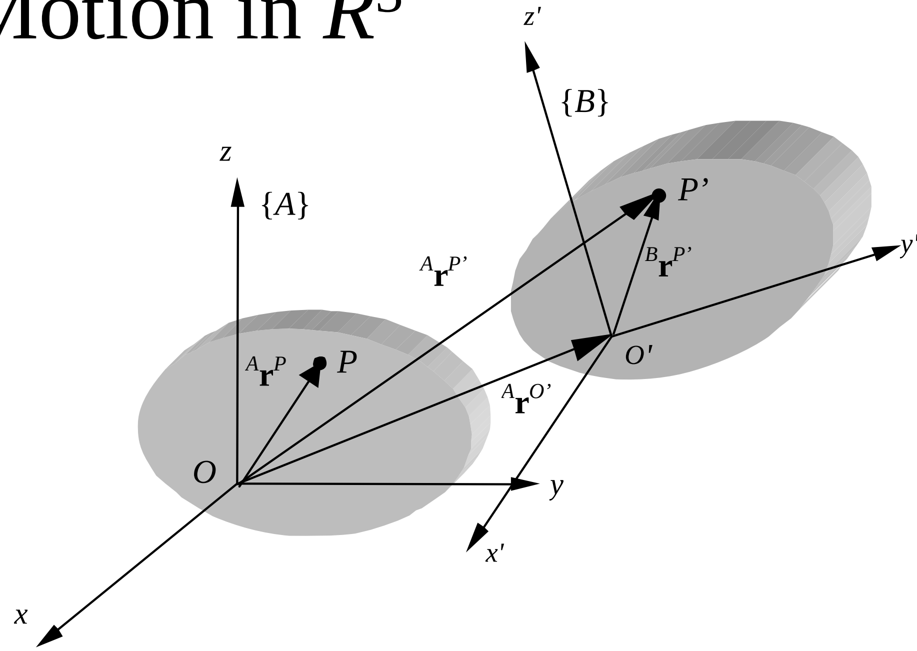


Rigid Motion in R^3



$$^A\mathbf{r}^P = {}^A\mathbf{R}_B\,{}^B\mathbf{r}^P + {}^A\mathbf{r}^{O'}$$

Coordinate transformation from $\{B\}$ to $\{A\}$



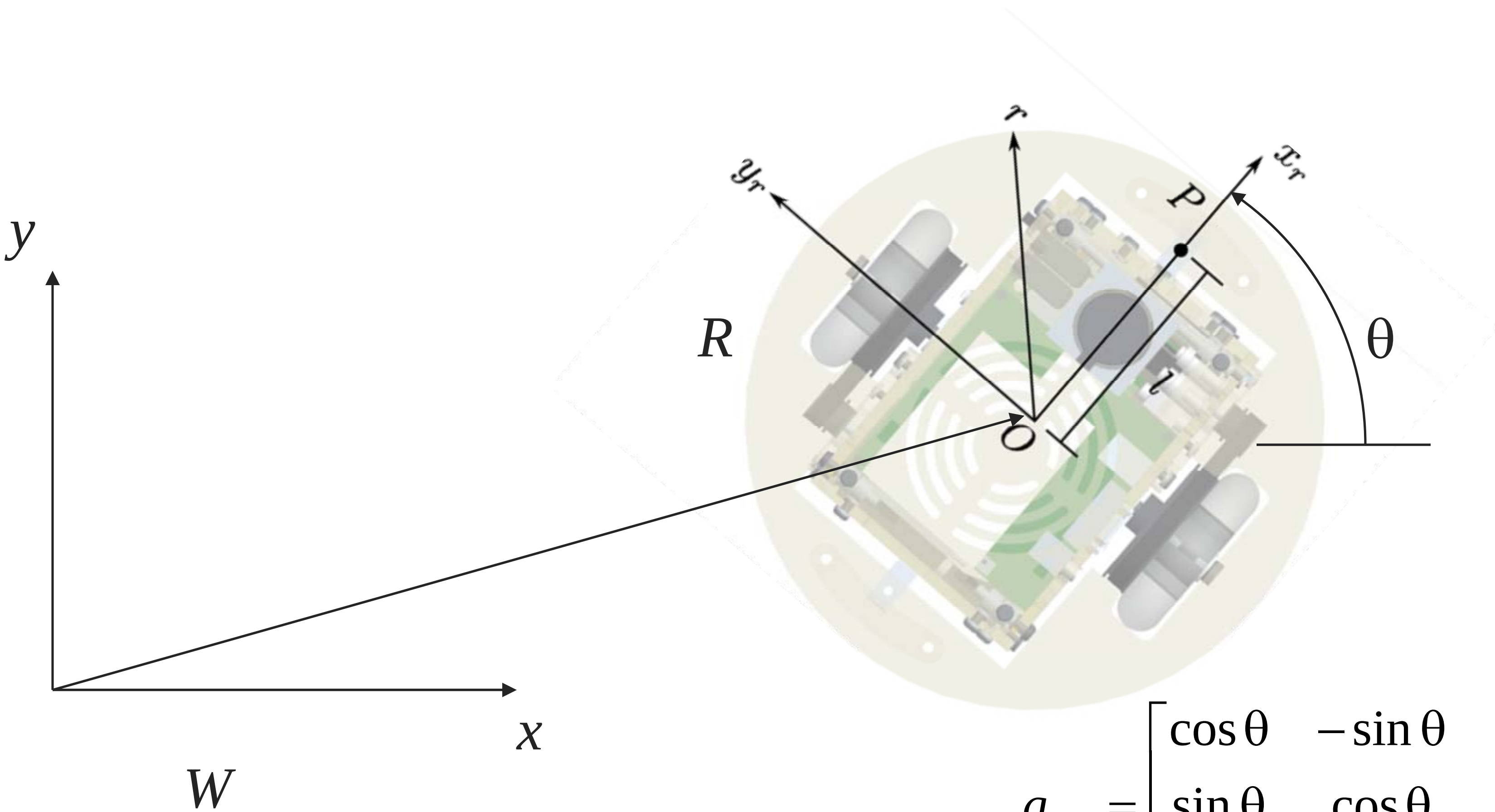
$$^A\mathbf{r}^{P'} = {}^A\mathbf{R}_B\,{}^A\mathbf{r}^P + {}^A\mathbf{r}^{O'}$$

Displacement of a body-fixed frame from $\{A\}$ to $\{B\}$

The same equation can have two interpretations:

- It transforms the position vector of any point in $\{B\}$ to the position vector in $\{A\}$
- It transforms the position vector of any point in the first position/orientation (described by $\{A\}$) to its new position vector in the second position orientation (described by$\{B\}$).

Mobile Robots



$$g_{WR} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

${}^{W}A_{R}$

The Lie group $SE(3)$

$$SE(3) = \left\{ \mathbf{A} \;\middle|\; \mathbf{A} = \left[\begin{array}{c|c} \mathbf{R} & \mathbf{r} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array} \right], \mathbf{R} \in R^{3\times 3}, \mathbf{r} \in R^3, \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I} \right\}$$

http://www.seas.upenn.edu/~meam520/notes02/RigidBodyMotion3.pdf

SE(3) is a Lie group

SE(3) satisfies the four axioms that must be satisfied by the elements of an *algebraic group*:

- The set is closed under the binary operation. In other words, if **A** and **B** are any two matrices in *SE*(3), **AB** ∈ *SE*(3).
- The binary operation is associative. In other words, if **A**, **B**, and **C** are any three matrices ∈ *SE*(3), then (**AB**) **C** = **A** (**BC**).
- For every element **A** ∈ *SE*(3), there is an identity element given by the 4×4 identity matrix, **I** ∈ *SE*(3), such that **AI = A**.
- For every element **A** ∈ *SE*(3), there is an identity inverse, $\mathbf{A}^{-1}$ ∈ *SE*(3), such that $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$.

SE(3) is a *continuous group.*

- the binary operation above is a continuous operation — the product of any two elements in *SE*(3) is a continuous function of the two elements
- the inverse of any element in *SE*(3) is a continuous function of that element.

In other words, *SE*(3) is a *differentiable manifold*. A group that is a differentiable manifold is called a *Lie group*[Sophus Lie (1842-1899)].

