The Lie group $SE(3)$

$$SE(3)=\left\{\mathbf{A}\ \middle|\ \mathbf{A}=\left[\begin{array}{c|c}\mathbf{R} & \mathbf{r}\\ \hline \mathbf{0}_{1\times 3} & 1\end{array}\right], \mathbf{R}\in R^{3\times 3}, \mathbf{r}\in R^{3}, \mathbf{R}^{T}\mathbf{R}=\mathbf{R}\mathbf{R}^{T}=\mathbf{I}, \left|\mathbf{R}\right|=1\right\}$$

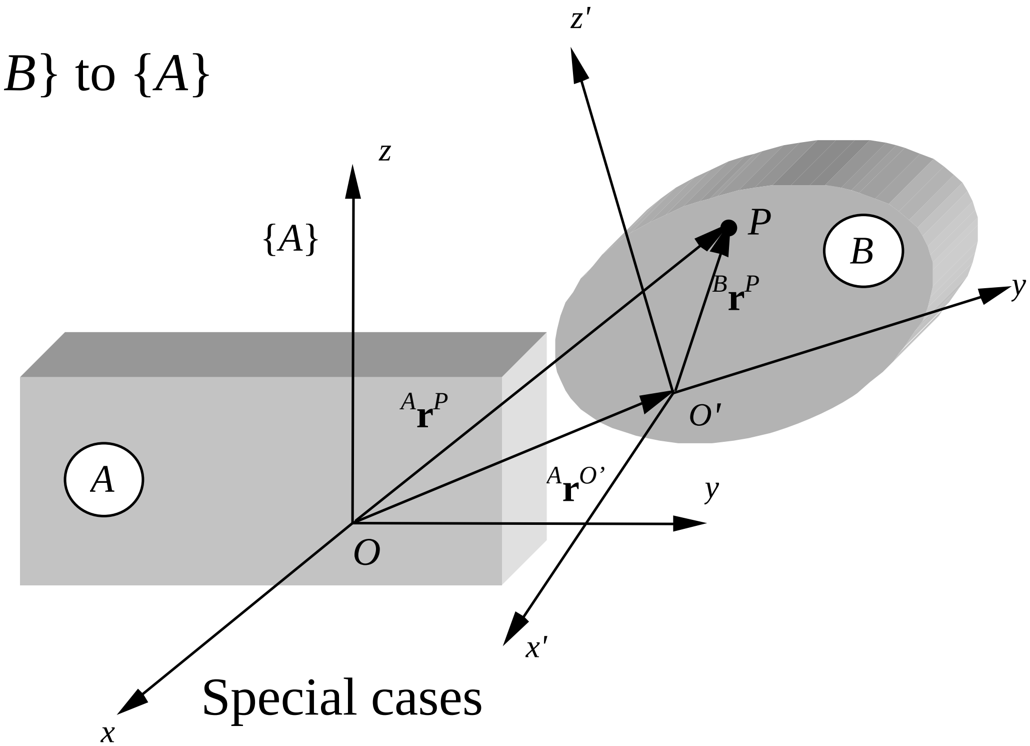
http://www.seas.upenn.edu/~meam620/notes/RigidBodyMotion3.pdf

Homogeneous Transformation Matrix

Coordinate transformation from $\{B\}$ to $\{A\}$

$$ {}^{A}\mathbf{r}^{P'} = {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{r}^{P} + {}^{A}\mathbf{r}^{O'} $$

$$ \begin{bmatrix} {}^{A}\mathbf{r}^{P} \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \begin{bmatrix} {}^{B}\mathbf{r}^{P} \\ \hline 1 \end{bmatrix} $$



Homogeneous transformation matrix

$$ {}^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right] $$

Special cases

1. $$ {}^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 1} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]. $$

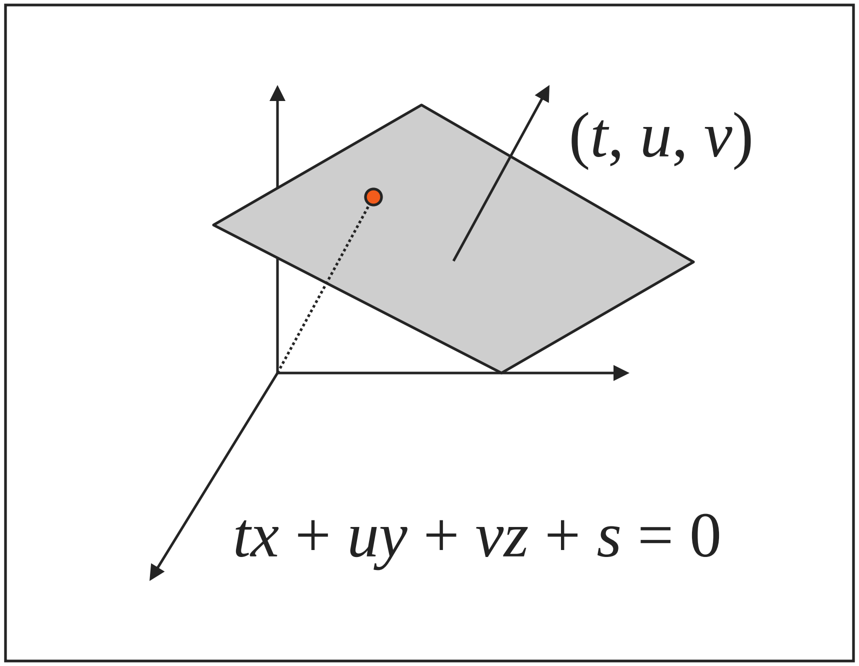
2. $$ {}^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]. $$

Homogeneous Coordinates

- Description of a point
  (x, y, z)
- Description of a plane
  (t, u, v, s)
- Equation of a circle
  $x^2 + y^2 + z^2 = a^2$

Homogeneous coordinates

- Description of a point
  (x, y, z, w)
- Equation of a plane
  $tx + uy + vz + sw = 0$
- Equation of a sphere
  $x^2 + y^2 + z^2 = a^2 w^2$



