2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

Distributed Leader Selection

Sérgio Pequito ⋆ Victor Preciado ⋆ George J. Pappas ⋆

***Abstract*— In this paper, we address the problem of distributed leader selection in a network of agents. We assume that each agent's state is defined by a scalar that evolves according to a linear dynamics involving the states of its neighbors and its own state. We propose a distributed algorithm, assuming a time-invariant communication graph, to determine: (*i*) those agents that should behave as leaders (i.e., agents whose state serves as a reference (or input) to the remaining agents – the followers); and (*ii*) a communication protocol to ensure some control-theoretical specifications and/or performance guarantees. Finally, an illustrative example using the main results of the paper is provided.**

I. INTRODUCTION

The analysis of multi-agent systems is a vibrant research field with applications in, for example, sociology (opinion dynamics), vehicle coordination, power systems, distributed clock synchronization, and distributed computing and sensor networks [1], [2]. In this field, consensus-like protocols have gained great popularity [3], [4]. These protocols can be either autonomous (i.e., leaderless) or driven by a subset of agents, referred to as *leaders*. The *leader selection* problem can be stated as follows: Given a collection of agents and a communication protocol (e.g., a linear update rule), determine the smallest subset of agents that will receive an *external* signal (i.e., an exogenous input), referred to as leaders, so that we can drive the collective of agents' states towards a given goal. The leader selection problem is often approached as a minimization of the energy cost [5], number of leaders [6-10], assignability cost [11-12], network coherence [13], mean square error with respect to the reference trajectory, or variants of the former [14-20].

Due to the combinatorial nature of the leader selection problems most of the literature relies in centralized (offline) methods to solve them, which limits its applicability in distributed scenarios where agents only have access to the information exchanged locally. Alternatively, some of the distributed methods for leader selection problems proposed to the date have strong assumptions, such as assuming that the communication topology (or the dynamics induced by it) is fully known by all the agents [19]. Other strong assumptions considered in the literature are the assumption of strong connectivity, balanced and/or acyclic communication graphs [14-18,20]. At last, we notice that leader selection problems can be seen as a particular case of input selection problems, with similar optimization objectives [5,7,21-25].

This work was supported in part by the TerraSwarm Research Center, one of six centers supported by the STARnet phase of the Focus Center Research Program (FCRP) a Semiconductor Research Corporation program sponsored by MARCO and DARPA. This work was supported in part by the NSF under grants CNS-1302222 and IIS-1447470.

⋆Department of Electrical and Systems Engineering, School of Engineering and Applied Science, University of Pennsylvania

Hereafter, we study and provide a solution to the problem of leader selection in a fully distributed setting, where agents not only have to determine if they should behave as leaders, but also the communication protocol (i.e., the weights used to perform a linear combination of their neighbors' states) – see Section II for a formal statement. Therefore, the main contributions of this paper are twofold: (*i*) determine the minimum subset of agents that should behave as leaders; and (*ii*) design a communication protocol (i.e., the weights to be used in the linear updates performed by the different agents) to ensure some control-theoretic specifications and/or performance guarantees when the communication graph is time-invariant. As mentioned before, we consider these two questions in a purely distributed setting.

The rest of the this paper is organized as follows. In Section II, we formally state the problem statement addressed in this paper. Section III provides some preliminary concepts and results. In Section IV we present the main technical results, when the communication graph is assumed time-invariant. Subsequently, we provide an illustrative example in Section V. Conclusions and discussion avenues for further research are presented in Section VI.

II. PROBLEM STATEMENT

Let $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ denote a communication graph connecting a set of agents, represented by the vertices in $\mathcal{V}$, and a set of directed edges $\mathcal{E}$. A directed edge $(i, j) \in \mathcal{E}$ indicates that agent i is able to transmit information to agent j. We denote by $\mathcal{N}_i^-$ the in-neighbors of agent i, i.e., all the agents $j \neq i$ such that $(j, i) \in \mathcal{E}$. We assume that an agent can always communicate with itself; thus, $(i, i) \in \mathcal{E}$ for all i. In the context of this paper, a *communication protocol* is understood as the following linear update of the agents' states:

$$x_i[k+1] = w_{ii} x_i[k] + \sum_{j \in \mathcal{N}_i^-} w_{ij} x_j[k], \qquad (1)$$

where $x_i[k]$ is agent i state at time k, and $\{w_{ij} : j \in \mathcal{N}_i^- \cup \{i\}\}$ is the set of weights that determines the protocol run by agent i. The network communication protocol can be re-written as a discrete-time linear time-invariant dynamical system:

$$x[k+1] = W(\mathcal{D}) x[k], \qquad (2)$$

where $W(\mathcal{D})$ is the dynamics induced by the communication graph $\mathcal{D}$, with $[W(\mathcal{D})]_{i,j} = 0$ if $(j, i) \notin \mathcal{E}$. Whereas (2) represents an autonomous dynamical system, we are interested in the case where some of the agents are driven by exogenous input signals. We call these agents leaders and describe the resulting non-autonomous dynamics by:

$$x_i[k+1] = x_i[k] + \sum_{j \in \mathcal{N}_i} w_{ij} x_j[k] + b_i u_i[k], \tag{3}$$

where $b_i = 1$ if agent i is a leader, and $b_i = 0$ otherwise. Similarly, we can rewrite (3) in matrix form as

$$x[k+1] = W(\mathcal{D})x[k] + \mathbb{I}_n(\mathcal{J})u[k], \tag{4}$$

where $\mathbb{I}_n(\mathcal{J})$ corresponds to the collection of columns of the $n \times n$ identity matrix with index in $\mathcal{J}$, where $\mathcal{J}$ is the set of indices corresponding to the leaders.

The main problem addressed in this paper can be stated as follows: Given the communication graph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, determine the communication protocol $W^*(\mathcal{D})$ and the set of leaders $\mathcal{J}^* \subseteq \{1, \ldots, n\}$, in a *fully distributed fashion*, such that

$$\begin{aligned}(W^*(\mathcal{D}), \mathcal{J}^*) = \arg \min_{W(\mathcal{D}) \in \mathbb{R}^{n \times n}, \mathcal{J} \subseteq \{1,\ldots,n\}} & \quad |\mathcal{J}| \\ \text{s.t.} & \quad (W(\mathcal{D}), \mathbb{I}_n(\mathcal{J})) \text{ is controllable.}\end{aligned} \tag{5}$$

Notice that, at first glance, (5) is a hard combinatorial problem involving a rank constraint. Nevertheless, despite these challenges, we shall show that (5) can be solved by resorting to polynomial complexity algorithms.

Complementarily, after a solution to the leader selection problem have been determined, one needs to design the control law that drives the collective to a given goal. Nonetheless, we notice that inherently to the design of a control law it is (often) required to assume that $W(\mathcal{D})$, or its parametrization, is known. In particular, if only the parametrization of $W(\mathcal{D})$ is known, then one can resort to robust adaptive control strategies [26-27]. Nevertheless, due to the distributed nature of our problem, we can only assume that leaders only have access to the local structure and state of the network; in other words, the leaders do not known $W(\mathcal{D})$, neither its parametrization.

Subsequently, we consider the particular case in which all leaders are fed with the same static input signal; i.e., $u[k] = [u^* \ \ldots \ u^*]$. We show that, in this case, we can distributedly determine $(W^*(\mathcal{D}), \mathcal{J}^*)$ that satisfies (5) such that $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} x_j(t)$ for all $i, j \in \mathcal{V}$. In other words, all agents in the network *synchronize* asymptotically to the same state.

III. PRELIMINARIES AND TERMINOLOGY

In this section, we review some results from structural systems theory [28] to be used in our derivations, since *structural controllability* plays a key role in our analysis.

Definition 1 ([28]): (Structural controllability) Consider a discrete linear time-invariant system

$$x[k+1] = Ax[k] + Bu[k], \ x[0] = x_0 \in \mathbb{R}^n,$$

and let $(\bar{A}, \bar{B})$ denote the sparsity (or structural pattern) of the pair (A, B), with $\bar{A} \in \{0, \star\}^{n \times n}$ and $\bar{B} \in \{0, \star\}^{n \times p}$, where 0 corresponds to zero entries and $\star$ with an arbitrary parameter. The pair $(\bar{A}, \bar{B})$ is said to be structurally controllable if there exists a controllable pair (A, B), with the same sparsity pattern as $(\bar{A}, \bar{B})$. $\diamond$

In fact, a stronger characterization of structural controllability holds, as stated in the following result.