Composition of Displacements

Displacement from $\{A\}$ to $\{B\}$

$$ {}^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right], $$

Displacement from $\{B\}$ to $\{C\}$

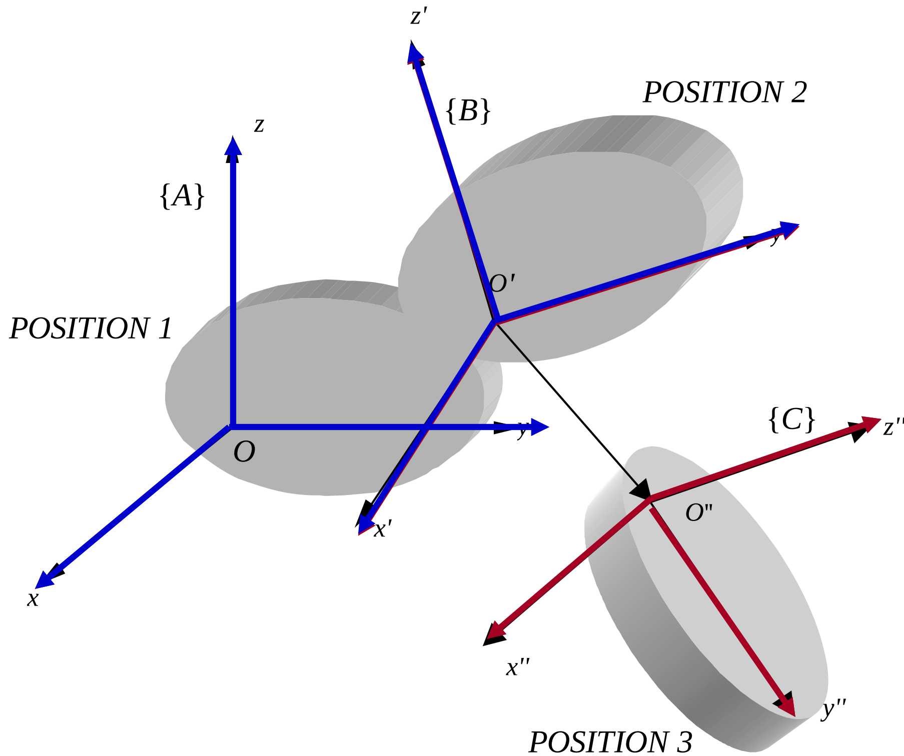
$$ {}^{B}\mathbf{A}_{C} = \left[\begin{array}{c|c} {}^{B}\mathbf{R}_{C} & {}^{B}\mathbf{r}^{O''} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right], $$

Displacement from $\{A\}$ to $\{C\}$

$$ {}^{A}\mathbf{A}_{C} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{C} & {}^{A}\mathbf{r}^{O''} \\ \hline \mathbf{0} & 1 \end{array}\right] $$

$$ = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0} & 1 \end{array}\right] \times \left[\begin{array}{c|c} {}^{B}\mathbf{R}_{C} & {}^{B}\mathbf{r}^{O''} \\ \hline \mathbf{0} & 1 \end{array}\right] $$

$$ = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} & {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{r}^{O''} + {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0} & 1 \end{array}\right] $$

