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Periodic Event-Triggered Average Consensus over Directed Graphs

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Abstract— **This paper considers a multi-agent consensus problem over strongly connected and balanced directed graphs. Unlike many works that consider continuous or periodic communication and control strategies, we are interested in developing an event-triggered algorithm to reduce the overall load of the network in terms of limited communication and control updates. Furthermore, we focus on a sampled-data implementation that allows agents in a communication network to determine whether locally sampled information should be discarded or broadcasted to neighbors. This formulation allows us to automatically rule out Zeno behavior that is often a challenge in distributed event-triggered systems. We show that all agents eventually rendezvous at the centroid of their initial formation given an appropriate selection of the local sampling period and event-triggering parameters. We demonstrate the effectiveness of the proposed communication and control law through simulations.**

I. Introduction

A multi-agent system usually consists of a group of intelligent agents, such as robots, vehicles, or sensors, that work cooperatively to tackle problems which are hard or impossible for an individual system to solve [1], [2], [3], [4]. Each agent is an autonomous entity powered by batteries. Agents often interact with one another via wireless communication; however, they typically have limited resources available for computation, communication, actuation, etc. By reducing the amount of communication required by the network, we are able to reduce communication energy costs as well as decrease the chance of congestion issues such as delays or packet losses. This is our main motivation for conducting the present study in this paper.

Event-triggered sampling techniques [5] have proven to be efficient in reducing communication for distributed optimization in sensor networks [6] and they have been applied to multi-agent systems for reducing the frequency of control updates in [7], [8]. However, this generally requires agents to continuously communicate with neighboring agents in order to detect potential events. The authors in [7], [8] also considered self-triggered techniques to avoid continuous communication between neighboring agents. Under the event-triggered framework, continuous and periodic communication were relaxed in [9] and [6], respectively, by designing local event triggers that determine when messages should be broadcasted to neighbors. A challenging issue for event-triggered control of multi-agent systems is to seek a strictly positive lower bound of the lengths of inter-event times. It was shown that the lengths of inter-event times for at least one agent is strictly greater than a positive constant in [7]. Instead, the authors in [9] were able to show that inter-event times for all agents are strictly positive; however, the possibility of an infinite number of events being triggered in a finite time period was not ruled out. A strictly positive lower bound of the lengths of inter-event intervals was derived in [10], where a time-dependent triggering threshold is used based on the knowledge of the algebraic connectivity of the underlying graph. Close to our treatment here, [11] and [12] use a sampled-data approach in order to guarantee the absence of Zeno executions [13], but this was only done for undirected graphs.

There are a few results on event-triggered control for multi-agent systems over directed networks [14], [15], [16]. Event-triggered control laws are adopted in [14], [15] to reduce the control updating frequency relying on continuous communication. The absence of accumulation points is shown in [16], in which each agent monitors the difference between its local state and the state average of all agents in the network. Analogous to the undirected network case, to find a strictly positive lower bound of the lengths of inter-event times remains challenging for directed networks.

In this paper, we study the sampled-data average consensus problem for multi-agent systems under the event-triggered framework to mitigate the communication frequency between neighboring agents. Different from classic results on sampled-data consensus [17], a communication logic unit is configured on each agent, which is known as an event detector, to mediate the data transmission between neighboring agents. More specifically, some locally sampled data are first forwarded to the local event detector. If the communication logic condition is not violated, the sampled data will not be communicated. Otherwise, an agent broadcasts this data to its neighbors. The agent and its neighbors will then update their control signals using this new information. The control signal is piecewise constant, which is realized by a zero-order hold between updates. We develop this framework for both directed and undirected graphs in contrast to many previous works. It is shown that average consensus is reached

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if each agent chooses an appropriate detection parameter and a sampling period.