Proposition 1 ([29]): For a structurally controllable pair $(\bar{A}, \bar{B})$, a numerical realizations (A, B) with the same sparsity pattern as $(\bar{A}, \bar{B})$ that is non-controllable lies in a proper variety in $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p}$. Therefore, almost all realizations respecting the structural pattern of a structurally controllable pair are controllable. $\diamond$

We now introduce some graph theoretic concepts. A digraph is defined as a set of *vertices* $\mathcal{V}$ and a set of *directed edges* $\mathcal{E}$ of the form (v_i, v_j), where $v_i, v_j \in \mathcal{V}$. Given $(\bar{A}, \bar{B})$, we can define the following digraphs: (i) the *state digraph*, denoted by $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, which is the digraph that comprises only the state variables as vertices (i.e., $\mathcal{X} = \{x_1, \ldots, x_n\}$ as *state vertices*) and a set of directed edges between the state vertices (i.e., $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : x_i, x_j \in \mathcal{X} \text{ and } \bar{A}_{j,i} \neq 0\}$); (ii) the *system digraph*, denoted by $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$, where $\mathcal{U} = \{u_1, \ldots, u_p\}$ denote the *input vertices* and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_i, x_j) : u_i \in \mathcal{U}, \ x_j \in \mathcal{X} \text{ and } \bar{B}_{j,i} \neq 0\}$ the set of edges connecting input to the state vertices. In addition, the edges in $\mathcal{E}_{\mathcal{X},\mathcal{X}}$ and $\mathcal{E}_{\mathcal{U},\mathcal{X}}$ are referred to as *state edges* and *input edges*, respectively.

A digraph $\mathcal{D}$ is said to be *strongly connected* if there exists a directed path[1] between any two vertices. A subgraph $\mathcal{D}_S$ of $\mathcal{D}$ is a digraph whose vertex and edge sets are subsets of those of $\mathcal{D}$. A *strongly connected component* (SCC) is a maximal[2] subgraph $\mathcal{D}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of $\mathcal{D}$ such that for every $u, v \in \mathcal{V}_S$ there exists a path from u to v and from v to u. We can create a *directed acyclic graph* (DAG) by visualizing each SCC as a virtual vertex, in which a directed edge between two virtual vertices (SCCs) exists if and only if there exists a directed edge between the states from the corresponding SCCs in the digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, i.e., the original digraph. A *directed tree* $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ is a directed acyclic graph that is *rooted* in a vertex without incoming edges on it, and where there exists exactly one incoming edge in each of the remaining vertices. A *directed spanning forest* of a digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ is a disjoint union of directed trees $\mathcal{T}_i = (\mathcal{V}_i, \mathcal{E}_i)$, with $i = 1, \ldots, n$, such that $\cup_{i=1,\ldots,n} \mathcal{V}_i = \mathcal{V}$. At last, the SCCs in the DAG may be further categorized as follows.

Definition 2 ([7]): An SCC is said to be linked if it has at least one incoming (respectively, outgoing) edge from (respectively, to) another SCC. In particular, an SCC is *non-top linked* if it has no incoming edge to its vertices from the vertices into another SCC. $\diamond$

The characterization of the SCCs provided in Definition 2 is particularly useful to characterize structural controllability as presented in the next result.

Corollary 1 ([10]): Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the state digraph such that $\{(x_i, x_i) : i \in \{1, \ldots, n\}\} \subset \mathcal{E}_{\mathcal{X},\mathcal{X}}$. The pair $(\bar{A}, \mathbb{I}_n^{\mathcal{J}'})$ is structurally controllable if and only if at least

[1] A directed path is a sequence of directed edges where the end-vertex of one edge is the start-vertex of the other.