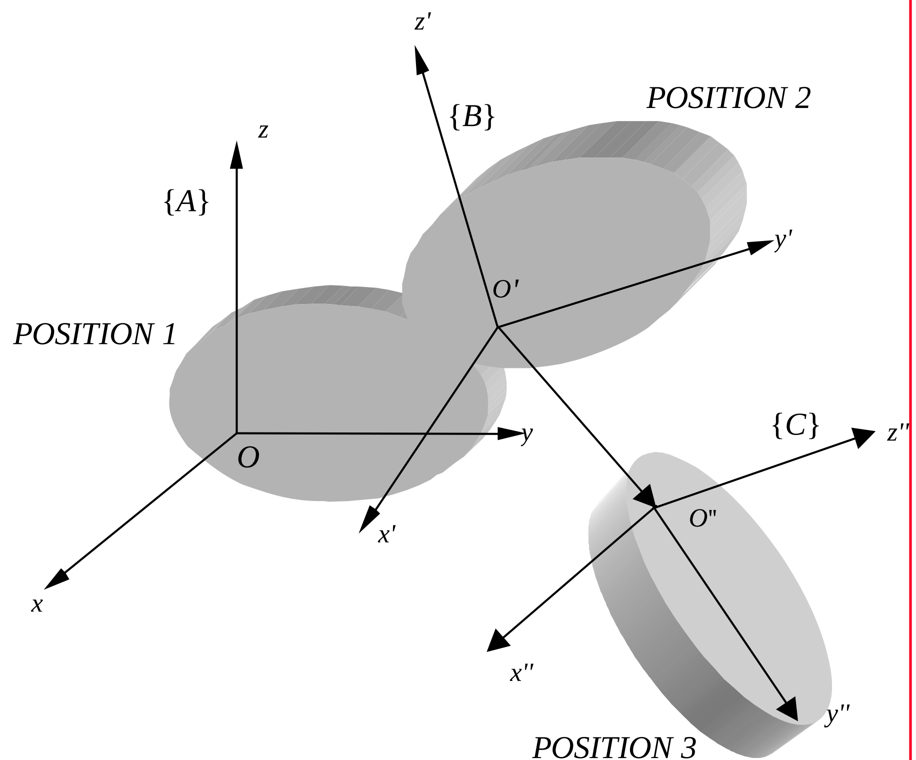


Note ${}^{X}\mathbf{A}_{Y}$ describes the displacement of the body-fixed frame from $\{X\}$ to $\{Y\}$ in reference frame $\{X\}$

Composition (continued)

Composition of displacements

- Displacements are generally described in a body-fixed frame
- Example: ${}^{B}\mathbf{A}_{C}$ is the displacement of a rigid body from B to C *relative* to the axes of the "first frame" B.



Composition of transformations

- Same basic idea
- ${}^{A}\mathbf{A}_{C} = {}^{A}\mathbf{A}_{B}\ {}^{B}\mathbf{A}_{C}$

Note that our description of transformations (e.g., ${}^{B}\mathbf{A}_{C}$) is *relative* to the "first frame" (B, the frame with the leading superscript).

Note ${}^{X}\mathbf{A}_{Y}$ describes the displacement of the body-fixed frame from $\{X\}$ to $\{Y\}$ in reference frame $\{X\}$

Rigid Body Kinematics

Subgroups of $SE(3)$

Subgroup	Notation	Definition	Significance
The group of rotations in three dimensions	$SO(3)$	The set of all proper orthogonal matrices. $SO(3)=\{\mathbf{R} \mid \mathbf{R} \in R^{3\times 3}, \mathbf{R}^T\mathbf{R}=\mathbf{R}\mathbf{R}^T=\mathbf{I}\}$	All spherical displacements. Or the set of all displacements that can be generated by a spherical joint (*S*-pair).
Special Euclidean group in two dimensions	$SE(2)$	The set of all 3×3 matrices with the structure: $\begin{bmatrix} \cos\theta & \sin\theta & r_x \\ -\sin\theta & \cos\theta & r_y \\ 0 & 0 & 1 \end{bmatrix}$ where θ, r_x, and r_y are real numbers.	All planar displacements. Or the set of displacements that can be generated by a planar pair (*E*-pair).
The group of rotations in two dimensions	$SO(2)$	The set of all 2×2 proper orthogonal matrices. They have the structure: $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where θ is a real number.	All rotations in the plane, or the set of all displacements that can be generated by a single revolute joint (*R*-pair).
The group of translations in *n* dimensions.	$T(n)$	The set of all $n\times 1$ real vectors with vector addition as the binary operation.	All translations in *n* dimensions. $n = 2$ indicates planar, $n = 3$ indicates spatial displacements.
The group of translations in one dimension.	$T(1)$	The set of all real numbers with addition as the binary operation.	All translations parallel to one axis, or the set of all displacements that can be generated by a single prismatic joint (*P*-pair).
The group of cylindrical displacements	$SO(2)\times T(1)$	The Cartesian product of $SO(2)$ and $T(1)$	All rotations in the plane and translations along an axis perpendicular to the plane, or the set of all displacements that can be generated by a cylindrical joint (*C*-pair).
The group of screw displacements	$H(1)$	A one-parameter subgroup of $SE(3)$	All displacements that can be generated by a helical joint (*H*-pair).