The contributions of this paper are threefold. Firstly, we propose an event-triggered sampled-data communication and control algorithm for average consensus on balanced and connected graphs, which guarantees a strictly positive lower bound of the inter-event intervals for all agents. Secondly, we analyze the correctness of the proposed algorithm and show that the states of all agents on a balanced and connected graph will asymptotically converge to the initial state average of all agents with an appropriate choice of the sampling period and detection parameters. Finally, we are able to characterize the maximum allowable sampling period for our algorithm and show that it relaxes the requirement found in [11] for undirected graphs. Furthermore, our maximum allowable sampling period is in agreement with the maximum allowable sampling period of the standard periodic sample-and-hold algorithm in [17]. In other words, it is guaranteed that our algorithm will not require more broadcasts than a periodic communication and control algorithm.

Notation: We denote by $\mathbb{R}^p$ the set of $p \times 1$ real vectors. We define $\mathbf{1}_N$ and $\mathbf{0}_N$ to be $N \times 1$ column vectors of all ones and zeros, respectively. The norm $\|x\|$ is the 2-norm of vector x.

II. Problem Formulation

A. Algebraic Graph Theory

Here we collect basic definitions about graphs and their algebraic properties. Further details can be found in [18], [19].

A directed graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ which consists of a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. The vertex set with N elements is represented by $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$. The edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ consists of ordered pairs (v_i, v_j), where (v_i, v_j) means that agent v_j can receive information from agent v_i. For undirected graphs, edges are unordered pairs of distinct vertices, that is, the edge $(v_i, v_j) \in \mathcal{E}$ if and only if $(v_j, v_i) \in \mathcal{E}$. If there exists an edge $(v_i, v_j) \in \mathcal{E}$, then we say that agent v_i is a neighbor of agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i$. A directed path from v_i to v_j in a directed graph is a sequence of edges starting with v_i and ending with v_j. An undirected path in an undirected graph is defined analogously. A directed graph is called strongly connected if for every pair of agents there is a directed path between them. An undirected graph is called connected if any two distinct agents are linked by an undirected path.

The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ of a directed graph is defined such that $a_{ij} = 1$ if $(v_j, v_i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The adjacency matrix of an undirected graph is defined analogously except that $a_{ij} = a_{ji}$ for all $i \neq j$. A directed graph is called balanced if for all i, $\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji}$. Every undirected graph is balanced. Define the Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ as

$$l_{ii} = \sum_{j=1, j \neq i}^{N} a_{ij}, \qquad l_{ij} = -a_{ij}, i \neq j.$$

For an undirected graph, $\mathcal{L}$ is symmetrical. However, for a directed graph, $\mathcal{L}$ is not necessarily symmetrical.

For both the undirected and directed cases, the vector of all ones is the eigenvector associated with the zero eigenvalue of $\mathcal{L}$ because $\mathcal{L}$ has zero row sums. The Laplacian matrix $\mathcal{L}$ of a balanced graph has zero column sums too. For an undirected graph, $\mathcal{L}$ is positive semidefinite, whereas, for a directed graph, all nonzero eigenvalues have positive real parts. For an undirected graph, 0 is a simple eigenvalue of $\mathcal{L}$ if and only if the undirected graph is connected. For a directed graph, 0 is a simple eigenvalue of $\mathcal{L}$ if the directed graph is strongly connected.

B. System Model

The dynamics of each agent obeys a single integrator model

$$\dot{x}_i(t) = u_i(t), \qquad i = 1, \ldots, N \tag{1}$$

where $x_i \in \mathbb{R}$ denotes the scalar state, and $u_i \in \mathbb{R}$ the control input of agent v_i. The work in this paper can easily be extended to the vector state case but we do not consider it here for simplicity.

A time sequence $\{kh, k \in \mathbb{Z}^+\}$, known as *sampling instants*, is pre-determined, where $h > 0$ is a common sampling period. Note that all agents in a synchronous way take sampling commands from their embedded clocks periodically. An event detector for each agent decides whether the sampled data $x_i(kh)$ should be broadcasted to its neighbors or not. The time instants at which agent v_i broadcasts its sampled data to its neighbors are denoted as $\{t_k^i, k \in \mathbb{Z}^+\}$, also known as *event instants*. Note that the event instants are asynchronous in general, and are a subset of the sampling instants.