Central ideas

- Equivalence class
- Projective space P^3, and not Euclidean space R^3

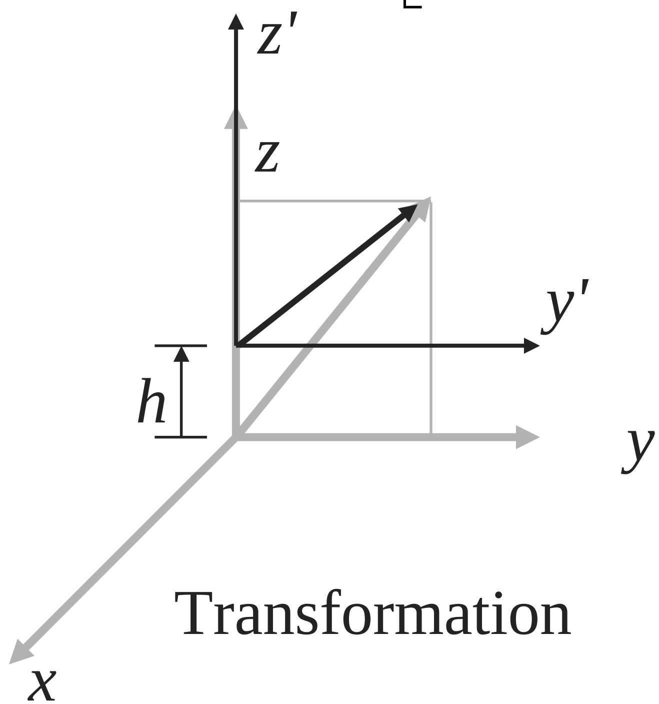
Mathematical and practical advantages

Example: Translation

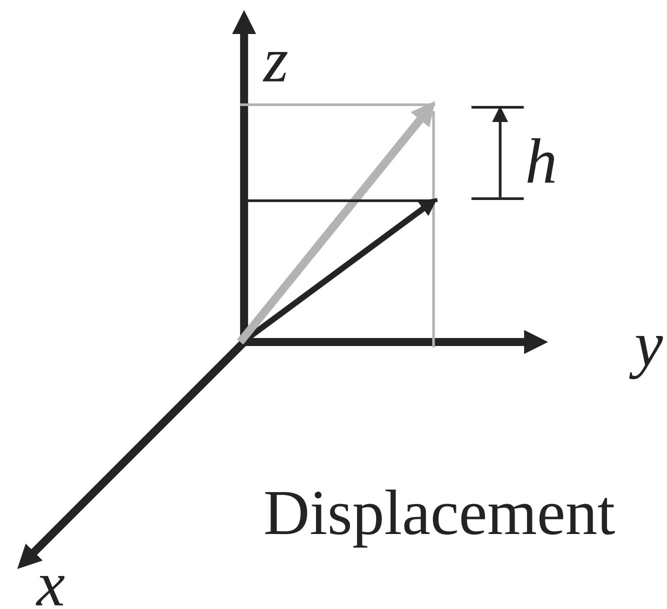
- Translation along the z-axis through h

$$Trans(z,h)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & h\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & h\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x'\\ y'\\ z'\\ 1\end{bmatrix}$$



Transformation

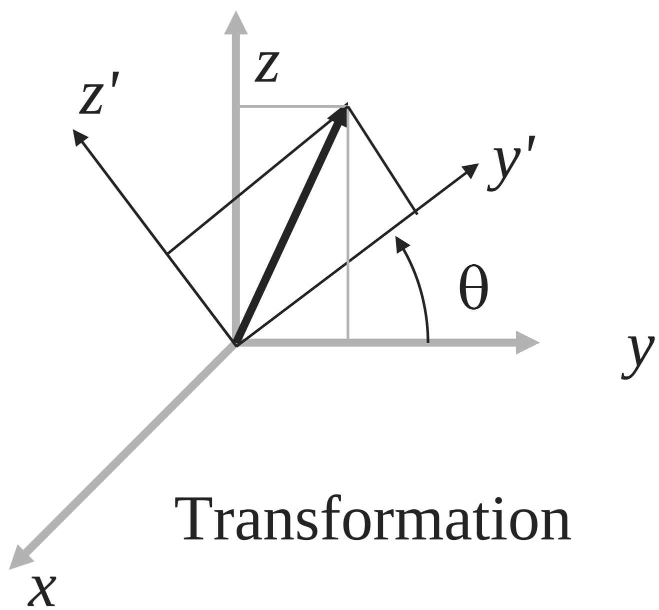


Displacement

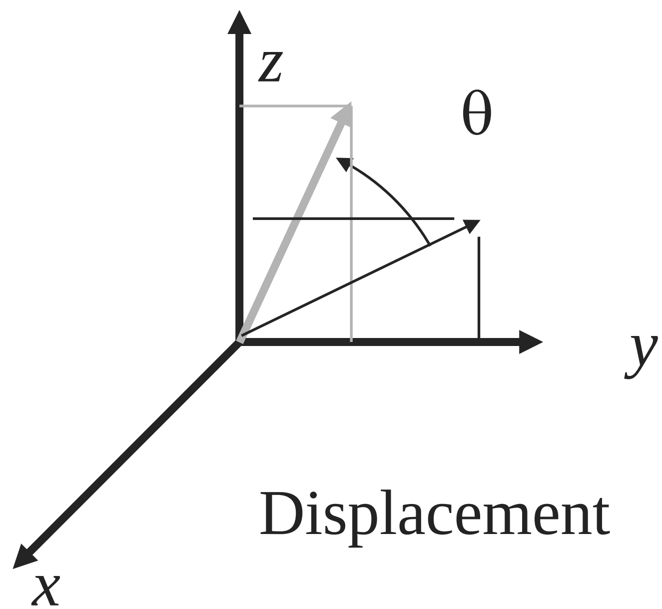
Example: Rotation

- Rotation about the x-axis through θ

$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad \begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x'\\ y'\\ z'\\ 1\end{bmatrix}$$



