Abstract—We study the problem of stabilizing a linear system over a wireless control network. We propose a scheme where each wireless node maintains a scalar state, and periodically updates it as a linear combination of neighboring plant outputs and node states. We make connections to decentralized fixed modes and structured system theory to provide conditions on the network topology that allow the system to be stabilized. Our analysis provides the minimal number of feedback edges that have to be introduced to stabilize the system over a network, and shows that as long as the network connectivity is larger than the geometric multiplicity of any unstable eigenvalue, stabilizing controllers can be constructed at each actuator. A byproduct of our analysis is that by co-designing the network dynamics with the controllers, delays in the network are not a factor in stabilizing the system.

I. INTRODUCTION

The widespread availability of low-cost wireless networking technology promises to bring a shift in the architecture of industrial control systems. Traditional wired interconnections between the plant sensors, controllers and actuators can be replaced by wireless multi-hop mesh networks, yielding cost and space savings for the plant operator [1], [2].

While the introduction of wireless communications into the feedback loop has several benefits, it also presents several challenges for real-time feedback control. For instance, delays may be introduced if a multi-hop wireless network is used to route information between the plant sensors, actuators and controllers. Furthermore, transmissions in the network must be scheduled carefully to avoid packet dropouts due to collisions between neighboring nodes. These issues can be detrimental to the goal of maintaining stability of the closed loop system if not explicitly accounted for, and substantial research has been devoted to understanding the limitations of performance in such settings (e.g., [3], [4], [5], [6], [7], [8]). These works typically adopt the convention of having one or more dedicated controllers or state estimators located somewhere in the network, and study the stability of the closed loop system assuming that the sensor-estimator and/or controller-actuator communication channels are unreliable (dropping packets with a certain probability, for example).

In a recent paper [9], [10], we asked the following question: is it possible to do away with the standard “sensor → channel → controller → channel → actuator” architecture and have the computation of the control law be performed in network? In other words, is it possible to formulate a distributed algorithm for the (resource constrained) wireless nodes to follow so that the network itself acts as a controller for the plant?

To answer this question, we considered a setup where a network of wireless nodes is deployed in the proximity of a plant, with some nodes having access to the sensor measurements (outputs) of the plant, and some nodes placed within the listening range of the plant’s actuators. Each node is capable of maintaining only a limited internal state. We then presented a distributed algorithm in the form of a linear iterative strategy for each node to follow, where each node periodically updates its state to be a linear combination of the states of the nodes in its immediate neighborhood. The actuators of the plant also apply linear combinations of the states of the nodes in their neighborhood. Given a linear plant model and the topology of the wireless network, we devised a numerical design procedure that produced the coefficients of the linear combinations for each node and actuator to apply in order to stabilize the plant (if such a stabilizing set of coefficients exist). We showed that our method could also handle a sufficiently low rate of packet dropouts in the network to maintain mean square stability. We referred to this paradigm (where the computation of the control law is done in a distributed fashion by the wireless nodes) as a Wireless Control Network (WCN). The proposed scheme has several benefits, including easy scheduling of wireless transmissions, allowing compositional design, and its ability to handle geographically separated sensors and actuators.

In this report, we continue our investigation of how the dynamics and topology of the wireless network can be leveraged to stabilize large scale plants. Specifically, we provide topological conditions for the network to satisfy in order to determine whether it is possible to stabilize a plant via the wireless control network. We also show how to design the wireless network so that we can stabilize the plant. To do this, we make connections with the idea of decentralized fixed modes [11], [12], [13], [14], and with structured system theory, which allows us to use graph-theoretic tools to analyze linear systems [15], [16], [17]. Our analysis reveals that as long the plant is stabilizable and detectable from the set of all its inputs and outputs taken together, and as long as the wireless network provides paths from certain plant sensors to certain other plant actuators, then for almost
any choice of coefficients in the linear iterative strategy employed by the network nodes, a stabilizing compensator can be designed at each actuator. An interesting byproduct of our results is that stabilization is possible despite the length of the paths between the sensors and actuators, as long as compensators of larger order are allowed at the actuators. Furthermore, our scheme requires each node in the wireless network to transmit its state only once per time-step of the plant, thereby maintaining the beneficial scheduling and compositionality aspects described in [9], [10].

II. NOTATION AND TERMINOLOGY

We use \( e_i \) to denote the column vector (of appropriate size) with a 1 in its \( i \)-th position and 0’s elsewhere. The symbol \( I_N \) denotes the \( N \times N \) identity matrix, and \( A' \) indicates the transpose of matrix \( A \). For a square matrix \( M \), \( \Lambda(M) \) denotes the set of eigenvalues of \( M \). The notation \( \text{blkdiag} \) denotes a block-diagonal operator with the quantities inside the brackets on the diagonal. The cardinality of a set \( S \) is denoted by \(|S|\), and for two sets \( S \) and \( R \), we use \( S \setminus R \) to denote the set of elements in \( S \) that are not in \( R \). Finally, we denote the sets \( M = \{1, 2, ..., m\} \) and \( P = \{1, 2, ..., p\} \).

A. Graph Theory

A graph is an ordered pair \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{v_1, v_2, \ldots, v_N\} \) is a set of vertices (or nodes), and \( \mathcal{E} \) is a set of ordered pairs of different vertices, called directed edges. The vertices in the set \( \mathcal{V}_{vi} = \{v_j | (v_j, v_i) \in \mathcal{E}\} \) are the neighbors of vertex \( v_i \). A subgraph of \( G \) is a graph \( H = (\mathcal{V}, \mathcal{E}) \), with \( \mathcal{V} \subseteq \mathcal{V} \) and \( \mathcal{E} \subseteq \mathcal{E} \) (where all edges in \( \mathcal{E} \) are between vertices in \( \mathcal{V} \)). A subgraph \( H \) of \( G \) is said to be induced if \( (v_i, v_j) \in \mathcal{E} \Leftrightarrow (v_i, v_j) \in \mathcal{E} \) whenever \( v_i, v_j \in \mathcal{V} \). A path \( P \) from vertex \( v_i \) to vertex \( v_k \) is a sequence of vertices \( v_{i_0}, v_{i_1}, \ldots, v_{i_t} \) such that \( (v_{i_t}, v_{i_{t+1}}) \in \mathcal{E} \) for \( 0 \leq j \leq t - 1 \). The nonnegative integer \( t \) is the length of the path. A path is called a cycle if its start vertex and end vertex are the same, and no other vertex appears more than once in the path. We will call a graph disconnected if there exists at least one pair of vertices \( v_i, v_j \in \mathcal{V} \) such that there is no path from \( v_j \) to \( v_i \). A graph is said to be strongly connected if there is a path from every vertex to every other vertex. An induced subgraph \( H \) of \( G \) is called a strongly connected component of \( G \) if \( H \) is strongly connected, and no other vertex of \( G \) can be added to \( H \) without making \( H \) disconnected.

Given two subsets \( \mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V} \), an \( r \)-linking from \( \mathcal{V}_1 \) to \( \mathcal{V}_2 \) is a set of \( r \) vertex disjoint paths, each with start vertex in \( \mathcal{V}_1 \) and end vertex in \( \mathcal{V}_2 \). Note that if \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are not disjoint, we will take their common vertices to be vertex disjoint paths between \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) of length zero.

B. Structured Linear Systems

Consider a system \( \Sigma \) of the form:

\[
\begin{align*}
x[k + 1] &= Ax[k] + Bu[k] \\
y[k] &= Cx[k],
\end{align*}
\]

where \( x[k] \in \mathbb{R}^n, u[k] \in \mathbb{R}^m, y[k] \in \mathbb{R}^p \) and the matrices are of the appropriate dimensions. For convenience, we will denote the system as \( \Sigma = (A, B, C) \).