Remark 1 (Common sampling period): We will provide justifications of the adoption of a common sampling period for all agents from both theoretical and practical perspectives. In the literature, an underlying fact behind discrete-time consensus algorithms is that the sampling behavior is synchronous among all agents [1], [2]. An event-triggered scheme for discrete-time multi-agent consensus is investigated in [20]. The same setup of sampled-data event detection as our paper is used in [21], which models the sampling behavior as an input delay to solve the consensus problem. From an implementation standpoint, continuous monitoring and detection might be an idealized assumption. It is more pragmatic to assume the sampled-data detection.

Holding the event data constant over the *event intervals* $\{[t_k^i, t_{k+1}^i), k \in \mathbb{Z}^+\}$ yields a piecewise constant signal

$$\widehat{x}_i(t) = x_i(t_k^i), \qquad \text{for } t_k^i \leq t < t_{k+1}^i.$$

For ease of notation, define the relative event state for agent v_i with respect to its neighbors as

$$\widehat{z}_i(t) = \sum\nolimits_{j \in \mathcal{N}_i} (\widehat{x}_i(t) - \widehat{x}_j(t)).$$

The control strategy for agent v_i is then given by

$$u_i(t) = -\widehat{z}_i(t). \tag{2}$$

It is well known that if the controller is implemented in continuous time, then all states converge to their initial average [1]. The control protocol uses only the broadcasted value of the local state. The motivation for this particular choice is twofold: first, make the consensus state time-invariant; second, relax the continuous communication and control requirements. Therefore, the above piecewise constant controller is used instead.

The algorithm in (2) is distributed in the sense that each agent only needs information from its neighbors in order to compute the relative state differences from its neighbors with respect to its own state. Note that the input signal for agent v_i is updated at its own event instants as well as the event instants of its neighbors, i.e., all times $t \in \left\{t_k^i, k \in \mathbb{Z}^+\right\} \cup_{j\in\mathcal{N}_i} \{t_k^j, k \in \mathbb{Z}^+\}$. For digital implementation, a strictly positive lower bound of the lengths of inter-event intervals for each agent is desirable because very fast sampling can cause excessive equipment wear or even infeasibility of the hardware being able to perform all required actions. Here the lengths of inter-event times are guaranteed by design to be lower bounded by the sampling period h so that the so called "Zeno behavior" is ruled out.

Define the average state of all agents as $\theta(t) = \frac{1}{N}\sum_{i=1}^N x_i(t)$. Then the disagreement vector is defined as

$$\delta = x - \theta \mathbf{1}_N,$$

where $x^T = [x_1, \ldots, x_N]$ and $\delta^T = [\delta_1, \ldots, \delta_N]$.

The purpose is to rendezvous for all agents in the network asymptotically, that is, for any $x(0) \in \mathbb{R}^N$, we want $\delta(t) \to 0$ as $t \to \infty$. Moreover, we want to ensure the average is preserved such that $x_i(t) \to \theta(0)$ for all $i \in \{1, \ldots, N\}$ as $t \to \infty$. Our goal in this paper is to determine a distributed event triggering mechanism for each agent such that this is achieved.

III. Main Results

The event condition for agent v_i is given by

$$e_i^2(kh) \le \sigma_i^2 \widehat{\eta}_i^2(kh), \qquad \text{for } i \in \{1, \ldots, N\}, \tag{3}$$

where $\sigma_i > 0$ is a scalar to be determined later,

$$\widehat{\eta}_i^2(kh) = \sum\nolimits_{j\in\mathcal{N}_i} \left(\widehat{x}_i(kh) - \widehat{x}_j(kh)\right)^2,$$

and $e_i(t)$ is the measurement error for agent v_i defined as

$$e_i(t) = x_i(t) - \widehat{x}_i(t).$$

Without loss of generality, let the initial event time $t_0^i = 0$ for all $i \in \{1, \ldots, N\}$. The event instants for agent v_i can then be determined iteratively by

$$t_{k+1}^i = h \inf\left\{l : lh > t_k^i, e_i^2(lh) > \sigma_i^2 \widehat{\eta}_i^2(lh)\right\}.$$