Transformation

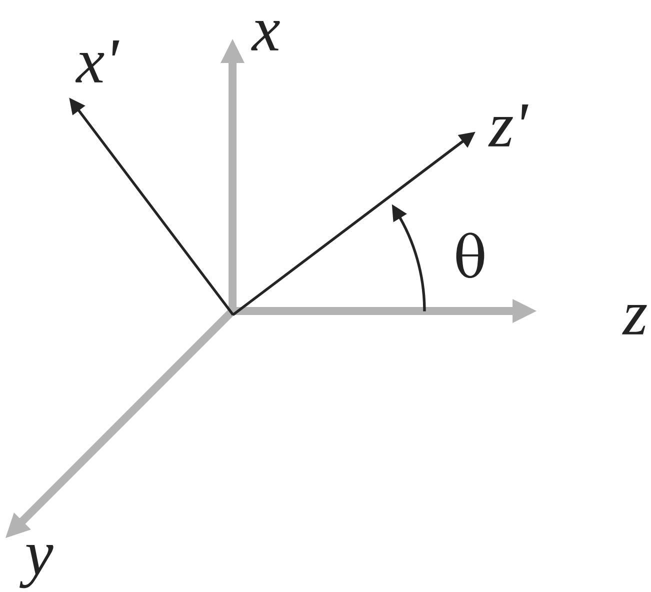


Displacement

Example: Rotation

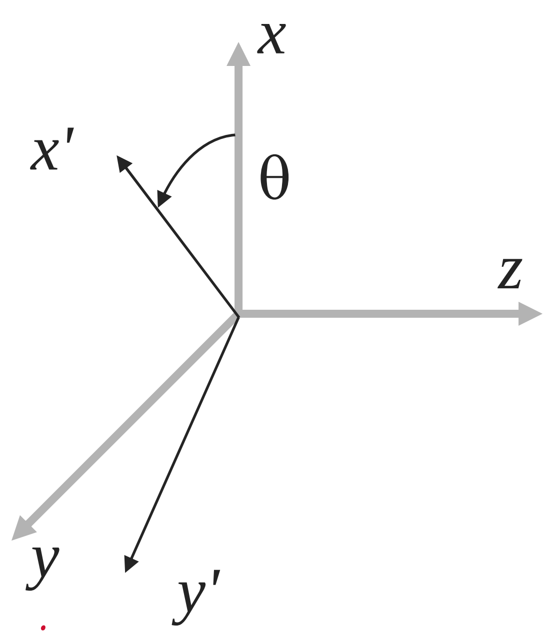
Rotation about the *y*-axis through θ

$$Rot(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

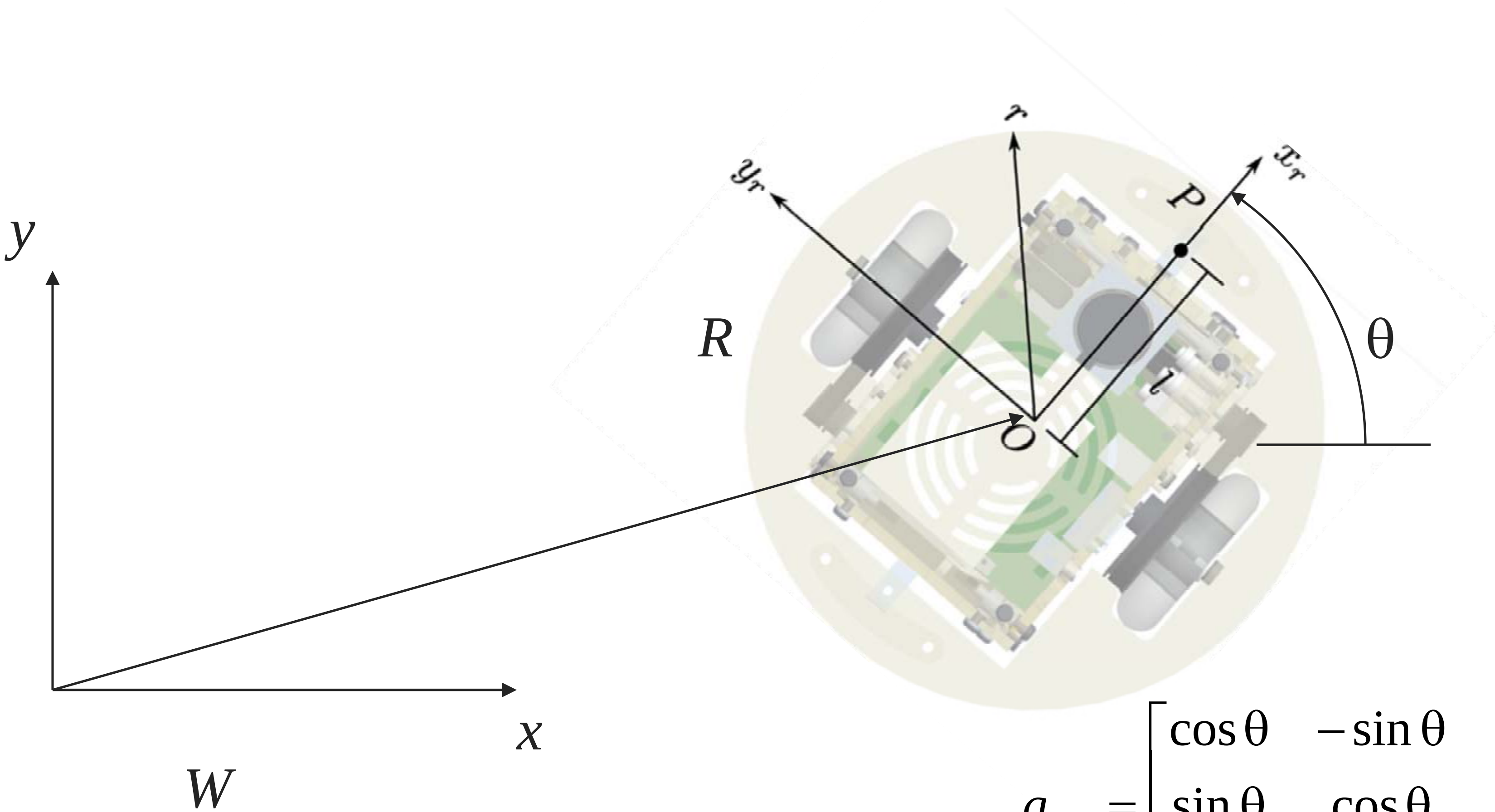


Rotation about the *z*-axis through θ

$$Rot(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



Mobile Robots



$$g_{WR} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

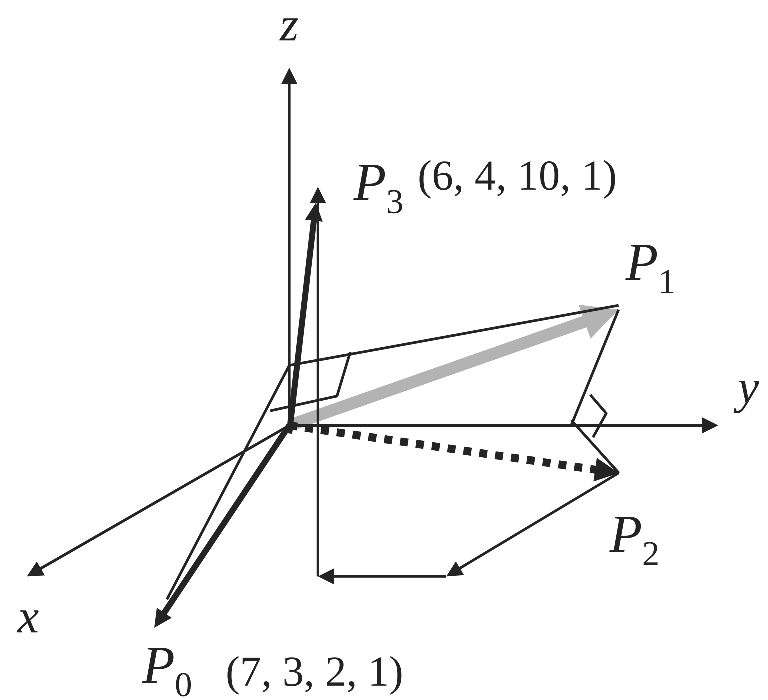
$^{W}A_{R}$

Example: Displacement of Points

Displace (7, 3, 2) through a sequence of:

1. Rot(z, 90)
2. Rot(y, 90)
3. Trans(4, -3, 7)

in the frame F: (x, y, z):



Trans(4, -3, 7) Rot(y, 90) Rot(z, 90)

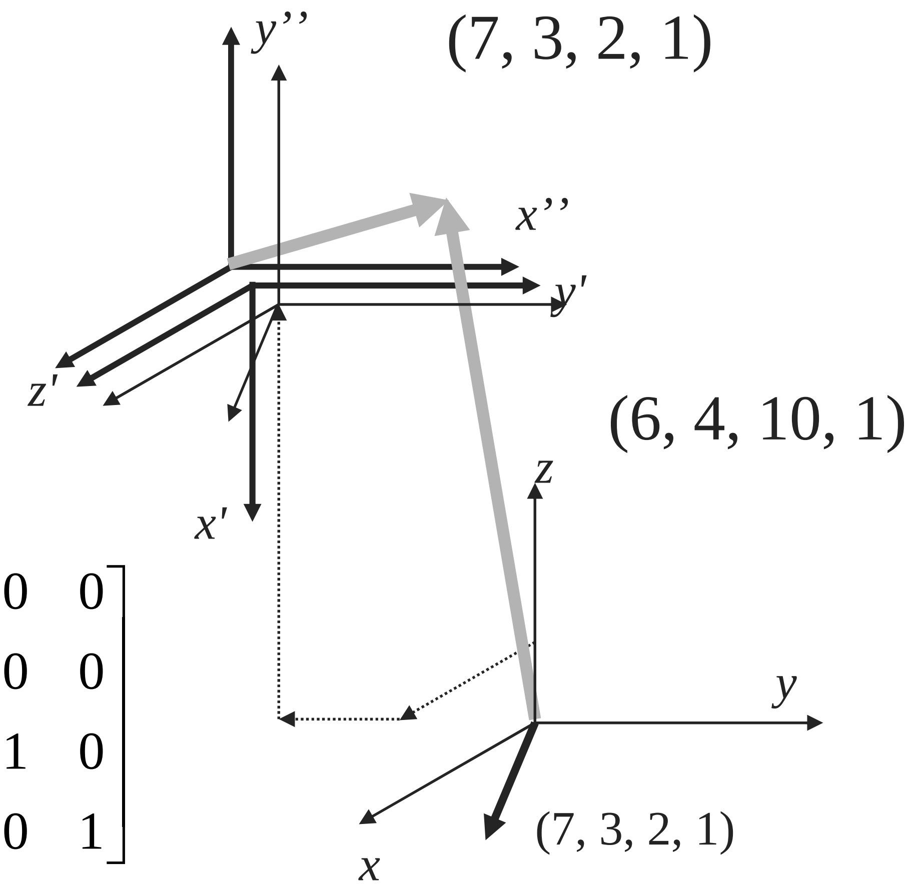
$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Transformation of Points