A linear system of the form (1) is said to be structured if each entry in the system matrices is either a fixed zero or an independent free parameter [15]. A structured system \( \Sigma \) can be represented via a directed graph \( G_{\Sigma} = (V_{\Sigma}, E_{\Sigma}) \).\(^1\) The vertex set is given by \( V_{\Sigma} = \{X \cup U \cup Y\} \) where \( X = \{x_1, \ldots, x_n\} \) denotes the set of state vertices, while \( U = \{u_1, \ldots, u_m\} \) and \( Y = \{y_1, \ldots, y_p\} \) denote the sets of input and output vertices, respectively. The edge set is given by \( E_{\Sigma} = E_A \cup E_B \cup E_C \) with \( E_A = \{(x_i, x_j) | a_{ji} \neq 0\} \), \( E_B = \{(u_i, y_j) | b_{ji} \neq 0\} \), \( E_C = \{(x_i, y_j) | c_{ji} \neq 0\} \).

For a structured system, a simple path is called a \( U \)-rooted path if the path has its begin vertex in \( U \). Also, a number of mutually disjoint \( U \)-rooted paths is called a \( U \)-rooted path family. Similarly, a simple path that has its end vertex in \( Y \) is called a \( Y \)-topped path, while a number of mutually disjoint \( Y \)-topped paths is called a \( Y \)-topped path family.

We will be interested in properties of a structured system that can be inferred purely from the zero/nonzero structure of the system matrices. These properties will hold almost everywhere - for almost any choice of free parameters (i.e., the set of parameters for which the property does not hold has Lebesgue measure zero [15]). Thus, two systems will be called structurally equivalent if they have the same number of states, inputs and outputs, and their system matrices have zeros in the same locations.

III. WIRELESS CONTROL NETWORK

Consider the system in Fig. 1, where a wireless network is used to control a system \( \Sigma = (A, B, C) \) with state \( x \in \mathbb{R}^n \), input \( u \in \mathbb{R}^m \) and output \( y \in \mathbb{R}^p \). The output vector \( y[k] \) contains measurements of the plant state vector \( x[k] \) provided by the sensors from the set \( S = \{s_1, s_2, \ldots, s_p\} \), while the input vector \( u[k] \) corresponds to the signals applied to the plant by actuators from the set \( A = \{a_1, a_2, \ldots, a_m\} \).

The wireless network is described by a graph \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{v_1, v_2, \ldots, v_N\} \) is the set of \( N \) nodes and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) represents the radio connectivity (communication topology) in the network (i.e., edge \( (v_i, v_j) \in \mathcal{E} \) if node \( v_i \) can receive information directly from node \( v_j \)). We

\(^1\)We will sometimes refer to these graphs as structural graphs.
also define \( V_s \subset V \) as the set of nodes that can receive information directly from at least one sensor, and \( V_A \subset V \) as the set of nodes whose transmissions can be heard by at least one actuator. In addition, we define a new graph \( \hat{G} = \{V \cup S \cup A, E \cup E_{in} \cup E_{out}\} \) that includes the initial graph \( G \), the plant’s sensors and actuators and the edge sets:

\[
E_{out} = \left\{ (s_i, v_j) \middle| s_i \in S, v_j \in V_s, v_i \text{ can receive values from sensor } s_i \right\}, \tag{2}
\]

\[
E_{in} = \left\{ (v_i, a_j) \middle| a_j \in A, v_i \in V_A, a_i \text{ can receive values from } v_i \right\}. \tag{3}
\]

The WCN scheme proposed in [9] requires that each wireless node maintains a (possible vector) state and implements a simple, lightweight linear iterative procedure. At every time step (i.e., once every communication frame) each node in the network updates its state to be a linear combination of values from the nodes in its neighborhood. Denoting node \( v_i \)'s state at time step \( k \) by \( z_i[k] \), the update procedure is given by:

\[
z_i[k+1] = w_{ii}z_i[k] + \sum_{v_j \in N_{v_i}} w_{ij}z_j[k] + \sum_{s_j \in N_{s_i}} h_{ij}y_j[k]. \tag{4}
\]

For the scheme in [9], each plant input \( u_i[k], i \in \mathcal{M} \) is taken to be a linear combination of values from the nodes in actuator \( a_i \)'s neighborhood:

\[
u_i[k] = \sum_{j \in N_{a_i}} g_{ij}z_j[k]. \tag{5}
\]

Aggregating the state values of all nodes at time step \( k \) into the value vector \( z[k] \), the state maintained by the network evolves according to the dynamics given by

\[
z[k+1] = Wz[k] + Hy[k], \tag{6}
\]

\[
u[k] = Gz[k], \tag{7}
\]

where \( z[k] \in \mathbb{R}^N \) if each node maintains a scalar state\(^4\) and matrices \( W, H \) and \( G \) are of the appropriate dimensions. In the above equation, for all \( i \in \{1, \ldots, N\}, w_{ij} = 0 \) if \( v_j \notin N_{v_i} \cup \{v_i\} \), \( h_{ij} = 0 \) if \( s_j \notin N_{s_i} \), and \( g_{ij} = 0 \) if \( v_j \notin N_{a_i} \). Therefore, the matrices \( W, H \) and \( G \) are structured, with sparsity constraints determined by the WCN topology. Thus, the linear strategy employed by all nodes causes the entire network itself to behave as a structured dynamical compensator.

If the overall system state is denoted by \( \hat{x}[k] = [x[k]’ \; z[k]’]’ \), the closed-loop system can be described as:

\[
\hat{x}[k+1] = \begin{bmatrix} A & BG \\ HC & W \end{bmatrix} \begin{bmatrix} x[k] \\ z[k] \end{bmatrix} = \hat{A}x[k]. \tag{8}
\]

Choosing \( W, G \) and \( H \) to obtain a stable \( \hat{A} \) can be cast in the form of a static output feedback problem with sparsity constraints on the gain matrix; this is a nonconvex problem, but various numerical procedures have been proposed in the literature (e.g., [18], [19]). In [9], [10], we adapted some of these numerical procedures to find values for the nonzero parameters in \( W, H \) and \( G \) so that the matrix \( \hat{A} \) is stable, given a network topology and a predefined state size maintained by each node. However, the proposed procedure is iterative in nature, and convergence depends on the initialization point for the algorithm. Thus, even in cases when a stabilizing configuration\(^5\) exists, the procedure might not be able to find it.

In this paper, we study topological conditions on the network that guarantee the existence of a stabilizing configuration. To facilitate our investigation, we focus on a WCN architecture where each wireless node maintains a scalar state, but where more computation can be assigned to the actuators (essentially causing them to act as reduced order controllers). This scenario is motivated by practical reasons, since actuators are usually placed in 'fixed' positions and are not power constrained, allowing them to utilize more powerful CPUs. On the other hand, wireless nodes in the network are usually battery-operated low-power microcontrollers, which are not computationally powerful. To find conditions that guarantee stabilization in this case, we will make use of the concept of fixed modes in decentralized control systems.

### IV. Decentralized Fixed Modes

The decentralized fixed modes of a linear dynamical system are eigenvalues of the plant that cannot be moved by static output feedback, where the feedback gain matrix potentially has some sparsity constraints. The concept of fixed modes was initially introduced for decentralized continuous-time systems in [11] (where the gain matrix has a block diagonal structure), and was generalized in [17] to handle arbitrary feedback patterns, and to enable a graph-theoretic analysis of the problem. In addition, for the discrete-time case, the basic problem of system stabilization using decentralized feedback controllers (i.e., a problem equivalent to [11]) was considered in [20], [21], where only algebraic conditions were derived.