Clearly, event instants form a subset of the sampling instants, that is, $\left\{t_k^i : k \in \mathbb{Z}^+\right\} \subseteq \{kh : k \in \mathbb{Z}^+\}$ for all $v_i \in \mathcal{V}$. In general, event instants are asynchronous for different agents although sampling instants are synchronous, i.e., it is not required that $\left\{t_k^i : k \in \mathbb{Z}^+\right\} = \{t_k^j : k \in \mathbb{Z}^+\}$ for $i \neq j$.

In addition to the triggering condition, we need to ensure that the controller used by each agent guarantees that the initial average of all agents is preserved throughout the evolution of the system. There are two key factors to guarantee this fact. One is the data chosen to update the controller. Note that although the local state $x_i(kh)$ is available to agent v_i at every sampling instant, only the last broadcast state $\hat{x}_i(kh)$ is used to update the controller. The other is the communication pattern. After each event, the local state of agent v_i will be broadcasted to all of its neighbors, and all of its neighbors will update their controllers instantaneously; thus the controller updates are synchronized for all neighbors of v_i with the same information. Under this control law, the average of the agents' state $\theta(t)$ is time invariant if the graph is balanced because

$$\begin{aligned}\dot{\theta}(t) &= \frac{1}{N}\sum\nolimits_{i=1}^N \dot{x}_i(t) = \frac{1}{N}\sum\nolimits_{i=1}^N u_i(t)\\ &= -\frac{1}{N}\sum\nolimits_{i=1}^N \widehat{z}_i(t) = -\frac{1}{N}\mathbf{1}_N^T \mathcal{L}\widehat{x}(t) = 0.\end{aligned}$$

The last equality follows from the fact that the corresponding Laplacian matrix $\mathcal{L}$ of a balanced graph has zero column sums.

Consider the Lyapunov function

$$V(x(t)) = \frac{1}{2}\delta^T(t)\,\delta(t).$$

Then taking the time derivative of the Lyapunov function along the dynamics (2), we have

$$\dot{V}(x(t)) = -x^T(t)\,\mathcal{L}\widehat{x}(kh), \quad \text{for } t \in [kh, kh+h),$$

where $\widehat{x}^T = [\widehat{x}_1, \ldots, \widehat{x}_N]$. Also denote

$$e^T = [e_1, \ldots, e_N], \ \widehat{z}^T = [\widehat{z}_1, \ldots, \widehat{z}_N].$$

Note that

$$x(kh) = e(kh) + \widehat{x}(kh) \tag{4}$$

and

$$x(t) = x(kh) - (t - kh)\,\widehat{z}(kh),$$

for all $t \in [kh, kh+h)$. Then, we can rewrite the time derivative of V as

$$\begin{aligned}\dot{V}(x(t)) = &-\left[\widehat{x}(kh) + e(kh) - (t-kh)\,\widehat{z}(kh)\right]^T \mathcal{L}\widehat{x}(kh)\\ = &-\widehat{x}^T(kh)\,\mathcal{L}\widehat{x}(kh) - e^T(kh)\,\mathcal{L}\widehat{x}(kh)\\ &+ (t-kh)\,\widehat{x}^T(kh)\,\mathcal{L}^T\mathcal{L}\widehat{x}(kh).\end{aligned} \tag{5}$$

In the following, we will specify the results for directed and undirected graphs, respectively.

A. Directed Graphs

First, let us introduce two lemmas below. The first lemma gives the relationship between the singular values of a square matrix A and those of $\frac{A+A^T}{2}$.

Lemma 2: For any square matrix A, let $\alpha_1, \ldots, \alpha_N$, be the eigenvalues of $\frac{1}{2}\left(A^T + A\right)$ with the ordering $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_N$, and let $\beta_1, \ldots, \beta_N$, be the eigenvalues of $A^T A$

with the ordering $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_N$. Then, the following relationships hold:

$$\alpha_i \leq \sqrt{\beta_i}, \qquad \text{for } i = 1, 2, \ldots, N.$$

This lemma can be proved by the celebrated *Courant-Fischer min-max* theorem [22]. The details are omitted due to space limitations.