Successive transformations of (7, 3, 2):

1. Trans(4, -3, 7)
2. Rot(y, 90)
3. Rot(z, 90)

in a body fixed frame



Trans(4, -3, 7) Rot(y, 90) Rot(z, 90)

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$SE(3)$ is a Lie group

$SE(3)$ satisfies the four axioms that must be satisfied by the elements of an *algebraic group*:

- The set is closed under the binary operation. In other words, if **A** and **B** are any two matrices in $SE(3)$, $\mathbf{AB} \in SE(3)$.
- The binary operation is associative. In other words, if **A**, **B**, and **C** are any three matrices $\in SE(3)$, then $(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$.
- For every element $\mathbf{A} \in SE(3)$, there is an identity element given by the 4×4 identity matrix, $\mathbf{I} \in SE(3)$, such that $\mathbf{AI} = \mathbf{A}$.
- For every element $\mathbf{A} \in SE(3)$, there is an identity inverse, $\mathbf{A}^{-1} \in SE(3)$, such that $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$.

$SE(3)$ is a *continuous group.*

- the binary operation above is a continuous operation — the product of any two elements in $SE(3)$ is a continuous function of the two elements
- the inverse of any element in $SE(3)$ is a continuous function of that element.

In other words, $SE(3)$ is a *differentiable manifold*. A group that is a differentiable manifold is called a *Lie group*[Sophus Lie (1842-1899)].

Composition of Displacements

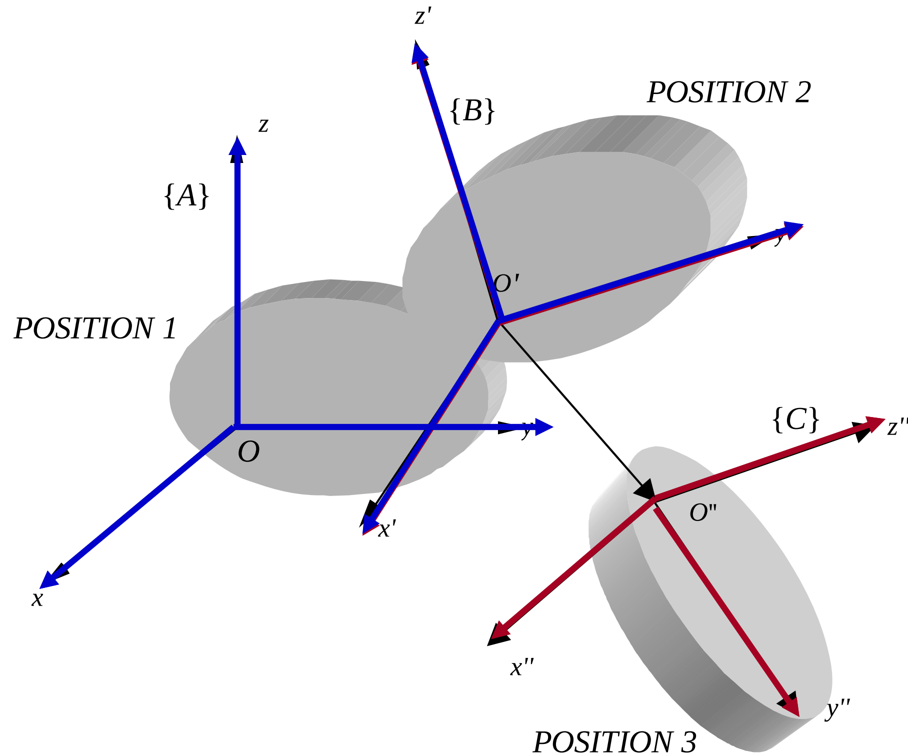
Displacement from $\{A\}$ to $\{B\}$

$$^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} ^{A}\mathbf{R}_{B} & ^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right],$$

Displacement from $\{B\}$ to $\{C\}$

$$^{B}\mathbf{A}_{C} = \left[\begin{array}{c|c} ^{B}\mathbf{R}_{C} & ^{B}\mathbf{r}^{O''} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right],$$

Displacement from $\{A\}$ to $\{C\}$



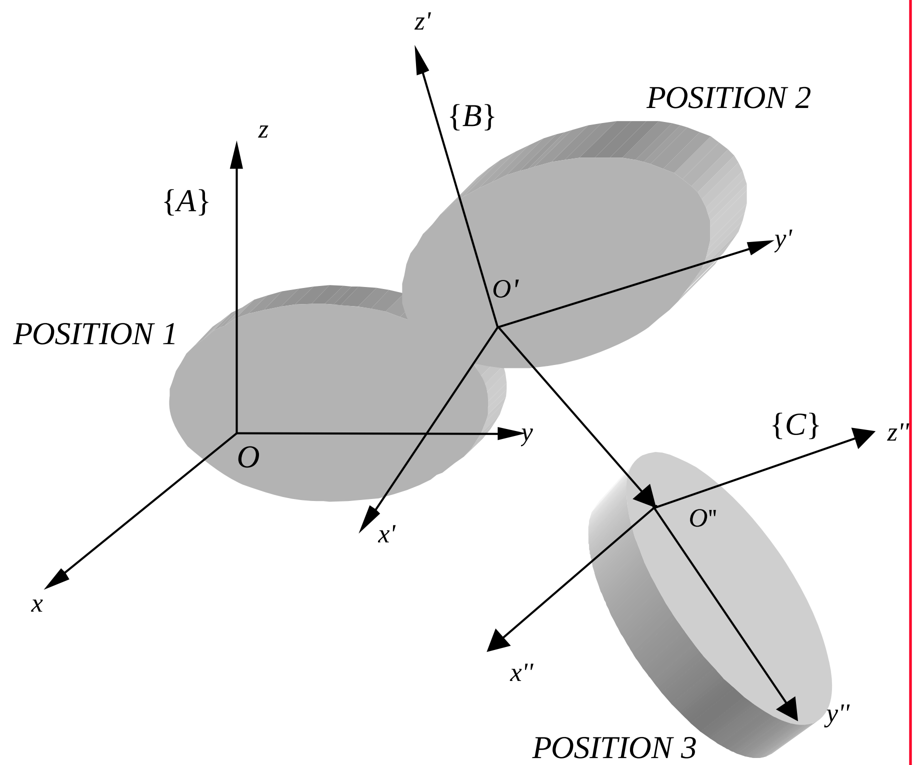
$$
\begin{aligned}
^{A}\mathbf{A}_{C} &= \left[\begin{array}{c|c} ^{A}\mathbf{R}_{C} & ^{A}\mathbf{r}^{O''} \\ \hline \mathbf{0} & 1 \end{array}\right] \\
&= \left[\begin{array}{c|c} ^{A}\mathbf{R}_{B} & ^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0} & 1 \end{array}\right] \times \left[\begin{array}{c|c} ^{B}\mathbf{R}_{C} & ^{B}\mathbf{r}^{O''} \\ \hline \mathbf{0} & 1 \end{array}\right] \\
&= \left[\begin{array}{c|c} ^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} & ^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{r}^{O''} + {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0} & 1 \end{array}\right]
\end{aligned}
$$

Note $^{X}\mathbf{A}_{Y}$ describes the displacement of the body-fixed frame from $\{X\}$ to $\{Y\}$ in reference frame $\{X\}$

Composition (continued)

Composition of displacements

- Displacements are generally described in a body-fixed frame
- Example: ${}^{B}\mathbf{A}_{C}$ is the displacement of a rigid body from B to C *relative* to the axes of the "first frame" B.