Consider a discrete-time system \( \Sigma = (A, B, C) \) controlled by a set of \( m \) controllers where each controller is located at a different actuator, and has direct access to only a subset of the plant outputs.

**Definition 1:** The decentralized feedback structure constraints (i.e., patterns) are specified as \( m \) sets \( J_1, J_2, \ldots, J_m \subseteq \mathcal{P} \) such that for each \( i \in \mathcal{M}, j \in J_i \) and if only if output \( y_j \) can be directly used to calculate input \( u_i \).

Using the above definition, \( m \) linear time-invariant dynamical feedback compensators are described as \( (i = 1, \ldots, m) \):

\(^2\)The neighborhood \( N_v \) of a vertex \( v \) is with respect to the graph \( \hat{G} \).

\(^3\)Eq. (5) models the situation where the plant sensors and actuators are geographically separated, preventing the plant input from directly depending on any of the plant’s outputs.

\(^4\)In the general case \( z[k] \in \mathbb{R}^{N_v} \), where \( N_v \) is a sum of state sizes of all nodes in the network. For more details see, [9], [10].

\(^5\)In this work, matrices \( W, H \) and \( G \) that satisfy the topological constraints and guarantee stability of \( \hat{A} \) are referred to as a stabilizing configuration.
\[ z_i[k + 1] = F_i z_i[k] + \sum_{j \in J_i} q_{ij} y_j[k] \]
\[ u_i[k] = h_i z_i[k] + \sum_{j \in J_i} k_{ij} y_j[k], \]  
(9)

where \( z_i \in \mathbb{R}^{n_i} \) is the controller’s state vector (of size determined by the nature of the plant and the feedback patterns), while matrix \( F_i \) and vectors \( q_{ij}, h_i \) are of the appropriate dimensions. Based on the feedback patterns \( J_1, J_2, \ldots, J_m \), define the set

\[ K_f = \{ K \in \mathbb{R}^{m \times p} | k_{ij} = 0 \text{ if } j \notin J_i \}. \]  
(10)

**Definition 2 ([11], [17]):** For the system \( \Sigma = (A, B, C) \), the set

\[ \Lambda_f = \bigcap_{K \in K_f} \Lambda (A + BK) \]  
(11)

is called the set of fixed modes with respect to the feedback structure constraints specified by \( J_1, J_2, \ldots, J_m \).

In words, the fixed modes are the eigenvalues of \( A + BK \) that remain fixed despite the choice of matrix \( K \in K_f \). The following classical result explains the vital role that fixed modes play in the stabilizability analysis of linear dynamical systems.

**Theorem 1 ([11]):** The system \( \Sigma \) can be stabilized using the set of controllers defined in (9) if and only if all fixed modes are stable.

For any subset \( I \subseteq M \) we define \( J = \bigcup_{i \in M \setminus J_i} J_i \). The following theorem characterizes the fixed modes of a given system with respect to the feedback pattern \( J_1, J_2, \ldots, J_m \).

**Theorem 2 ([13]):** A complex number \( \lambda \) is a fixed mode of the system \( \Sigma = (A, B, C) \) if and only if there exists a subset \( I \subseteq M \) such that

\[ \text{rank} \begin{bmatrix} A - \lambda I & B_I \\ C_J & 0 \end{bmatrix} < n, \]  
(12)

where \( B_I \) and \( C_J \) are the columns and rows of \( B \) and \( C \) indexed by the elements in sets \( I \) and \( J \), respectively.

Various other algebraic tests have been proposed to determine if a given system \( \Sigma \) has unstable fixed modes with respect to a given feedback pattern \( [22], [23], [24], [12], [25], [14] \). These numerical tests are usually computationally intensive, and require calculation of the rank of a large number of matrices. In [25], [26] graphical tests were provided to test if certain numerically specified eigenvalues of the system (with multiplicity equal to 1) are fixed modes. In an effort to get away from numerical calculations and to analyze fixed modes of large-scale systems with uncertain parameters, a purely graph-theoretic test was provided in [17] to test whether a given system with a certain sparsity structure would have any fixed modes under a given feedback pattern. This approach will serve us well in analyzing and designing the wireless control network, and so we will review some of the relevant concepts here.

As described in [12], there are two distinct sources for a fixed mode. A fixed mode can either arise from a loss of rank due to a ‘perfect cancellation’ of the numerical parameters (which is a degenerate case), or it can be caused by deeper issues relating to the system structure. The later type of fixed modes are called structural fixed modes.

**Definition 3 ([12]):** The system \( \Sigma \) has structural fixed modes with respect to the feedback constraints from (9) if every system structurally equivalent to \( \Sigma \) has fixed modes with the same feedback constraints.

As described in Section II-B, one can associate a graph \( G_\Sigma = (V_\Sigma, E_\Sigma) \) with a given system \( \Sigma \). The graph can be augmented to capture a given feedback pattern \( J_1, J_2, \ldots, J_m \) via a set of edges \( E_J = \{(y_j, u_i) | i \in M, j \in J_i \} \). This produces the graph \( G_{\Sigma, J} = (V_\Sigma, E_\Sigma \cup E_J) \).

From this graphical representation of the closed-loop system, and using the same approach as in [17] we can state the following theorem that specifies graph-theoretic characterization of the conditions for nonexistence of structural fixed modes.

**Theorem 3:** When a feedback structure from (9) is used the system \( \Sigma \) has no structural fixed modes if and only if both of the following conditions hold:

i. each state vertex \( x_k \in X \) is contained in a strong component of \( G \) that includes an edge from \( E_J \).

ii. there exist a set of disjoint cycles \( C_k \) that covers all state vertices.

The second condition from the above theorem ensures that the system \( \Sigma \) does not have any fixed modes at zero. Although such modes are a concern for continuous-time systems, they are not an issue for stabilization of discrete-time plants. Thus, we state the following corollary.

**Corollary 1:** When a feedback structure \( J_1, J_2, \ldots, J_m \) is used, the system \( \Sigma \) has no structural fixed modes (other than at the origin) if and only if each state vertex \( x_k \in X \) contained in a strong component of \( G_{\Sigma, J} \) with an edge from \( E_J \).

Since a system can have stable fixed modes outside of zero, the above corollary specifies sufficient conditions for the existence of a set of stabilizing feedback controllers for almost every plant that has the given structure, with the given feedback pattern. A couple of caveats are in order. First, the theorem does not specify the size of the stabilizing controllers (i.e., the values for \( n_i, i = 1, \ldots, m \), from (9)); only that sufficiently large controllers can be found at each actuator to jointly stabilize the system. This could be an issue when resource constrained processors are used as controllers (e.g., when wireless nodes in the WCN are used to compute the control laws). The second major caveat is that the existing analysis of decentralized feedback control systems assumes that each actuator has direct access to at least one of the plant outputs (i.e., the quantities \( q_{ij} \) and \( k_{ij} \) in (9) are nonzero). This leads to a nonempty set \( K_f \) in (10), and this assumption is utilized in the proof of sufficiency from [11] to show that all non-fixed modes can be stabilized.

These caveats prevent Corollary 1 from being directly used to analyze whether the system can be stabilized using a WCN. We would like the wireless nodes to maintain only...
small state vectors (ideally scalars). Even more importantly, from (5) it can be noticed that plant inputs do not contain a direct feedthrough relation from plant outputs. Instead, each node in the network uses the values received from other neighboring nodes, with only a few nodes incorporating sensor measurements in their updates. Furthermore, each actuator generally only has access to the transmissions of nearby wireless nodes, and not the plant outputs directly. As a result, $K_f$ from (9), (11) contains only the zero matrix, although the network imposes a set of structural constraints determined by the underlying network topology. Therefore, in this case, the role of fixed modes in stabilization over a network must be carefully studied. This is the objective of the rest of this work.