The following result compares the Laplacian matrix $\mathcal{L}$ and $\mathcal{L}^T\mathcal{L}$ in the sense that the difference between the two matrices is a semidefinite matrix.

Lemma 3: Let $\mathcal{L}$ be a Laplacian matrix of a strongly connected and balanced graph. Then the following relationship holds for any $x \in \mathbb{R}^N$

$$x^T\mathcal{L}x \geq \frac{\alpha_2}{\beta_N} x^T\mathcal{L}^T\mathcal{L}x, \tag{6}$$

where α_2 is the smallest positive eigenvalue of $\frac{1}{2}\left(\mathcal{L}^T + \mathcal{L}\right)$, and β_N is the largest eigenvalue of $\mathcal{L}^T\mathcal{L}$.

This lemma can be proved by the utilization of the Rayleigh quotient [22]. The details are omitted due to space limitations.

Now, we are ready to state and prove the main result of this subsection.

Theorem 4: Consider the system (1), control law (2) and event condition (3). Assume that the network topology is strongly connected and balanced. Then, the states of all agents eventually rendezvous at the initial state average if for given $0 < \rho < 1$, the following conditions hold

$$h = \frac{\alpha_2}{\beta_N}\left(1 - \rho\right), \qquad 0 < \sigma_i < \frac{\rho}{2\sqrt{|\mathcal{N}_i|}},$$

where α_2 and β_N are defined in Lemma 3, and $|\mathcal{N}_i|$ is the cardinality number of the set $\mathcal{N}_i$.

Proof: Applying the inequality in (6) to (5), we have

$$\dot{V}\left(x(t)\right) \leq -\rho\widehat{x}^T\left(kh\right)\mathcal{L}\widehat{x}\left(kh\right) - e^T\left(kh\right)\mathcal{L}\widehat{x}\left(kh\right). \tag{7}$$

Using Young's inequality, we obtain

$$\begin{aligned}\dot{V}\left(x(t)\right) \leq & -\frac{\rho}{2}\sum_{i=1}^{N}\widehat{\eta}_i^2\left(kh\right) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left[\frac{e_i^2\left(kh\right)}{\sigma_i\sqrt{|\mathcal{N}_i|}}\right. \\ & \left. +\sigma_i\sqrt{|\mathcal{N}_i|}\left(\widehat{x}_i\left(kh\right) - \widehat{x}_j\left(kh\right)\right)^2\right] \\ = & -\frac{1}{2}\sum_{i=1}^{N}\left[\left(\rho - \sigma_i\sqrt{|\mathcal{N}_i|}\right)\widehat{\eta}_i^2\left(kh\right)\right. \\ & \left. -\sigma_i^{-1}\sqrt{|\mathcal{N}_i|}e_i^2\left(kh\right)\right].\end{aligned}$$

Enforcing the event condition in (3), we obtain

$$\dot{V}\left(x(t)\right) \leq -\frac{1}{2}\sum_{i=1}^{N}\left(\rho - 2\sigma_i\sqrt{|\mathcal{N}_i|}\right)\widehat{\eta}_i^2\left(kh\right), \tag{8}$$

which is negative when $\hat{\eta}_i\left(kh\right) \neq 0$ for any $i \in \{1, \ldots, N\}$, and $t \in [kh, kh + h)$.

In the sequel, we will establish that the Lyapunov function is converging to zero exponentially along sampling instants.

Integrating both sides of the equation in (8) with respect to t from kh to $kh + h$, we have

$$V\left(x(kh + h)\right) - V\left(x(kh)\right) \leq -h\epsilon\widehat{x}^T(kh)\mathcal{L}\widehat{x}(kh), \tag{9}$$

where $\epsilon = \min\left(\rho - 2\sigma_i\sqrt{|\mathcal{N}_i|}, i = 1, \ldots, N\right)$.