Composition of transformations

- Same basic idea
- ${}^{A}\mathbf{A}_{C} = {}^{A}\mathbf{A}_{B}\ {}^{B}\mathbf{A}_{C}$

Note that our description of transformations (e.g., ${}^{B}\mathbf{A}_{C}$) is *relative* to the "first frame" (B, the frame with the leading superscript).

Note ${}^{X}\mathbf{A}_{Y}$ describes the displacement of the body-fixed frame from $\{X\}$ to $\{Y\}$ in reference frame $\{X\}$

Transformations associated with the Lie group $SE(3)$

Differentiable Manifold

Definition

A manifold of dimension *n* is a set *M* which is locally homeomorphic* to R^n.

Homeomorphism:

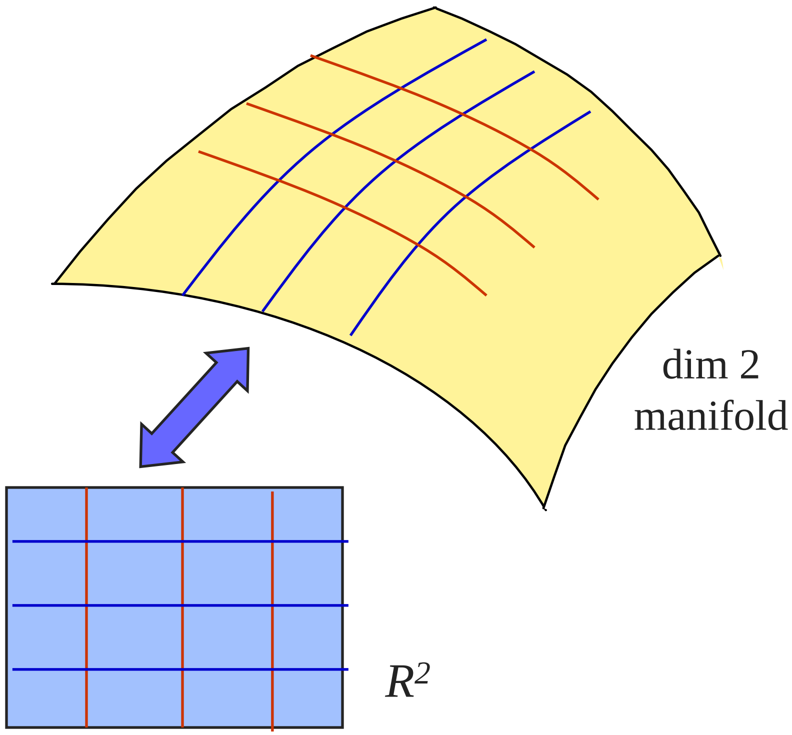
A map *f* from *M* to *N* and its inverse, f^{-1} are both continuous.

Smooth map

A map *f* from $U \subset R^m$ to $V \subset R^n$ is smooth if all partial derivatives of *f*, of all orders, exist and are continuous.

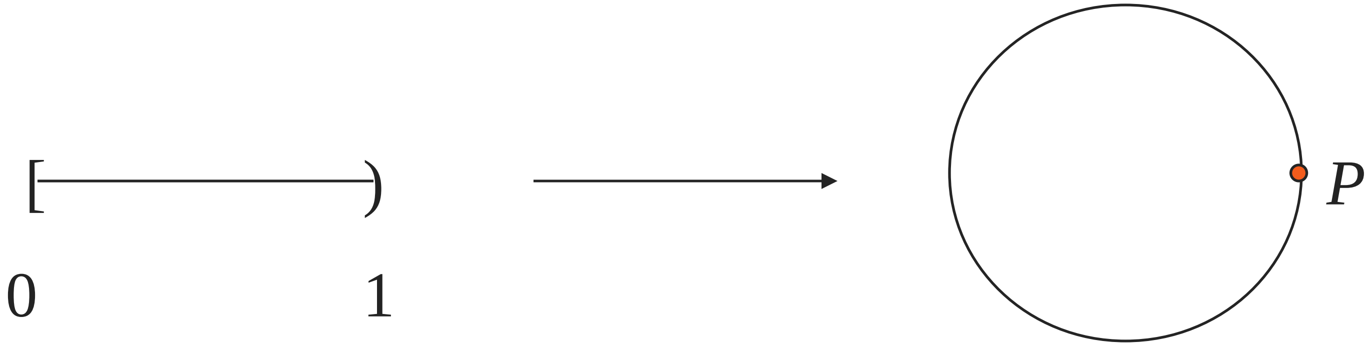
Diffeomorphism

A smooth map *f* from $U \subset R^n$ to $V \subset R^n$ is a diffeomorphism if all partial derivatives of f^{-1}, of all orders, exist and are continuous.



Examples

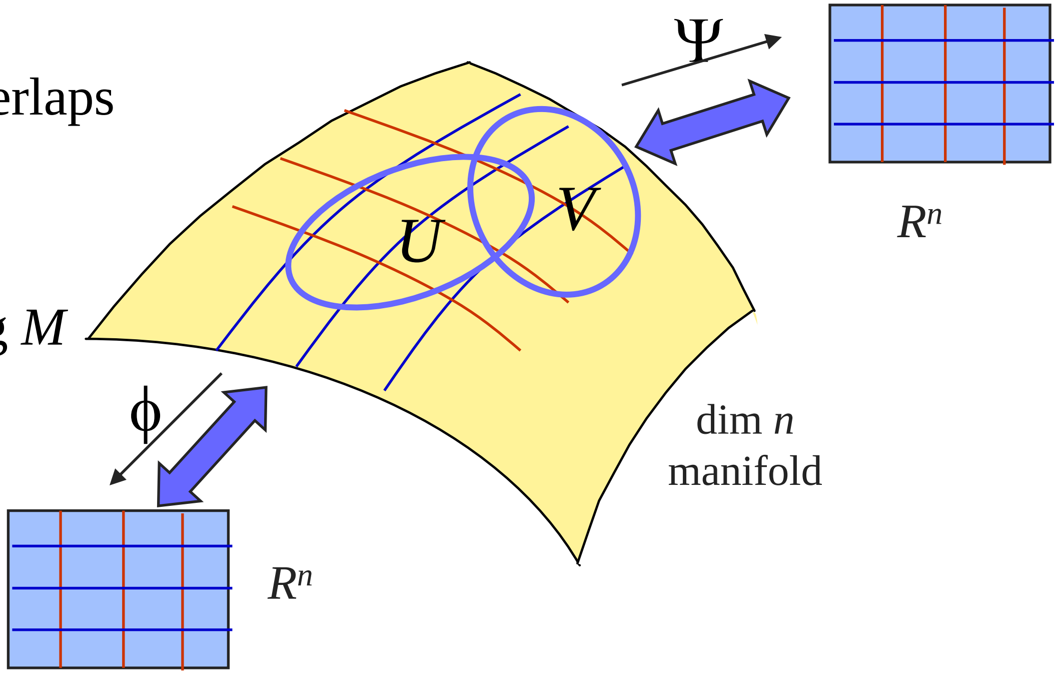
$$f:(-1,1)\to R,\quad f(x)=\frac{x}{1-x^2}$$