To illustrate the limitations of the topological conditions from Corollary 1 when a WCN is used, consider the example from Fig. 2(a) where the plant described by the matrices

$$
A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.5 & 1 \end{bmatrix},
$$

is to be controlled using a WCN consisted of two nodes. The graph $G_{\Sigma, J}$ of the system is presented in Fig. 2(b). In this case, the WCN, which acts a dynamic compensator, has access to all of the plant outputs and inputs (i.e., $J_1 = P = \{1\}$), and the conditions from Corollary 1 are satisfied. However, since the WCN is a structured controller (due to its sparsity constraints) it can be shown that in this case there is no stabilizing configuration for the WCN when each node maintains a scalar state. Thus, the necessary and sufficient conditions from Corollary 1 do not appear to hold in this case; later, we will show how to address this by allowing the actuator to maintain a larger state, while the other nodes maintain scalar states.

V. WIRELESS CONTROL NETWORK WITH DYNAMIC COMPENSATORS AT ACTUATORS

In this section, we provide conditions for a given system to not have structural fixed modes when controlled over a WCN, where each node in the network maintains only a scalar state, the actuator nodes maintain vector states, and no actuator has direct access to any plant output.

A. Introducing a Dynamical Compensator at Actuators

Consider the plant $\Sigma = (A, B, C)$ and the WCN together as a linear system $\hat{\Sigma}$, where the outputs of the plant are injected into the WCN. At each time-step the plant actuators receive the transmissions from the wireless nodes in the set $V_A$, but suppose for now that the plant actuators do not use these transmissions to close the loop (as done in (7)). If we view the transmissions of the nodes in $V_A$ as the output of the system $\Sigma$, the system can be specified as:

$$
\dot{x}[k + 1] = \begin{bmatrix} x[k + 1] \\ z[k + 1] \end{bmatrix} = \begin{bmatrix} A & 0 \\ HC & W \end{bmatrix} \begin{bmatrix} x[k] \\ z[k] \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u[k],
$$

$$
y[k] = \begin{bmatrix} 0 \\ E_{V_A} \end{bmatrix} \begin{bmatrix} x[k] \\ z[k] \end{bmatrix}, \quad (14)
$$

with $E_{V_A} = [e_i' e_i' \ldots e_i']$, selecting the state values from the set $V_A = \{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$ (where $t = |V_A|$).

The structural graph $G_{\Sigma} = (V_{\Sigma}, E_{\Sigma})$ of the system $\Sigma$ is obtained by composing the structural graph of the initial plant $\Sigma$ and the network graph $G = (V, E)$:

$$
V_{\Sigma} = V \cup U \cup \Gamma, \quad E_{\Sigma} = E_A \cup E \cup E_\Gamma
$$

where (for $E_{\Gamma}$ defined in (2))

$$
E_\Gamma = \{(x_i, v_j) \in \Gamma \times V_{\Sigma} \mid y_{ik}, (x_i, y_k) \in E_C, (y_k, v_j) \in E_{\Gamma}\}
$$

is the edge set between the state vertices connected to a plant output and all network nodes in the neighborhood of the corresponding plant sensor.

In the basic WCN scheme each actuator updates the corresponding plant input to be a scalar linear combination of its neighboring nodes’ states (i.e., see Eq. (5)). However, suppose that we allow each actuator $a_i$ ($1 \leq i \leq m$) to maintain a possible vector state denoted by $z_{a_i} \in \mathbb{R}^{n_i}$. The procedure implemented by the actuator can be described as:

$$
z_{a_i}[k + 1] = W_{a_i} z_{a_i}[k] + \sum_{v_j \in N_{a_i}} g_{ij} z_j[k]
$$

$$
u_i[k] = t_{a_i} z_{a_i}[k] + \sum_{v_j \in N_{a_i}} k_{ij} z_j[k], \quad (15)
$$

8For technical reasons, we assume that the matrices $H$ and $C$ satisfy the property that either $H$ has a single nonzero entry in each column, or $C$ has a single nonzero entry in each row. This assumption guarantees that each nonzero entry in the product $HC$ will be an independent free parameter, if each nonzero entry in $H$ and $C$ is treated as independent free parameter.

9This assumption can be satisfied by having a dedicated wireless node for each plant output.

10While $G = (V, E)$ refers to the ‘physical’ graph, when all the nodes in the network maintain a scalar state there is a one-to-one correspondence between this graph and a structural graph of the WCN (viewed as a structured controller). Thus, we will also use $G$ as a structural graph.

11These edges appear as a result of the product $HC$. Since all nonzero entries of $HC$ are independent, the set of plant output vertices $Y$ do not appear in the graph, and the wireless node state vertices are directly connected to the plant state vertices.

12For each actuator $a_i$ and each node $v_j \in N_{a_i}$, there exists some row $l$ of $y[k]$ in (14) such that $z_l[k] = y_l[k]$. Therefore, the terms $\sum_{v_j \in N_{a_i}} g_{ij} z_j[k]$ and $\sum_{v_j \in N_{a_i}} k_{ij} z_j[k]$ correspond to linear combinations of the WCN outputs $y[k]$. 
for some matrices $W_{a_i}$, vectors $g_i$, $t_{a_i}$, and scalars $k_{ij}$.

In this setup, the overall system $\Sigma$ in (14) is to be controlled with a set of $m$ decentralized feedback controllers described by (15). In addition, the feedback pattern is specified with the edge set $E_{in}$ from (3) (i.e., in this case $E_j = E_{in}$). The key insight is the following: by having each wireless node run a linear strategy, the WCN and the plant together form a linear system $\Sigma$. Then, by viewing the transmissions of the wireless nodes closest to the actuators as the new ‘outputs’ of the system $\Sigma$, the problem of stabilizing the system with compensators at the actuators fits within the classical decentralized control formulation described in Section IV (where each input has direct access to some outputs). Therefore, with this insight, we can apply Lemma 1 to obtain the following topological condition that guarantees the existence of a stabilizing configuration.

**Theorem 4:** Consider system $\Sigma = (A, B, C)$ and a WCN where each actuator acts as a dynamical compensator. For each plant state vertex $x_i \in X$ in the structural graph $G_\Sigma$, let $A_i$ denote the set of input vertices from which $x_i$ is reachable in the initial system, while $V_{A_i}$ denotes the set of the WCN nodes that are neighbors of the actuators in $A_i$. Then, almost any system structurally equivalent to $\Sigma$ can be stabilized with the WCN if in the corresponding structural graph $G_\Sigma = \{V_\Sigma, E_\Sigma\}$ every plant state vertex $x_i$ has a path to a WCN state vertex from $V_{A_i}$. \qed

**Proof:** Consider the graph $G_\Sigma = \{V_\Sigma, E_\Sigma\}$ of the structured system (14) composed of the plant and the WCN. If for a plant state vertex $x_i$, there exists a WCN state vertex $z_j \in V_{A_i}$ reachable from $x_i$, then $x_i$ belongs to a strong component with an edge from $E_{in}$. Since this holds for all plant state vertices, if all network state vertices belong to a strong component with an edge from $E_{in}$, Corollary 1 will be satisfied and the system will not have structured fixed modes outside of the origin.