Note that

$$V\left(x\right) \leq \frac{1}{2\alpha_2}x^T\mathcal{L}x = \frac{1}{2\alpha_2}\left(\widehat{x}^T\mathcal{L}\widehat{x} + 2e^T\mathcal{L}_s\widehat{x} + e^T\mathcal{L}e\right),$$

with $\mathcal{L}_s = \frac{\mathcal{L}^T + \mathcal{L}}{2}$. Using the inequalities

$$e^T\mathcal{L}_s\widehat{x} \leq \|e\|\,\|\mathcal{L}_s\widehat{x}\| \leq \sigma_{\max}\sqrt{2\alpha_N}\widehat{x}^T\mathcal{L}\widehat{x},$$

where α_N is the largest eigenvalue of $\mathcal{L}_s$, $\sigma_{\max} = \max\{\sigma_i, i = 1, \ldots, N\}$, and

$$e^T\mathcal{L}e \leq \alpha_N e^T e \leq 2\alpha_N\sigma_{\max}^2\widehat{x}^T\mathcal{L}\widehat{x},$$

we have

$$V\left(x\right) \leq \frac{\left(1 + \sqrt{2\alpha_N}\sigma_{\max}\right)^2}{2\alpha_2}\widehat{x}^T\mathcal{L}\widehat{x}.$$

Applying the above inequality to (9), we obtain

$$V\left(x(kh + h)\right) - V\left(x(kh)\right) \leq -\frac{2\alpha_2 h\epsilon V\left(x(kh)\right)}{\left(1 + \sqrt{2\alpha_N}\sigma_{\max}\right)^2}.$$

Let $\gamma = 2\alpha_2 h\epsilon / \left(1 + \sqrt{2\alpha_N}\sigma_{\max}\right)^2$. It is easy to see that $0 < \gamma < 1/2$ by the use of Lemma 2. By recursion, we have $V\left(x(kh)\right) \leq (1 - \gamma)^k V\left(x(0)\right)$, where $1/2 < 1 - \gamma < 1$. Between the inter-sampling interval $[kh, kh + h)$, $V\left(x(t)\right)$ is strictly decreasing if $\hat{\eta}_i\left(kh\right) \neq 0$, for any $i \in \{1, \ldots, N\}$. Therefore, we can get the conclusion that $V\left(x(t)\right) \to 0$ as $t \to \infty$, which completes the proof. ■

B. Undirected Graphs

Now we will give the main result for undirected graphs, which specifies the feasible ranges of sampling periods and parameters of event detectors. The feasible ranges of tuning parameters for an undirected graph are larger than those for a directed graph due to the symmetric property of Laplacian matrices.

In the above derivation, the Lyapunov function $V\left(x(t)\right)$ is required to decrease all the time. We will show that the maximum sampling period for event detection can be the same as the maximal transmission period of the traditional periodic sample-and-hold method [17] if we only require $V(x(kh + h)) \leq V(x(kh))$ for all k and do not care what happens in between.

Hence, the following corollary can be concluded.

Corollary 5: Consider the system (1) with the control law (2) and the event condition (3). Assume that the network topology is connected. Then, the states of all agents eventually rendezvous at the initial state average of all agents if for given $0 < \rho < 1$, the following conditions hold

$$h = \frac{2\left(1 - \rho\right)}{\alpha_N}, \qquad 0 < \sigma_i < \frac{\rho}{2\sqrt{|\mathcal{N}_i|}},$$

where α_N is the largest eigenvalue of the Laplacian matrix $\mathcal{L}$.

