

**ALTERNATIVE APPROACH TO DERIVING THE OPTIMALITY OF
UNIFORM SPEED TO MINIMIZE VEHICLE WORK AND FUEL
CONSUMPTION**

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This brief note provides an alternative derivation of the optimality of a uniform speed in minimizing fuel consumption on a wide range of guideway conditions (combination of tangents, curves, gradients, etc). The derivation below involves use of only one variable, in comparison to the Lagrangian approach given in the paper that is to be published. The single variable approach has the advantage of being more accessible to those not familiar with Lagrangian methods, though it is more tedious. This is the reason for its placement on this website. The paper and abstract are:

**VEHICLE SPEED PROFILES TO MINIMIZE WORK AND FUEL
CONSUMPTION**

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ABSTRACT

This paper addresses the question of what speed profile will minimize fuel consumption of a land transport vehicle (road or rail) in traversing a path or route. Numerous previous studies, using a control theory approach applied to specific profiles, have suggested that fuel consumption is approximately minimized by operation at constant speed. This result is derived much more directly here, along with boundaries on the conditions under which this result holds. The derivation relies on (1) the approximate proportionality between fuel consumption and propulsive work, (2) the inherent resistance of a vehicle having the usual quadratic form (of road and rail vehicles), (3) the energy conversion characteristics of on-board gasoline or diesel (or diesel-electric) propulsion systems, and (4) relatively long distances between stops. Tests, using a train performance simulator, confirm the theoretical results. The results are discussed from the standpoint of basic principles in transportation engineering, and as guidance for designing systems that conserve fuel.

The text below begins with the same introduction as the paper with the Lagrangian approach, but then continues with the single variable calculus approach:

OPTIMAL SPEED ON LEVEL TANGENT PATHS

First, we examine the case when the vehicle is traveling on level tangent path. Assume s is the total length of the segment, and that it is divided into two equal sections, 1 and 2, with length s_1 and s_2 respectively. It should be noted that the assumption of equal length sections facilitates the derivation; it is later relaxed. Starting from equation (8) in the paper, we have

$$s_1 = s_2 = s/2 \tag{8}$$

It is assumed that the sections are greater than zero length, for otherwise there is no problem:

$$s_1 > 0 \tag{9a}$$

$$s_2 > 0 \tag{9b}$$

It is assumed that the sections will be traversed at constant speeds, v_1 and v_2 respectively. For this derivation, the effect on total propulsive work due to acceleration and deceleration will be ignored. This could reflect the corresponding assumption that there is only a small difference between these two speeds, or alternatively, it can be assumed that the distances are great, so that the effect of even a substantial difference in

speeds is small. (The effect of acceleration and deceleration on the optimal speed profile will be addressed at length in the paper.) Then the total travel time is:

$$t = t_1 + t_2 = s_1 / v_1 + s_2 / v_2 = s / v \quad (10)$$

where

t total segment travel time

v average segment speed

Note that in the context of comparing different combinations of v_1 and v_2 the total travel time t should be a constant, thus Eqtn (9) forms a constraint on the solution.

The speeds are further constrained to be greater than zero, since a zero speed makes satisfying Eqtn (10) with finite speeds impossible. Also, the speeds will be limited by an upper bound that depends on the vehicle and guideway characteristics. Thus:

$$v_1 > 0 \quad (11a)$$

$$v_2 > 0 \quad (11b)$$

$$v_1 \leq V_{\max} \quad (12a)$$

$$v_2 \leq V_{\max} \quad (12b)$$

where

V_{\max} is the maximum speed at which the vehicle can travel on the segment in question.

This upper bound, assuming it applies over the entire distance, must be greater than the average speed required over the segment (s/t):

$$V_{\max} \geq v = s/t \quad (13)$$

The fuel consumed in traversing the two level tangent sections, z , is:

$$z = r(a + bv_1 + cv_1^2)s_1 + r(a + bv_2 + cv_2^2)s_2 \quad (14)$$

For purposes of minimization, the constant r can be dropped, yielding:

$$W = (a + bv_1 + cv_1^2)s_1 + (a + bv_2 + cv_2^2)s_2 \quad (15)$$

(W is thus the propulsive work required to traverse the two level tangent sections.)