[2] A subgraph is maximal with respect to a property if there is no other subgraph, strictly containing it, with the same property.

one state variable in each non-top linked SCC of the state digraph $\mathcal{D}(\bar{A})$ has an incoming edge from an input vertex in system digraph $\mathcal{D}(\bar{A}, \mathbb{I}_n^{\mathcal{J}'})$. $\diamond$

Given a communication graph $\mathcal{D}(\mathcal{V}, \mathcal{E})$, the min-consensus algorithm consists in the following update rule for each agent $i \in \mathcal{V}$:

$$x_i[k+1] = \min_{j \in \bar{\mathcal{N}}_i^-} x_j[k], \tag{6}$$

where $\bar{\mathcal{N}}_i^- = \mathcal{N}_i^- \cup \{i\}$. Further, we say that min-consensus is achieved [30], if there exists an instance of time l for which $x_i[l'] = x_j[l']$, for all $i, j \in \{1, \ldots, n\}$ with $l' \geq l$, and for all initial conditions $x[0] = [x_1[0] \ \ldots \ x_n[0]]^\intercal$. In particular, the value of l suffices to be equal to the length of the longest shortest path between any pair of agents in the digraph.

Lemma 1: [30] If $\mathcal{D}$ is strongly connected, then min-consensus is achieved. $\diamond$

Further, we say that min-consensus is achieved in a subgraph $\mathcal{D}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of $\mathcal{D}$ if there exists an instance of time l for which $x_i[l'] = x_j[l']$, for all $i, j \in \mathcal{V}_S$ and $l' \geq l$ for all initial conditions $x[0]$. Subsequently, if $\mathcal{D}$ is not strongly connected, then we have the following result.

Proposition 2: Let $\mathcal{D}^* = (\mathcal{V}^*, \mathcal{E}^*)$ denote the DAG decomposition of $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, and $\mathcal{N}_j^T = (\mathcal{V}_j, \mathcal{E}_j)$ with $j = 1, \ldots, \beta$ are the non-top linked SCCs. Then, the min-consensus is achieved in each non-top linked SCC $\mathcal{N}_j^T$ with $j = 1, \ldots, \beta$. $\diamond$

In fact, as a by-product of Proposition 2, the reason why min-consensus may not be achieved in a top-linked SCCs is described by the following result.

Proposition 3: Let $\mathcal{D}^* = (\mathcal{V}^*, \mathcal{E}^*)$ denote the DAG decompostion of $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, and $\mathcal{N}_i$, with $i = 1, \ldots, \alpha$, correspond to the different SCCs of $\mathcal{D}^*$. Then, by performing the min-consensus protocol we obtain that $x_m[l'] = x_i[l']$, with $m, i \in \mathcal{N}_j$, and l' greater than the length of the longest shortest path from any state that have a directed path to $k \in \mathcal{N}_j$. In addition, $x_m[l'] = x^*[0]$ where $x^*[0]$ is the lowest initial state among the states associated with the vertices that have a directed path to m. $\diamond$

IV. MAIN RESULTS

In this section, we present the main results of this paper. First, in Algorithm 1, we present a distributed algorithm to compute the minimum number of leaders. Further, the complexity and correctness of Algorithm 1 are presented in Theorem 1. Next, we introduce a fully distributed scheme to obtain a communication protocol, for a given collection of leaders, to ensure controllability of the system (see Algorithm 2). The proof of correctness and complexity of Algorithm 2 are given in Theorem 2. Subsequently, resorting to Algorithm 1 and Algorithm 2 a solution to (5) is obtained, as stated in Theorem 3. In fact, the solution obtained ensures asymptotic stability of the agents' dynamics to a common steady-state reference value given to the leaders, as described in Theorem 4. In order to obtain these results, the only assumption we need is that each agent has a unique id, which hereafter we assume to be given by a prime number (for simplicity).