Proof: Integrating both sides of the equation in (5) with respect to t, we have

$$\begin{aligned}V\left(x(kh + h)\right) - V\left(x(kh)\right) \leq & -h\rho\widehat{x}^T\left(kh\right)\mathcal{L}\widehat{x}\left(kh\right) \\ & - he^T\left(kh\right)\mathcal{L}\widehat{x}\left(kh\right).\end{aligned}$$

By following a similar procedure to the proof of Theorem 4, we obtain

$$V\left(x(kh+h)\right)-V\left(x(kh)\right)\leq-\gamma V\left(x(kh)\right), \qquad (10)$$

where

$$\gamma=\frac{2h\alpha_2\epsilon}{\left(1+\sqrt{2\alpha_N}\sigma_{\max}\right)^2},$$

with $\epsilon=\min\left(\rho-2\sigma_i\sqrt{|\mathcal{N}_i|}, i=1,\ldots,N\right)$, and α_2 the smallest eigenvalue of $\mathcal{L}$. It is not difficult to show that $0<\gamma<1$, which completes the proof. ■

Remark 6: The finite time consensus is possible for the algorithm in Corollary 5 for some special cases. For simplicity, consider the case of two agents. For the algorithm in Corollary 5, there is a possibility that two agents may pass by each other. If a sampling instant happens to coincide with the moment that two agents meet each other, then the two agents reach consensus at that location.

Remark 7: Roughly speaking, the convergence rate of the Lyapunov function is related to the sampling period, parameters of event detectors, and the smallest and largest eigenvalues of the Laplacian matrix. For instance, a smaller $\sigma_{\max}$ may lead to a faster convergence rate. The parameter $1-\gamma$ in the proof of Theorem 4 and Corollary 5 may be considered as an upper bound of the convergence rate. Larger $\sigma_{\max}$ leads to larger $1-\gamma$, which means a slower convergence rate. The same assertion for the sampling period may not be true. In addition, there is a trade-off between the sampling period and upper bounds of parameters of event detectors, which is linked by the parameter ρ. Larger ρ means a smaller detection period h, but a larger feasible range of $\sigma_i, i=1,\ldots,N$.

IV. From Sampled-Data Event Detection to Continuous Event Detection

It can be imagined that when the sampling period h approaches 0, the sampled-data event detection will be reduced to the case of continuous event detection. Thus, the event condition in (3) becomes

$$e_i^2\left(t\right)\leq\sigma_i^2\widehat{\eta}_i^2\left(t\right), \qquad \text{for } i=1,2,\ldots,N. \qquad (11)$$

The results in Theorem 4 and Corollary 5 can be extended to the case of continuous event detection simply by letting $h=\rho=0$. Therefore, we have the following results:

Corollary 8: Consider the system (1) with the control law (2) and the event condition (11). Assume that the network topology is strongly connected and balanced, or undirected and connected. Then, the states of all agents eventually rendezvous at the initial state average of all agents if

$$0<\sigma_i<\frac{1}{2\sqrt{|\mathcal{N}_i|}}.$$

By following a similar procedure to existing results, we can show that the lower bound of the lengths of inter-event intervals is greater than 0, but not greater than a strictly positive number due to jumps of the threshold. Detailed analysis can be found in [23].

TABLE I: Number of Events

Agent	v_1	v_2	v_3	v_4	v_5	v_6	v_7
No. Events	21	21	23	23	25	24	23

Remark 9: For continuous detection, it is possible to choose the parameters $\sigma_i, i=1,\ldots,N$, distributively. For sampled-data detection, the sampling period has to be designed in a centralized manner. That is to say, the topology must be known in advance to design an appropriate sampling period for the whole network.