On the other hand, a fixed mode will be introduced with each WCN state vertex $z_i$ that does not belong to a strong component in the graph $G_{\Sigma_{in}} = \{V_\Sigma, E_{\Sigma, E_{in}}\}$ with an edge from $E_{in}$ (this might happen if the network is disconnected). However, by setting to zero all the weights associated with the links outgoing from $z_i$, it is ensured that this WCN state vertex is effectively removed from the network. In this case, due to the state vertex $z_i$, the system will have an additional structured fixed mode in the origin. Thus, in both cases the closed-loop system does not have structured fixed modes outside of zero, meaning that almost every system with this structure will be stabilizable using the WCN.

**B. Extracting a Stabilizing Configuration**

If the condition in Theorem 4 is satisfied, a stabilizing configuration for the WCN with dynamical compensators can be found via a simple modification of the numerical procedure for the basic WCN described in [9]. Aggregating the state values of all actuators at time step $k$ into the vector $z_a = [z_{a_1}', ..., z_{a_m}'] \in \mathbb{R}^{N_a}$, ($N_a = \sum_{i=1}^{m} n_{a_i}$), the actuators act as a dynamical compensator described with:

\[
\begin{align*}
    z_a[k+1] &= W_a z_a[k] + G z[k] \\
    u[k] &= T_a z_a[k] + K z[k],
\end{align*}
\]

where matrices $G \in \mathbb{R}^{N_a \times N}$ and $K \in \mathbb{R}^{m \times N}$ have sparsity constraints imposed by the connections between the actuators and the nodes in the network (e.g., $k_{ij} = 0$ if $v_j \notin N_{a_i}$). In addition, $W_a = blkdiag((W_{a_i})_{i=1}^{m})$, and $T_a = blkdiag((t_{a_i}')_{p}^{m})$.

Therefore, as in the basic WCN case from (6) and (7), the closed-loop system can be modeled in state-space form with state $x_d[k] = [x[k]', z[k]', z_a[k]']'$ and

\[
\begin{align*}
    x_d[k+1] &= \begin{bmatrix} A & BG_d & W_d \end{bmatrix} x_d[k] + \begin{bmatrix} \hat{A}_d \hat{x}_d[k] \end{bmatrix},
\end{align*}
\]

where

\[
\begin{align*}
    G_d &\triangleq \begin{bmatrix} K & T_a \end{bmatrix}, & H_d &\triangleq \begin{bmatrix} H \ 0 \end{bmatrix}, & W_d &\triangleq \begin{bmatrix} W & 0 \\ 0 & G \end{bmatrix}.
\end{align*}
\]

The closed-loop system described by (17) is stable if the matrix $\hat{A}_d = \hat{A}_d(W_d, H_d, G_d)$ has all eigenvalues inside the unit circle. Since matrices $W_d, H_d, G_d$ are structured, a stabilizing configuration can be obtained using the numerical procedure for the basic WCN described in [9], [10]. In addition, a procedure similar to the one from from [9] can be used to extract a stabilizing configuration for the closed-loop system with unreliable communication links. If the links can be modeled as independent Bernoulli processes, the stabilizing configuration guarantees mean square stability of the system.

For example, consider the system presented in Fig. 2 and (13). If each node maintains a scalar state and the actuator (acting as a dynamical compensator) maintains a state from $\mathbb{R}^2$, using the aforementioned algorithm we have obtained the following stabilizing configuration:

\[
\begin{align*}
    W &= \begin{bmatrix} 1.94 & -0.26 \\ 37.67 & -3.35 \end{bmatrix}, & W_a &= \begin{bmatrix} -2.28 & 5.12 \\ -1.83 & 3.75 \end{bmatrix}, \\
    H &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, & G &= \begin{bmatrix} 0 & 1.0 \\ 0 & 0.54 \end{bmatrix}, & K &= \begin{bmatrix} 0 \\ 0.12 \end{bmatrix}, & T_a &= \begin{bmatrix} -0.39 \\ 0.61 \end{bmatrix}.
\end{align*}
\]

**VI. MINIMAL STABILIZING FEEDBACK CONNECTIONS**

The previous section describes conditions that guarantee that a system does not have structural fixed modes when controlled over a WCN. In this section, we investigate the minimal connectivity that the WCN should provide to ensure that the conditions from the previous section hold.

For decentralized continuous-time systems, [27] considered the problem of determining the minimal number of direct connections between plant outputs and inputs to ensure that the system does not have structured fixed modes. Leveraging the fact that fixed modes at zero do not cause problems for the stabilization of discrete-time systems, we now present a simplified procedure that can be used to determine a minimal set of feedback edges that guarantee the absence of nonzero structural fixed modes (again, when direct plant output-input connections are allowed). We will
Consider a system $\Sigma = (A, B, C)$. For all sets $I \subseteq \mathcal{M}$ and $J \subseteq \mathcal{P}$ we denote with $B_I$ and $C_J$ submatrices of $B$ and $C$ consisting of columns of $B$ and rows of $C$ with indices in $I$ and $J$, respectively. A system $\Sigma_{IJ} = (A, B_I, C_J)$ can be described with a graph $G_{\Sigma_{IJ}} = \{V_{\Sigma_{IJ}}, E_{\Sigma_{IJ}}\}$, which can be obtained from $G_S = \{V_S, E_S\}$ by keeping input vertices from the index set $I$ and output vertices associated with set $J$. These sets are denoted $U_I$ and $Y_J$, respectively. Since we consider structurally controllable and observable systems, we will use the following results that specify a set of conditions for structural controllability/observability.

**Theorem 5 ([15])**: A structured system is structurally controllable (observable) if and only if each state vertex in the corresponding graph is the end (beginning) of a Y-topped (Y-bottomed) path and there exists a disjoint union of a U-rooted (Y-rooted) path family and a cycle family that covers all state vertices.

Since structural fixed modes in zero are not an issue for stabilization of discrete-time systems, we will use the notion of structural detectability, rather than observability.

**Corollary 2**: A structured system is structurally detectable if each state vertex is a beginning of a Y-topped path.

**Definition 4**: A controllable subset of the plant inputs (actuators) is a set $I \subseteq \mathcal{M}$ such that $(A, B_I)$ is structurally controllable. Similarly, a detectable subset of the outputs (sensors) is a set $J \subseteq \mathcal{P}$ for which $(A, C_J)$ is structurally detectable.

For some controllable subsets $I$, it may be possible to find an even smaller controllable subset $I' \subseteq I$. Since we wish to investigate the minimal feedback connectivity requirements, we define the notion of essential input and output sets.

**Definition 5**: A controllable subset $I$ is called an essential input set if there is no structurally controllable (strict) subset $I' \subseteq I$. Similarly, a detectable subset $J$ is called an essential output set if there is no structurally detectable (strict) subset $J' \subseteq J$.

It is worth noting here that for a particular system $\Sigma = (A, B, C)$ there might exist several different essential input and output sets. In addition, these sets could have different numbers of elements. We use the essential input and output sets to determine the minimal number of feedback connections that would guarantee that a system does not have nonzero structural fixed modes. This brings us to the following result.

**Theorem 6**: For a structurally controllable and detectable system $\Sigma = (A, B, C)$, let $I$ and $O$ be essential input and output sets, respectively. Then the system can be stabilized by introducing $\max(|I|, |O|)$ feedback connections (directly between certain appropriately chosen outputs and inputs).

**Proof**: A directed graph $G_{\Sigma} = \{V_{\Sigma}, E_{\Sigma}\}$, representing the structured system $\Sigma$ can be uniquely decomposed into $k$ strongly connected components $C = \{C_1, ..., C_k\}$. A component $C_i$ is referred to as a root component if no vertex in the component has incoming edges from vertices in any other component. Furthermore, $C_j$ is called a leaf component if no vertex in $C_j$ has an outgoing edge to a vertex in any other component.