Rearranging the total travel time equation (10) enables replacing v_1 with v_2 :

$$v_2 = \frac{s_2}{t - \frac{s_1}{v_1}} \quad (16)$$

This yields equation (17)

$$W = (a + bv_1 + cv_1^2)s_1 + \left(a + b \left(\frac{s_2}{t - \frac{s_1}{v_1}} \right) + c \left(\frac{s_2}{t - \frac{s_1}{v_1}} \right)^2 \right) s_2 \quad (17)$$

Taking the derivative of equation (17), and setting $s_1 = s_2$, yields:

$$\frac{dW}{dv_1} = (b + 2cv_1)s_1 - b \left(\frac{s_1^3}{\left(t - \left(\frac{s_1}{v_1}\right)\right)^2 v_1^2} \right) - 2c \left(\frac{s_1^4}{\left(t - \left(\frac{s_1}{v_1}\right)\right)^3 v_1^2} \right) \quad (18)$$

Setting equation (13) equal to zero and solving the characteristic equation for roots of v_1 yields the stationary points of the function. The four roots are:

$$\text{Root 1: } 0$$

$$\text{Root 2: } 2 \frac{s_1}{t}$$

$$\text{Root 3: } \frac{1 - bt + 2cs_1 + \sqrt{b^2 t^2 + 4btcs_1 - 12c^2 s_1^2}}{4ct}$$

$$\text{Root 4: } \frac{1 - bt + 2cs_1 - \sqrt{b^2 t^2 + 4btcs_1 - 12c^2 s_1^2}}{4ct}$$

These lead to the following four stationary point solutions for v_1 and v_2 :

$$\text{Solution 1: } v_1=0 \quad \text{and} \quad v_2 = \frac{s}{t},$$

This is clearly not a feasible solution, with speed equal to zero on one section.

$$\text{Solution 2: } v_1 = 2\frac{s_1}{t} \quad \text{and} \quad v_2 = 2\frac{s_1}{t},$$

This solution calls for the speed to remain constant for both sections of the segment traveled, and is clearly feasible.

$$\begin{aligned} \text{Solution 3: } v_1 &= \frac{1}{4} \frac{-bt + 2cs_1 - \sqrt{b^2t^2 + 4btcs_1 - 12c^2s_1^2}}{ct} \quad \text{and} \\ v_2 &= \frac{s_1 \left(-bt + 2cs_1 - \sqrt{b^2t^2 + 4btcs_1 - 12c^2s_1^2} \right)}{t \left(-bt + 2cs_1 - \sqrt{b^2t^2 + 4btcs_1 - 12c^2s_1^2} - 4s_1c \right)}, \end{aligned}$$

In order to examine the implications of Solution 3, it is useful to focus on the ratio of v_2/v_1 . It is:

$$\frac{4s_1c}{-bt - 2cs_1 - \sqrt{b^2t^2 + 4btcs_1 - 12c^2s_1^2}} \quad (19)$$

This ratio is always negative, as can be readily seen. Clearly the term above the bar is always > 0 , given the definition of s , and c . The denominator in Equation (19) is clearly less than zero, and hence v_2/v_1 , in Solution 3 is negative. Thus solution 3 is unfeasible.

$$\text{Solution 4: } v_1 = \frac{1}{4} \frac{-bt + 2cs_1 + \sqrt{b^2t^2 + 4btcs_1 - 12c^2s_1^2}}{ct} \quad \text{and}$$

$$v_2 = \frac{s_1 \left(-bt + 2cs_1 + \sqrt{b^2t^2 + 4btcs_1 - 12c^2s_1^2} \right)}{t \left(-bt + 2cs_1 + \sqrt{b^2t^2 + 4btcs_1 - 12c^2s_1^2} - 4s_1c \right)},$$

Solution 4 is the complex conjugate of Solution 3. In this solution the ratio of v_2/v_1 is:

$$\frac{4s_1c}{-bt - 2cs_1 + \sqrt{b^2t^2 + 4btcs_1 - 12c^2s_1^2}} \quad (20)$$

This ratio is always negative. Clearly the numerator is always greater than 0. The denominator is always less than zero, because the negative term before the square root, $-[bt + 2cs_1]$, is greater than the square root term. One can see this by squaring the two terms and comparing:

$$[bt + 2cs_1]^2 = b^2t^2 + 4btcs_1 + 4c^2s_1^2$$

$$\left(\sqrt{b^2t^2 + 4btcs_1 - 12c^2s_1^2} \right)^2 = b^2t^2 + 4btcs_1 - 12c^2s_1^2$$

Canceling like terms yields

$$4c^2s_1^2 > -12c^2s_1^2$$

Thus the denominator in equation (20) must be less than zero, and hence v_2/v_1 in

Solutions 4 is also negative. Thus solution 4 is unfeasible. Therefore the only feasible solution is solution 2, which specifies a uniform velocity throughout the entire distance.