Smooth Manifold

- Differentiable manifold is locally homeomorphic to R^n
- Parametrize the manifold using a set of local coordinate charts
  - (U, ϕ), (V, Ψ), …
- Require compatibility on overlaps

  C^∞ related
- Collection of charts covering M



Actions of $SE(3)$

M any smooth manifold

- Think R^3, $SE(3)$, subgroups of $SE(3)$

A *left action* of $SE(3)$ on M is a smooth map, $\Phi: SE(3) \times M \to M$, such that

- $\Phi(\mathbf{I}, x) = x, \quad x \in M$
- $\Phi(\mathbf{A}, \Phi(\mathbf{B}, x)) = \Phi(\mathbf{AB}, x), \qquad \mathbf{A}, \mathbf{B} \in SE(3), \; x \in M$

Actions of $SE(3)$

1. Action of $SE(3)$ on R^3
   - Displacement of points, $\mathbf{p} \rightarrow \mathbf{Ap}$

2. Action of $SE(3)$ on itself

   $\Phi: SE(3) \times SE(3) \rightarrow SE(3)$

   $\Phi_{\mathbf{Q}}: \mathbf{A} \rightarrow \mathbf{Q\,A\,Q}^{-1}$

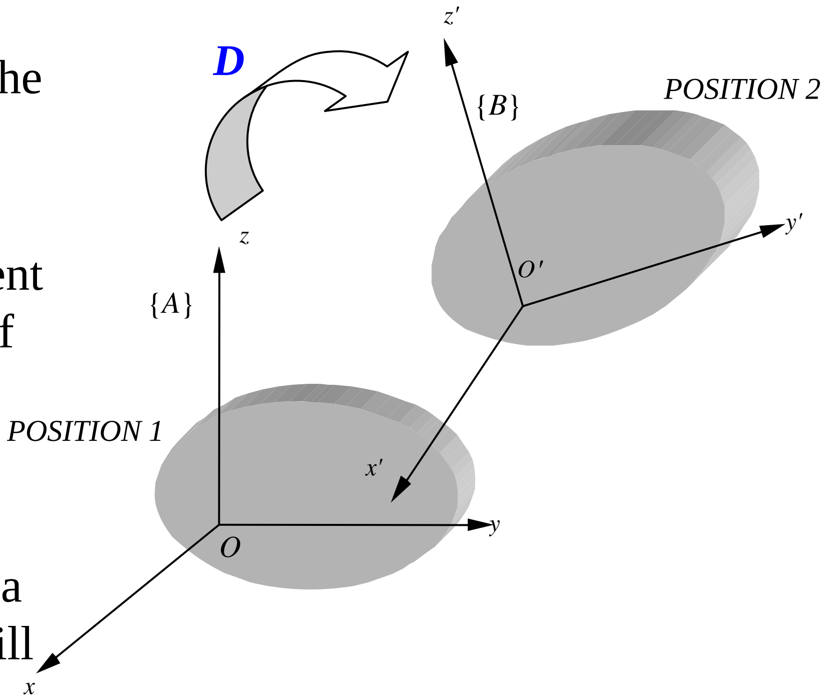
   - $\mathbf{A}$ denotes a displacement of a rigid body
   - $\Phi_{\mathbf{Q}}$ transforms the displacement $\mathbf{A}$

Actions on the Lie algebra (later)

Rigid Body Kinematics

What happens when you want to describe the displacement of the body-fixed frame from $\{A\}$ to $\{B\}$ in reference frame $\{F\}$?

Displacement is described in $\{A\}$ by the homogeneous transform, ${}^{A}\mathbf{A}_{B}$.



Want to describe the same displacement in $\{F\}$. The position and orientation of $\{A\}$ relative to $\{F\}$ is given by the homogeneous transform, ${}^{F}\mathbf{A}_{A}$.

The same displacement which moves a body-fixed frame from $\{A\}$ to $\{B\}$, will move another body-fixed frame from $\{F\}$ to $\{G\}$:

$$ {}^{F}\mathbf{A}_{G} = {}^{F}\mathbf{A}_{A}\,{}^{A}\mathbf{A}_{B}\,{}^{B}\mathbf{A}_{G} $$

$$ \boxed{{}^{F}\mathbf{A}_{G} = {}^{F}\mathbf{A}_{A}\,{}^{A}\mathbf{A}_{B}\,({}^{F}\mathbf{A}_{A})^{-1}} $$

Rigid Body Kinematics

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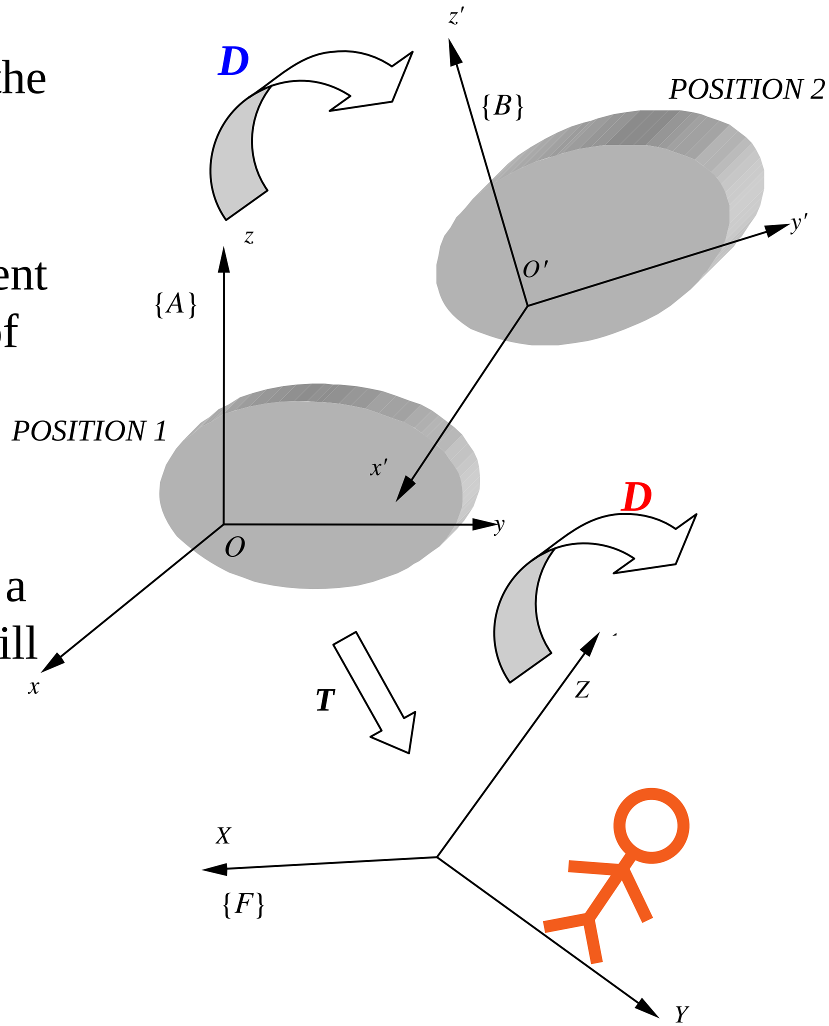
Want to describe the same displacement in $\{F\}$. The position and orientation of $\{A\}$ relative to $\{F\}$ is given by the homogeneous transform, ${}^{F}\mathbf{A}_{A}$.