Consider a directed acyclic graph $G_C = \{C \cup U_I \cup Y_J, \mathcal{E}_C \cup \mathcal{E}_{IC} \cup \mathcal{E}_{JC} \cup \mathcal{E}_{FC}\}$, where $(C_i, C_j) \in \mathcal{E}_C$ if and only if component $C_j$ has an incoming edge from a vertex in $C_i$, and

\[
\mathcal{E}_{IC} = \{(u_i, c_i) \mid i \in I, c_i \text{ is a root component from } C, \ c_i \text{ has an edge from input vertex } u_i \},
\]

\[
\mathcal{E}_{JC} = \{(c_j, y_j) \mid j \in J, c_j \text{ is a leaf component in } C_j, \ output vertex } y_j \text{ has an edge from } c_j \}.
\]

Since the system $\Sigma$ is structurally controllable and detectable each leaf component has to be connected to an output vertex $y_j \in Y_J$ and each root component is connected to an input vertex $u_i \in U_I$.

We will now introduce $\mathcal{E}_{F}$, a set of feedback links between output vertices from $Y_J$ and input vertices from $U_I$ by following Algorithm 1.

**Algorithm 1**: Creating a minimal set of feedback connections.

1. Select an input vertex $u_{i_1} \in U_I$ and a corresponding output vertex $y_{j_1} \in Y_J$ such that $y_{j_1}$ is reachable from $u_{i_1}$ in the graph $G_C$.

2. At iteration $k \geq 1$, select an input vertex $u_{i_{k+1}} \in U_I \setminus \{u_{i_1}, ..., u_{i_k}\}$ such that there exists an output vertex $y_{j_k+1} \in Y_J \setminus \{y_{j_1}, ..., y_{j_k}\}$ reachable from $u_{i_{k+1}}$ in the initial graph $G_C$. If such an input $u_{i_{k+1}}$ does not exist, add the edge $(y_{j_k}, u_{i_k})$ to $\mathcal{E}_F$, and go to the next step. Otherwise, add the edge $(y_{j_k}, u_{i_{k+1}})$ to the set $\mathcal{E}_F$, set $k \leftarrow k + 1$, and repeat step 2.

3. If $\{u_{i_1}, ..., u_{i_k}\} \neq I$ and $\{y_{j_1}, ..., y_{j_k}\} \neq J$ then select $u_{i_{k+1}} \notin \{u_{i_1}, ..., u_{i_k}\}$ and $y_{j_{k+1}} \notin \{y_{j_1}, ..., y_{j_k}\}$ and add the edge $(y_{j_{k+1}}, u_{i_{k+1}})$ to $\mathcal{E}_F$. Set $k \leftarrow k + 1$ and repeat step 3.

4. If $\{u_{i_1}, ..., u_{i_k}\} = I$ then for all $y_j \notin \{y_{j_1}, ..., y_{j_k}\}$ add the edge $(y_j, u_{i_1})$ to $\mathcal{E}_F$, where $u_{i_1}$ is an input vertex from which $y_j$ can be reached in the initial graph $G_C$.

5. If $\{y_{j_1}, ..., y_{j_k}\} = J$ then for all $u_k \notin \{u_{i_1}, ..., u_{i_k}\}$ add the edge $(y_{j_1}, u_k)$ to $\mathcal{E}_F$, where $y_{j_1}$ is an output vertex reachable from $u_k$ in the initial graph $G_C$.

In step 1, there has to exist an output $y_{j_1}$ as components connected to $u_{i_1}$ have to be connected to at least one output (since the system is detectable). Step 2 will create a cycle $L$ in the newly obtained graph $G_{C,F} = \{C \cup U_I \cup Y_J, \mathcal{E}_C \cup \mathcal{E}_{IC} \cup \mathcal{E}_{JC} \cup \mathcal{E}_F\}$ that contains the same number of input and output nodes. In step 3, pairs of input and output vertices are selected from all input and output vertices from $I$ and $J$ that are not a part of the cycle. If $(y_j, u_k)$ is such a pair, $y_j$ is not reachable from $u_k$ in the initial graph $G_C$ (otherwise vertex $u_k$ would be selected in step 2). In addition, there has to exist a vertex $u_r \in L$, from which vertex $y_j$ can be reached, since if that is not the case the vertices $u_r, y_j$ would be selected in step 2. Similarly, there exist a vertex $y_r \in L$ reachable from $u_r$. Therefore, in the newly created graph $G_{C,F}$ there would exist a cycle containing vertices $y_j, u_k, y_j, u_r$.
After step 3, \( \min(|I|, |O|) \) feedback links will be added to set \( C \). Finally, in steps 4, and 5, the remaining output or input vertices, respectively, will be connected to the vertices from which they can be reached. Thus, the Algorithm 1 will use \( \max(|I|, |O|) \) feedback connections and for each input and output vertices \( u_i \in I \) and \( y_j \in J \) such that \( y_j \) is reachable from \( u_i \) in the initial graph \( G_c \) there will exist a path from \( y_j \) to \( u_i \) in the new graph \( G_{c,F} \), that contains an edge from \( \mathcal{E}_F \). Thus, each system component \( C_i \) will belong to a strongly connected component with an edge from \( \mathcal{E}_F \), implying that the system will not have structural fixed modes.

We now apply the above result to the case where a WCN is used for control (so that direct edges between inputs and outputs cannot be introduced). As mentioned earlier, the key trick is to view the composition of the WCN and the plant as a new dynamical system. Here, a set of nodes \( V_A \) (in the neighborhood of the actuators) corresponds to the outputs of the new system. The new system will be structurally detectable if there exists a path between each essential plant output and a node from \( V_A \). Therefore, we introduce the following results.

**Definition 6:** A detectable set of WCN nodes \( V_{DET} \subseteq V \) is a set of nodes such that for each sensor \( s_j \) that corresponds to an output \( y_j \) from an essential output set \( J \), there exists a path from \( s_j \) to a node from \( V_{DET} \).

**Corollary 3:** Consider a structurally controllable and detectable system \( \Sigma = (A, B, C) \) with essential input and outputs sets \( I \) and \( O \). The system can be stabilized with a WCN described by a graph \( G = \{V, E\} \) using \( \max(|I|, |V_{DET}|) \) links between the nodes from the detectable set \( V_{DET} \) and actuators corresponding to the essential input set \( I \).

The proof of the above corollary is readily obtained by noticing that if such a detectable set of nodes \( V_{DET} \subseteq V \) exists, then due to structural detectability of the plant there would be a path between each plant state vertex and a vertex representing the state of a node from \( V_{DET} \). In addition, all network nodes that do not have a path to at least one node from \( V_{DET} \) can be disregarded as in the proof of Theorem 4 (this can be achieved by setting all weights related to them to be zero). Thus the ‘new’ system \( \Sigma \) that contains the plant and the network is structurally detectable and the proof follows from Theorem 6.

The following corollary introduces a straightforward condition for designing WCNs that guarantees stabilization of almost all systems having a certain structure.

**Corollary 4:** Almost every structurally controllable and detectable system \( \Sigma \) can be stabilized with a strongly connected WCN if the following conditions are met:

i. There exists an essential output set, where each sensor in the set is connected to the network.

ii. There exists an essential input set, where each actuator in the set is connected to the network.