V. Simulation Examples

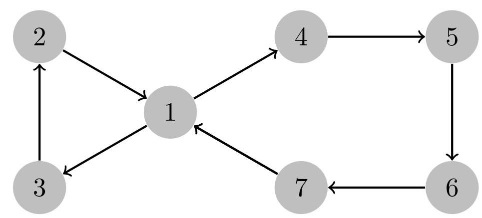

Fig. 1: A directed graph with 7 agents

A. Directed Graphs

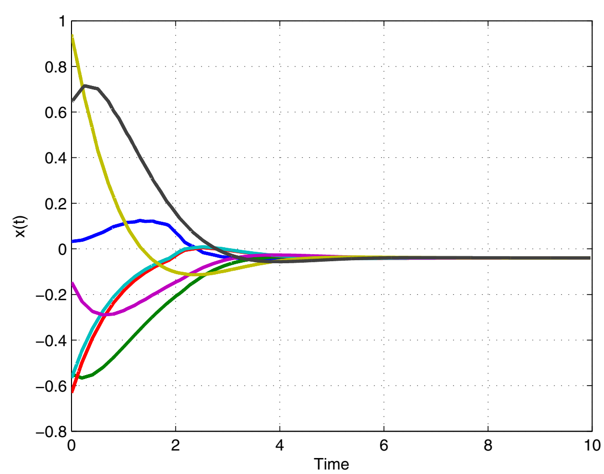

Fig. 2: State evolution of all agents

Assume that 7 agents aim to rendezvous to their initial average. They communicate according to the topology shown in Fig. 1. It is easy to see that the underlying graph is strongly connected and balanced. For this topology, we have $\alpha_2=0.3043$ and $\beta_N=7.8954$. Here the sampling period for all agents is chosen as $h=0.0077$, the parameter $\sigma_1=0.2828$, and all other σ_is are chosen as 0.4. The initial condition is randomly generated for all agents from the uniform distribution $[-1,1]$. The state evolution of all agents is shown in Fig. 2. The states of all agents rendezvous at their initial average -0.1601. Table I shows the number of events generated by each agent during a simulation time of

10 seconds. Note that the dynamical behavior of multi-agent systems is affected by many factors, such as, parameters of event detectors, sampling periods, and communication topologies. The numerical example verifies that the proposed method with event-triggered communication can still guarantee consensus with less communication and control updates for directed communication topologies. Also the inter-event intervals for all agents are guaranteed to be bounded from below by the positive sampling period $h = 0.0077$.

B. Undirected Graphs

Consider a multi-agent network, where the underlying communication topology is the same as Fig. 1 but with bidirectional communication links. The agents start from a random initial condition which is generated from the uniform distribution on the interval $[-10, 10]$, and evolve under the event conditions in (3) and control law in (2). The parameters $\sigma_i = 0.2$, for all $v_i \in \mathcal{V}$ and the sampling period for all agents is chosen as $h = 0.0774$. The simulation result shows that the states of all agents converge to their initial average. Fig. 3 shows the event instants of all agents. Even if the same parameter σ_i is used for all agents, the event instants are asynchronous in general. On the other hand, the simulation result provides a lower bound of the lengths of inter-event intervals for each agent. The minimum length of inter-event intervals for some agents is twice of the sampling period. There also exist agents whose minimum length of inter-event intervals is 3 times the sampling period.

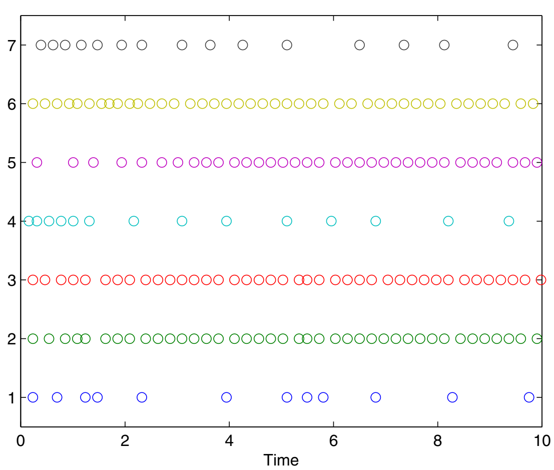

Fig. 3: Event instants of all agents

VI. Conclusion

The average consensus problem was studied over directed and undirected multi-agent networks, in which the communication between neighboring agents is triggered by events. We presented a sampled-data framework for event detection. Under our framework, the event condition for each agent is checked only at sampling instants instead of continuously, and we also showed that continuous event detection is a special case of sampled-data event detection. The Lyapunov approach enabled us to prove consensus of all agents to their initial average under the proposed conditions. The simulation results illustrated the effectiveness of the proposed methods in communication and control update reduction. Future work will consider general directed graphs, and asynchronous event detection. The main challenge of considering general directed graphs lies in the construction of a valid Lyapunov function.

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