The same displacement which moves a body-fixed frame from $\{A\}$ to $\{B\}$, will move another body-fixed frame from $\{F\}$ to $\{G\}$:

$$ {}^{F}\mathbf{A}_{G} = {}^{F}\mathbf{A}_{A}\, {}^{A}\mathbf{A}_{B}\, {}^{B}\mathbf{A}_{G} $$

$$ \boxed{{}^{F}\mathbf{A}_{G} = {}^{F}\mathbf{A}_{A}\, {}^{A}\mathbf{A}_{B}\, ({}^{F}\mathbf{A}_{A})^{-1}} $$



Rigid Body Kinematics

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Want to describe the same displacement in $\{F\}$. The position and orientation of $\{A\}$ relative to $\{F\}$ is given by the homogeneous transform, ${}^{F}\mathbf{A}_{A}$.

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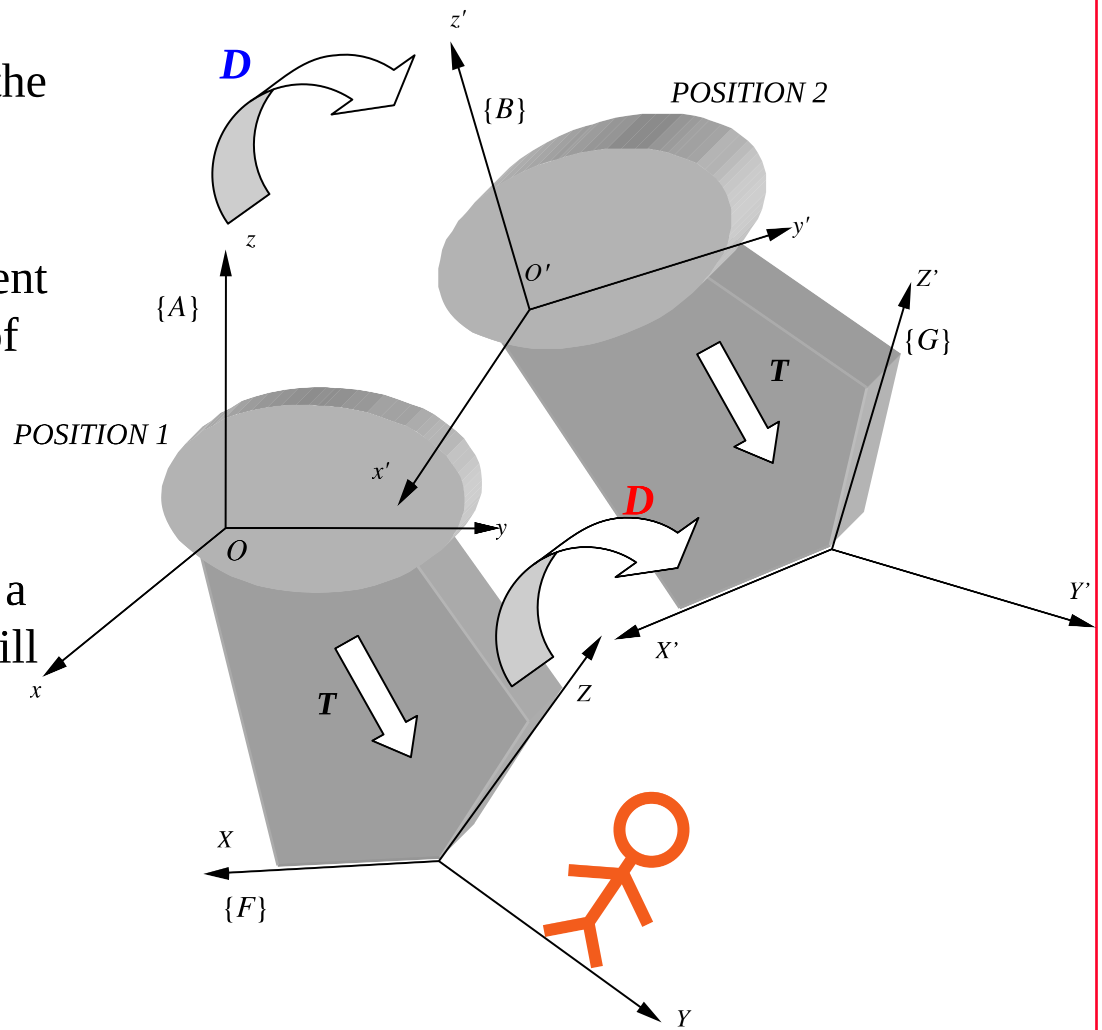
$$ {}^{F}\mathbf{A}_{G} = {}^{F}\mathbf{A}_{A}\,{}^{A}\mathbf{A}_{B}\,{}^{B}\mathbf{A}_{G} $$

$$ \boxed{{}^{F}\mathbf{A}_{G} = {}^{F}\mathbf{A}_{A}\,{}^{A}\mathbf{A}_{B}\,({}^{F}\mathbf{A}_{A})^{-1}} $$



Rotational motion in R^3

Specialize to $SO(3)$

Euler's Theorem for Rotations

Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O.

Later: Chasles' Theorem for Rotations

The most general rigid body displacement can be produced by a translation along a line followed (or preceded) by a rotation about that line.

Proof of Euler's Theorem for Spherical Displacements

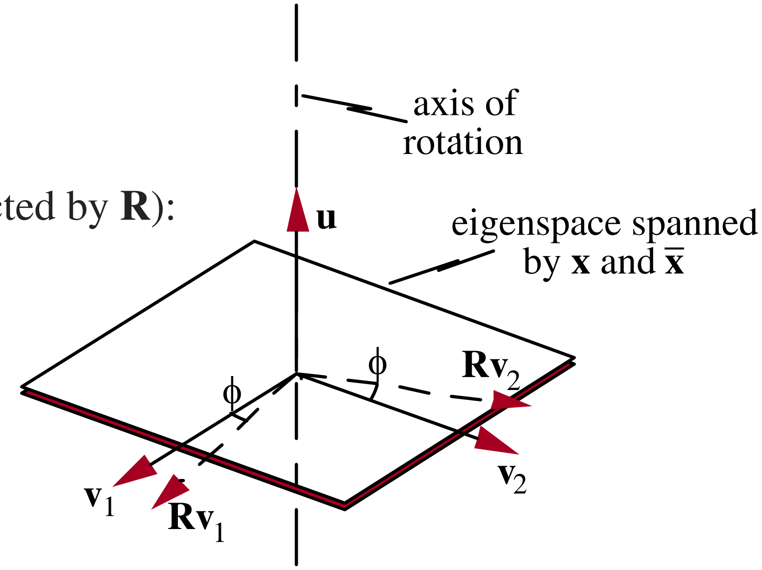
Displacement from $\{F\}$ to $\{M\}$

$$\mathbf{P} = \mathbf{Rp}$$

Solve the eigenvalue problem (find the vector that is unaffected by **R**):

$$\mathbf{Rp} = \lambda\,\mathbf{p}$$

$$\left|\mathbf{R} - \lambda\mathbf{I}\right| = 0$$



$$-\lambda^3 + \lambda^2\left(R_{11} + R_{22} + R_{33}\right) - \lambda\left[\left(R_{22}R_{33} - R_{32}R_{23}\right) + \left(R_{11}R_{33} - R_{13}R_{31}\right) + \left(R_{11}R_{22} - R_{12}R_{21}\right)\right] = 0$$

$$\Rightarrow \quad (\lambda - 1)\left(\lambda^2 - \lambda\left(R_{11} + R_{22} + R_{33} - 1\right) + 1\right) = 0$$