VII. DESIGNING THE WCN TOPOLOGY TO STABILIZE A NUMERICALLY SPECIFIED PLANT

In the previous sections, we have been focused on designing a WCN for a plant from a purely structural perspective; in other words, we considered only the interconnections between the plant state variables, but did not consider the numerical values of those interconnections. This allowed us to characterize WCN properties that would guarantee stabilization of almost any plant having a certain structure. However, one may be interested in designing a WCN for a given (numerically specified) system \( \Sigma = (A, B, C) \). If this system falls within the measure zero set that is not covered by the structural analysis, one has to be more careful in designing the WCN. Specifically, any plant that has eigenvalues of multiplicity larger than 1 will not be captured by the generic set [13], and we will show that the multiplicity of eigenvalues in the plant will require the WCN to contain linkings of a sufficiently large size. To the best of our knowledge this is the first work that studies the interplay between numerically specified systems (with eigenvalues of multiplicity larger than one), and structured systems (where graph-theoretic analysis dominates). Previous works that used graph-theory to analyze numerical systems were limited to the cases where all eigenvalues have multiplicity equal to one (e.g., [14]).

Consider a WCN used to control a given (numerically specified) system \( \Sigma = (A, B, C) \), where the pair \( (A, C) \) is detectable, and the pair \( (A, B) \) is stabilizable. As before, suppose that we do not have access to the outputs of the plant directly, but instead only have access to the transmissions of some nodes in the wireless control network. Recall that the update equation for the WCN is given by (6), where \( W \) and \( H \) have sparsity constraints imposed by the network topology. Assuming for now that the plant actuators do not close the loop via the transmissions of nearby wireless nodes, the overall system \( \Sigma = (\tilde{A}, \tilde{B}, \tilde{C}) \) (plant and wireless network) is given by (14). As in the previous sections, we consider the following problem. How should the WCN be designed to guarantee that a dynamic compensator can be designed at each actuator to stabilize the system, when each actuator only receives information about the output of the system via the WCN (and not directly)?

To answer this, for any actuator \( a_i \), \( i \in M \), let \( V_{a_i} \) denote the WCN nodes whose transmissions can be heard by \( a_i \). For any set \( I \subseteq M \), define:

\[
V_{M \setminus I} = \bigcup_{i \in M \setminus I} V_{a_i}.
\]

In words, \( V_{M \setminus I} \) is the set of all WCN nodes that are in the neighborhood of actuators not in \( I \). To show that the system (14) has no fixed modes with respect to the feedback structure \( V_{a_1}, \ldots, V_{a_m} \), we will use Theorem 2 to show that

\[
\begin{bmatrix}
A - \lambda I & 0 & B_f \\
HC & W - \lambda I & 0 \\
0 & E_F & 0
\end{bmatrix}_{M_{I,F}(\lambda)} \geq n + N \tag{18}
\]
for all unstable eigenvalues $\lambda$ of the matrices $A$ or $W$. Here, $E_F$ is a matrix with a single 1 in each row, selecting the portions of the WCN state vector $z[k]$ corresponding to the nodes in $V_M I$.

First, consider an unstable eigenvalue $\lambda$ of $A$. Assume that $\lambda$ is not an eigenvalue of $W$. Then, for any $I \subseteq M$, the matrix $M_{I,F}$ from (18) has rank

$$\text{rank} \left( M_{I,F}(\lambda) \right) = \text{rank} \begin{bmatrix} A - \lambda I & 0 & B_I \\ 0 & W - \lambda I & 0 \\ E_F(W - \lambda I)^{-1}HC & 0 & 0 \end{bmatrix} = N + \text{rank} \begin{bmatrix} A - \lambda I \\ 0 \\ E_F(W - \lambda I)^{-1}HC \end{bmatrix}. \quad (19)$$

Thus, $\lambda$ is a fixed mode of $\Sigma = (A, B, C)$ if and only if it is a fixed mode of $(A, B, E_F(W - \lambda I)^{-1}HC)$, with respect to the feedback pattern $V_{a_1}, V_{a_2}, \ldots, V_{a_m}$. For any set $I \subseteq M$, let

$$\text{rank} \begin{bmatrix} A - \lambda I & B_I \end{bmatrix} = n - d_I$$

where $d_I$ is a nonnegative integer. Thus, the matrix $[E_F(W - \lambda I)^{-1}HC 0]$ must provide $d_I$ rows that are linearly independent of all rows in $[A - \lambda I B_I]$. We will provide conditions on the WCN topology to guarantee this.

First, due to the assumption that the pair $(A, C)$ is detectable, it is the case that for any unstable eigenvalue $\lambda$ of $A$,

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n.$$}

This means that for any set $I \subseteq M$, there are $d_I$ rows in the matrix $[C 0]$ that are linearly independent of the rows in $[A - \lambda I B_I]$. Let $J'_1, J'_2, \ldots, J'_s$ be all possible sets of $d_I$ rows of $C$ that satisfy this linear independence property, and let $J_1, J_2, \ldots, J_s$ be the sets of $d_I$ outputs of the plant corresponding to those rows. If we can guarantee that the row space of $C_{J'_i}$ is contained in the row space of $E_F(W - \lambda I)^{-1}HC$ for some $i$, then the right hand side of (19) will be $N + n$.

To satisfy this condition, note that $E_F(W - \lambda I)^{-1}HC$ in (19) is the transfer function of the WCN (where the outputs are taken to be nodes in the set $V_{M(\lambda)}$) evaluated at $\lambda$. This matrix must have rank at least $d_I$ in order for the right hand side of (19) to have rank $n$. To analyze this condition, we can consider a general structured linear system $\Sigma$, and ask what the largest possible rank of the transfer function would be over all possible values of the nonzero free parameters and $\lambda$; this is called the generic rank of the transfer function matrix for the system. The following result from [28] relates this rank to a property of the graph associated with the system.

**Lemma 1:** [28] Let $\Sigma = (A, B, C)$ be a linear system, and let $\lambda$ be such that $A - \lambda I$ is invertible. Then

$$\text{rank} \begin{bmatrix} A - \lambda I & B_I \\ C & 0 \end{bmatrix} = \text{rank} (C(A - \lambda I)^{-1}B) + n.$$}

When $\Sigma$ is a structured linear system, one can ask what the largest possible rank of the transfer function would be, over all possible values of the nonzero free parameters and $\lambda$; this is called the generic rank of the transfer function matrix for the system. The following result relates this rank to a property of the graph associated with the system.

**Theorem 7 ([28]):** Let $\Sigma = (A, B, C)$ be a structured linear system, and let $G_\Sigma$ be its associated graph. Then, the generic rank of the transfer function matrix is equal to the size of the largest linking from the input vertices to the output vertices in $G_\Sigma$. \hfill $\square$

We now present a result guaranteeing that the transfer function matrix will be full rank when evaluated at certain values $\lambda$.

**Lemma 2:** Consider the structured system $\Sigma = (A, B, C)$, and suppose that the graph $G_\Sigma$ contains a linking of size $m$ from the input vertices to the output vertices. Let $L = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be a finite set of nonzero complex numbers. Then,

$$\text{rank}(C(A - \lambda_i I)^{-1}B) = m, \ i \in \{1, 2, \ldots, r\}. \quad (20)$$

for almost any choice of free parameters in $(A, B, C)$. \hfill $\square$

**Proof:** We will first show that there is some choice of free parameters for which $\text{rank}(C(A - \lambda_i I)^{-1}B) = m$ for $i \in \{1, 2, \ldots, r\}$. We will then show that this will be true for almost any choice of free parameters.

If the graph $G_\Sigma$ contains an $m$-linking, Theorem 7 and Lemma 1 tell us that there is a numerical choice of free parameters and $\lambda$ for which $\text{rank}(M(\lambda)) = n + m$. Thus, there must exist an $(n + m)$-th order minor of $M(\lambda)$ that is nonzero. If we replace $\lambda$ with a variable $z$, and revert all of the nonzero values in the system matrices to free parameters, this minor is a nonzero polynomial $f(z)$ in $z$ and the free parameters (we leave out the free parameters in the argument of $f(\cdot)$ for clarity).