It remains to show that solution 2 is a local minimum, and for this the second derivative of equation (17) is taken:

$$\begin{aligned} \frac{d^2W}{dv_1^2} = & (2c)s_1 + s_1 \left(2 \left(\frac{bs_1^3}{\left(t - \left(\frac{s_1}{v_1} \right) \right)^3 v_1^4} \right) + 2 \left(\frac{bs_1^2}{\left(t - \left(\frac{s_1}{v_1} \right) \right)^2 v_1^3} \right) \right. \\ & \left. + 6 \left(\frac{cs_1^4}{\left(t - \left(\frac{s_1}{v_1} \right) \right)^4 v_1^4} \right) + 4 \left(\frac{cs_1^3}{\left(t - \left(\frac{s_1}{v_1} \right) \right)^3 v_1^3} \right) \right) \end{aligned}$$

(21)

For the root 2 solution, $t > s_1/v_1$, and hence this second derivative is > 0 . Thus solution 2 is a local minimum. Being the only feasible stationary point, the solution $v_1 = v_2$ minimizes propulsive work. Thus if work is proportional to fuel consumption, this solution of constant speed will also minimize total fuel consumption.

Three issues remain to be addressed in connection with this result. One is to extend this result to sections of unequal length. The second is to show that the effect of acceleration and deceleration is small. The third is to derive the optimal speed around curves and on (non-zero) gradients.

Unequal Sections

While the conclusion that the optimal speed is a constant one on unequal sections also may seem obvious, a proof is necessary. This will be proven by contradiction.

Let x denote the propulsive work per unit distance. The longer of the two sections is of length s_3 and the other is of length s_2 . Assume that the work with two unequal speeds is less than that with equal speeds, the corresponding work per unit distance being x_3 , x_2 , and \bar{x} respectively. Then by assumption

$$x_3 s_3 + x_2 s_2 < \bar{x} s_3 + \bar{x} s_2 \quad (22)$$

The long section s_3 can be replaced with two shorter sections s_{31} and s_1 , where s_1 equals s_2 :

$$x_3s_{31} + x_3s_1 + x_2s_2 < \bar{x}s_{31} + \bar{x}s_1 + \bar{x}s_2 \quad (23)$$

Taking the limit as s_{31} approaches 0 results in

$$x_3s_1 + x_2s_2 < \bar{x}s_1 + \bar{x}s_2 \quad (24)$$

This clearly contradicts the prior analytical result. Thus, the hypothesis that work can be less with unequal speeds is not true. So in equation 22, 23, and 24 the inequality sign should be greater than to confirm with previous analysis. Therefore with unequal sections holding speed uniform over the entire two-section segment minimizes propulsive work.

LEVEL TANGENT FOLLOWED BY LEVEL CURVED SECTION

Now the case involving a curve will be examined. Assume the path is straight for the first section (length s_1) and then curved for the second section (length s_2). The propulsive work equation is similar to (14) but has the added resistance due to curvature (H) on section s_2 :

$$W = (a + bv_1 + cv_1^2)s_1 + (a + bv_2 + cv_2^2 + H)s_2 \quad (25)$$

As discussed in the paper, curvature resistance is not a function of velocity. Thus the addition of a constant H for added curvature resistance yields:

$$(a + bv_1 + cv_1^2)s_1 + (a + b \left(\frac{s_2}{t - \frac{s_1}{v_1}} \right) + c \left(\frac{s_2}{t - \frac{s_1}{v_1}} \right)^2 + H)s_2 \quad (26)$$

Taking the derivative of equation (31), and setting $s_1 = s_2$, to find the stationary points, yields

$$\frac{dW}{dv_1} = (b + 2cv_1)s_1 - b \left(\frac{s_1^3}{\left(t - \left(\frac{s_1}{v_1} \right) \right)^2 v_1^2} \right) - 2c \left(\frac{s_1^4}{\left(t - \left(\frac{s_1}{v_1} \right) \right)^3 v_1^2} \right) \quad (27)$$

Equation (27) is exactly the same as equation (18), because the resistance due to curvature is not a function of velocity. Thus, the results for the tangent path must hold for this case also. Namely, keeping the speed constant during the entire length s of level path including both tangent and curved sections will minimize propulsive work.

LEVEL TANGENT SECTION FOLLOWED BY UPGRADE TANGENT SECTION

For the case of level path for the first section s_1 and then an upgrade for the second section (length s_2), the analysis is similar. The total resistance will be changed in section 2, by the addition of F , the added resistance due to the upgrade.

$$W = (a + bv_1 + cv_1^2)s_1 + (a + bv_2 + cv_2^2 + F)s_2 \quad (28)$$

F is invariant with speed. Thus Equation (29) is identical to Equation (26), except for the substitution of the constant F for the constant H . Therefore the same result will apply in this case: The optimal speed is uniform over both the level and upgrade sections.

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