Three eigenvalues and eigenvectors are:

$$\lambda_1 = 1, \qquad \mathbf{p}_1 = \mathbf{u}$$

$$\lambda_2 = e^{i\phi}, \quad \mathbf{p}_2 = \mathbf{x}$$

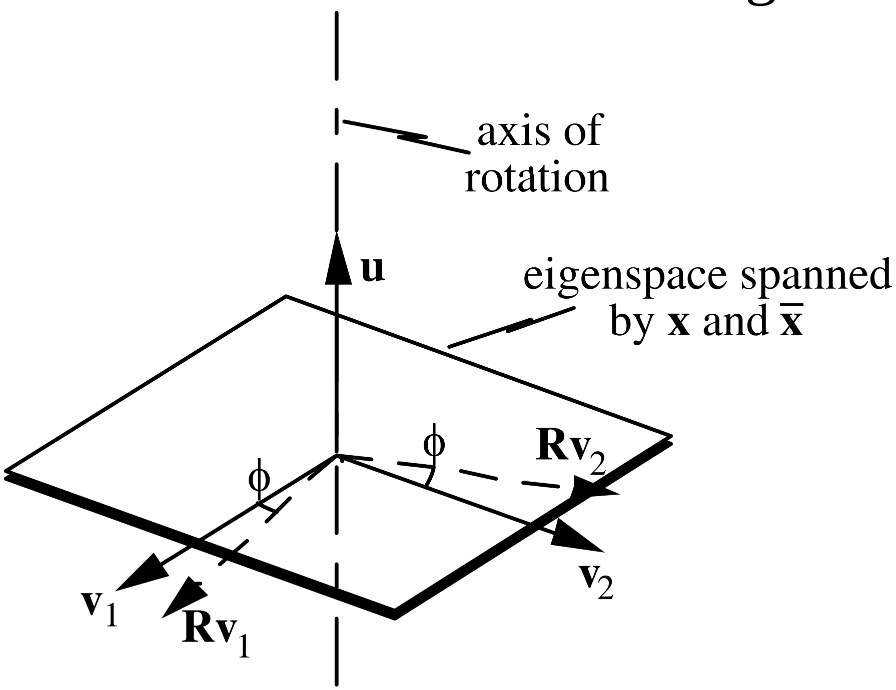
$$\lambda_3 = e^{-i\phi}, \quad \mathbf{p}_2 = \bar{\mathbf{x}}$$

where

$$\cos\phi = \frac{1}{2}\left(R_{11} + R_{22} + R_{33} - 1\right)$$

Rigid Body Kinematics

The Axis/Angle for a Rotation Matrix



$$\mathbf{R}\mathbf{v}_1 = \frac{1}{2}\left(\mathbf{R}\mathbf{x} + \mathbf{R}\bar{\mathbf{x}}\right) = \frac{1}{2}\left(\lambda_2 \mathbf{x} + \lambda_3 \bar{\mathbf{x}}\right)$$

$$= \frac{1}{2}\left(e^{i\phi}\mathbf{x} + e^{-i\phi}\bar{\mathbf{x}}\right)$$

$$= \mathbf{v}_1 \cos\phi + \mathbf{v}_2 \sin\phi$$

$$\mathbf{R}\mathbf{v}_2 = \frac{i}{2}\left(\mathbf{R}\mathbf{x} - \mathbf{R}\bar{\mathbf{x}}\right) = \frac{i}{2}\left(\lambda_2 \mathbf{x} - \lambda_3 \bar{\mathbf{x}}\right)$$

$$= \frac{i}{2}\left(e^{i\phi}\mathbf{x} - e^{-i\phi}\bar{\mathbf{x}}\right)$$

$$= -\mathbf{v}_1 \sin\phi + \mathbf{v}_2 \cos\phi$$

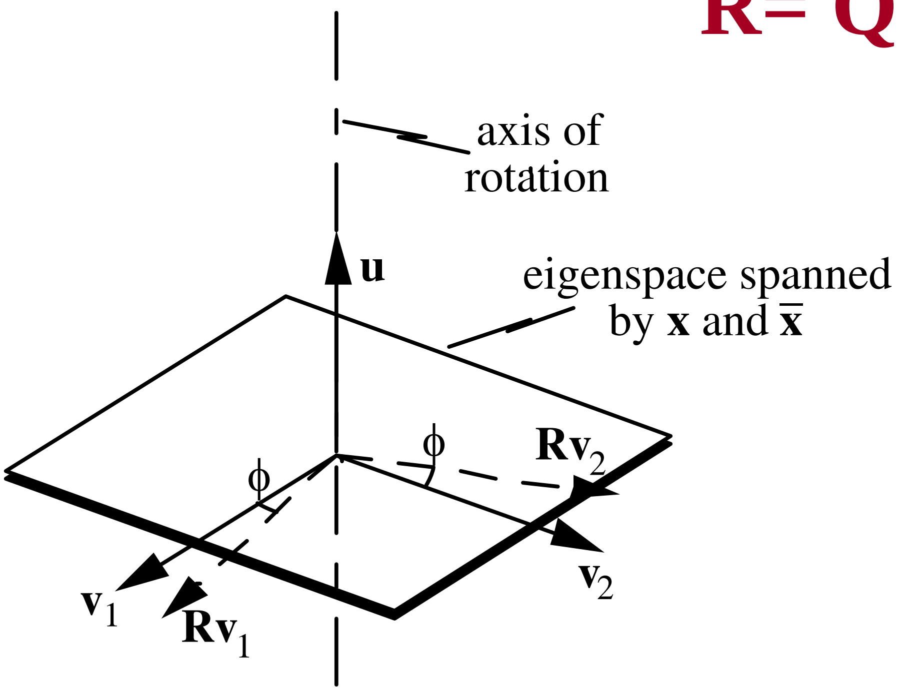
$$\mathbf{v}_1 = \frac{1}{2}\left(\mathbf{x} + \bar{\mathbf{x}}\right), \qquad \mathbf{v}_2 = \frac{i}{2}\left(\mathbf{x} - \bar{\mathbf{x}}\right)$$

$$\mathbf{x} = \frac{1}{2}\left(\mathbf{v}_1 - i\mathbf{v}_2\right), \quad \bar{\mathbf{x}} = \frac{1}{2}\left(\mathbf{v}_1 + i\mathbf{v}_2\right)$$

$$\left[\;\mathbf{R}\;\right]\underbrace{\left[\mathbf{v}_1 \;\vdots\; \mathbf{v}_2 \;\vdots\; \mathbf{u}\right]}_{\mathbf{Q}} = \underbrace{\left[\mathbf{v}_1 \;\vdots\; \mathbf{v}_2 \;\vdots\; \mathbf{u}\right]}_{\mathbf{Q}}\underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{\Lambda}}$$

Rigid Body Kinematics

$$\mathbf{R} = \mathbf{Q}\,\Lambda\,\mathbf{Q}^T$$



Define

$$^{F}\mathbf{R}_{F'} = \mathbf{Q} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{u} \end{bmatrix}$$

and look at the displacement in the new reference frame, $\{F'\}$.

Recall $\boxed{{}^{F}\mathbf{A}_G = {}^{F}\mathbf{A}_A\,{}^{A}\mathbf{A}_B\,({}^{F}\mathbf{A}_A)^{-1}}$

$$^{F'}\mathbf{R}_{M'} = \; ^{F'}\mathbf{R}_F\;{}^{F}\mathbf{R}_M\left[{}^{F'}\mathbf{R}_F\right]^T$$

$$\Lambda \qquad \mathbf{Q}^T \quad \mathbf{R} \quad \mathbf{Q}$$

Any rotation is a left action of $SO(3)$ on a canonical (?) element, Λ.

$$\Lambda = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$