For the specific choice of free parameters that guarantees that $\text{rank}(C(A - \lambda_i I)^{-1}B) = m$, if $f(z)$ has no roots in common with the set $L$, then we have shown that there exists one choice of free parameters for which (20) holds (because $f(\lambda_i) \neq 0$ for $i \in \{1, 2, \ldots, r\}$, which means that $\text{rank}(M(\lambda_i)) = n + m$). Otherwise let $\lambda_{\text{min}}$ be the nonzero root of $f(z)$ with smallest magnitude, and let $\alpha$ be a positive real number such that $\alpha \lambda_{\text{min}}$ has larger magnitude than the largest element of $L$. Then, if we scale $A, B$ and $C$ by $\alpha$, one can verify that the resulting $(n + m)$-th order minor of $M(z)$ becomes $\alpha^{n+m}f(\frac{z}{\alpha})$. The roots of this polynomial are the roots of $f(z)$ scaled by $\alpha$, and so all nonzero roots will have magnitude larger than any elements in $L$. Thus, there exists a choice of free parameters for which (20) holds.

Next, we will show that (20) will hold for almost any choice of free parameters. Let $g(z)$ be a polynomial whose roots are the elements of $L$ (extended to include complex conjugate roots if necessary). Then, by the above argument, $f(z)$ and $g(z)$ will have no roots in common for some choice of free parameters, or equivalently, the resultant of $f(z)$ and
polynomials have no roots in common. If we revert \( f(z) \) to be a polynomial in the free parameters, the resultant of \( g(z) \) and \( f(z) \) will also be a nonzero polynomial in the free parameters. The set of parameters which cause this resultant to be zero are the parameters for which \( f(z) \) has a root in \( \mathbb{C} \). Thus, the set of free parameters for which \( (20) \) does not hold lies on an algebraic variety, which proves the lemma.

Now that we have a handle on some rank properties of the matrix \( E_F(\mathbf{W} - \lambda I)^{-1}\mathbf{H} \), we return to the problem of ensuring that the row space of \( C_{J'} \) is contained in the row space of \( E_F(\mathbf{W} - \lambda I)^{-1}HC \), for some \( i \in \{1, 2, \ldots, s\} \). The following theorem provides topological conditions for the WCN to satisfy in order to guarantee that this condition holds.

**Theorem 8:** Consider the detectable and stabilizable (numerical) system \( \Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \), along with a WCN. Let \( \lambda \) be an unstable eigenvalue of \( \mathbf{A} \). For any subset \( I \subseteq \mathcal{M} \), let \( d_I = n - \text{rank } [\mathbf{A} - \lambda \mathbf{I} \mathbf{B}_I] \). If for every possible subset \( I \), there exists a subset \( J' \) of \( d_I \) plant outputs such that

\[
\text{rank } \begin{bmatrix}
\mathbf{A} - \lambda \mathbf{I} & \mathbf{B}_I \\
\mathbf{C}_{J'} & 0 
\end{bmatrix} = n,
\]

and the WCN contains a \( d_I \) linking from those outputs to \( \mathcal{V}_{\mathcal{M} \setminus I} \), then for almost any choice of free parameters in \( \mathbf{W} \) and \( \mathbf{H} \), \( \lambda \) is not a fixed mode of the system \( \Sigma \). Furthermore, if the above holds for every unstable eigenvalue of \( \mathbf{A} \), then for almost any choice of parameters in \( \mathbf{W} \) and \( \mathbf{H} \) such that \( \mathbf{W} \) is a stable matrix, system \( \Sigma \) will have no unstable fixed modes.

**Proof:** For a given subset \( I \) and corresponding set \( \mathcal{V}_{\mathcal{M} \setminus I} \), denote the graph of the structured system \( \Sigma_{\mathcal{W}CN} = (\mathbf{W}, \mathbf{H}, E_F) \) by \( \mathcal{G}_{\mathcal{W}CN} \). Noting that the inputs to the WCN are the outputs of the plant, the input vertices in \( \mathcal{G}_{\mathcal{W}CN} \) are given by \( \mathcal{Y} \). Furthermore, denote the output vertices of \( \mathcal{G}_{\mathcal{W}CN} \) by \( \mathcal{V}_{\mathcal{M} \setminus I} \). Consider any subset \( I \subseteq \mathcal{M} \) for which \( d_I > 0 \), and let \( \mathcal{Y}' \) be the set of \( d_I \) outputs corresponding to the set \( J' \) described in the theorem. According to the assumption in the theorem, the graph \( \mathcal{G}_{\mathcal{W}CN} \) contains a linking of size \( d_I \) from these outputs to \( \mathcal{V}_{\mathcal{M} \setminus I} \). Let \( \mathbf{H}_{J'} \) denote the matrix consisting of the columns of \( \mathbf{H} \) corresponding to the outputs in set \( \mathcal{Y}' \), and consider the system \( (\mathbf{W}, \mathbf{H}_{J'}, E_F) \). The graph of this system is obtained simply by removing the vertices that are not in \( \mathcal{Y}' \) from the graph \( \mathcal{G}_{\mathcal{W}CN} \). Since this reduced graph has an \( d_I \)-linking from the inputs to the outputs, Theorem 7 and Lemma 2 indicate that \( E_F(\mathbf{W} - \lambda I)^{-1}\mathbf{H}_{J'} \) will have rank \( d_I \) for almost choice of free parameters in \( \mathbf{W} \) and \( \mathbf{H}_{J'} \). Thus, \( E_F(\mathbf{W} - \lambda I)^{-1}\mathbf{H}_{J'} \mathbf{C}_{J'} \) will have rank \( d_I \), and

\[
\text{rank } \begin{bmatrix}
\mathbf{A} - \lambda \mathbf{I} & \mathbf{B}_I \\
E_F(\mathbf{W} - \lambda I)^{-1}\mathbf{H}_{J'} & 0 
\end{bmatrix} = n
\]

Now, consider the matrix

\[
\begin{bmatrix}
\mathbf{A} - \lambda \mathbf{I} & \mathbf{B}_I \\
E_F(\mathbf{W} - \lambda I)^{-1}\mathbf{H}_{J'} & 0 
\end{bmatrix}
\]

13The resultant of two polynomials is the determinant of the Sylvester matrix associated with those polynomials, and is nonzero if and only if the polynomials have no roots in common [7].
via a dynamic compensator at each actuator. An interesting byproduct of our analysis is the following observation: the network diameter does not enter into the conditions required for stabilization over the WCN. In other words, delays in the network are not a factor when considering the issue of stability. This is achieved by incorporating dynamics into the network via the linear iterative strategy, and then designing the controllers to capture the dynamics of the plant and network simultaneously. However, while stability is guaranteed despite the lengths of the paths in the network (as long as the network satisfies the appropriate connectivity requirements specified in Theorem 4 or Theorem 9), we conjecture that the robustness or performance of the closed-loop system will potentially suffer. A detailed analysis of this phenomenon will be the subject of future research.

In addition, the derived conditions for system stabilization using the WCN, allow us to consider another practical, network synthesis problem as an avenue of future work: ‘What is the minimal required number of wireless nodes to design a WCN that can stabilize a linear system with predefined sensor and actuator positions?’ Finally, we have shown that if $d$ is the largest geometric multiplicity of the system, then the system can be stabilized with a WCN with connectivity of at least $d$. This enable us to investigate conditions for a robust WCN design, in which the WCN can be used for system stabilization even if some of the wireless nodes fail.

REFERENCES