Now, we let the parameters of the dynamics $W(\mathcal{D})$, i.e., the communication protocol, to be undetermined, so to determine the minimum number of leaders in (5) we resort to structural systems theory. Let $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, and $\bar{W}(\mathcal{D})$ be the matrix such that $[\bar{W}(\mathcal{D})]_{j,i} = \star$ if $(i, j) \in \mathcal{E}$, and $[\bar{W}(\mathcal{D})]_{j,i} = 0$ otherwise, then we want to determine $\mathcal{J}^* \subseteq \{1, \ldots, n\}$ such that

$$\begin{aligned} \mathcal{J}^* = \arg \min_{\mathcal{J} \subseteq \{1,\ldots,n\}} & \quad |\mathcal{J}| \\ \text{s.t.} & \quad (\bar{W}(\mathcal{D}), \bar{\mathbb{I}}_n(\mathcal{J})) \text{ is struct. controllable,} \end{aligned} \tag{7}$$

where $\bar{I}_n$ has the structural pattern of the $n \times n$ identity matrix.

The fully distributed solution to (7) is presented in Algorithm 1. Briefly, agents have to determine if they belong to a non-top linked SCC, towards the satisfaction of Corollary 1 by the leaders. More specifically, each agent i has to determine the number of agents N_i able to route data to it; in other words, the number of different agents for which there exists a directed path starting at those agents and ending at agent i. Next, a min-consensus protocol is performed, where the initial state of agent i equals N_i; if the min-consensus value of agent i equals that of N_i, then the agent lies in a non-top linked SCC, since any agent in a top linked SCC receives data from the agents in at least one non-top linked SCC – see Proposition 2 and Proposition 3. Then, because each agent keeps a list of the identities of the agents from whom it receives information, those in the non-top linked SCCs can choose to be the leader, being the agent with the lowest *id* the one that declares itself as a leader.

The correctness and complexity of Algorithm 1 is presented next:

Theorem 1: Let $\mathcal{J}^*$ be the indices of the agents that declare themselves as leaders when Algorithm 1 is performed, then $\mathcal{J}^*$ is a solution to (7). In addition, let $\hat{n}$ be an upper-bound on the total number of agents in a single SCC, then each agent runs Algorithm 1 with complexity $\mathcal{O}(\hat{n}^2)$. $\diamond$

Proof: The proof consists in the following steps: first, we show that $\mathcal{J}^*$ contains the id of one agent from each non-top linked SCC (hereafter assumed to be the SCCs of the DAG representation of the communication digraph), and, secondly, $(\bar{W}(\mathcal{D}), \bar{\mathbb{I}}_n^{\mathcal{J}^*})$ is a minimal feasible solution to (7) by invoking Corollary 1.

By Proposition 3, all the agents with states in the top-linked SCCs, which ids we denote by $\mathcal{A}$, are such that $z_i[\hat{n}] < |\mathcal{L}_i|$ for $i \in \mathcal{A}$ (with $z_i[k]$ and $\mathcal{L}_i$ defined as in Algorithm 1), because $z_i[\hat{n}] = \min_{j \in \mathbb{D}_i[\hat{n}]} \{|\mathcal{L}_j|\}$ (by performing Steps 5-6), where $\mathbb{D}_i[\hat{n}]$ contains the ids of the agents from where a directed path starts and that ends in agent i. Further, $\hat{n}$ is an upper-bound on the number of agents in an SCC, and $\emptyset \neq \mathcal{L}_j \subsetneq \mathcal{L}_i$ for $j \in \mathbb{D}_i[\hat{n}]$ and $i \in \mathcal{A}$ after executing Steps 1-4.

On the other hand, by performing Steps 1-4, each non-top linked SCC achieve the min-consensus after $\hat{n}$ iterations, see

ALGORITHM 1: Distributed solution to (7)

Let each agent i have a list $\mathcal{L}_i$ describing the agents ids from which there exists a directed path to agent i; and $\hat{n}$ an upper-bound on the total number of agents in a single SCC of the network.

Set $\mathcal{L}_i = \{i\}$, and let agents transmit/receive the information about $\mathcal{L}_i$; more precisely, Γ_i^k that denotes the difference between the information received and in $\mathcal{L}_i$. Notice that Γ_i^k describes the agents' ids exactly k hops away from agent i, i.e., there exists a directed path starting in the agents with id in Γ_i^k and ending in agent i, but not at $k' < k$ hops away from agent i.
Set $\Gamma_i^1 = \mathcal{L}_i$;
For $k = 1, \ldots, \hat{n}$
1. Receive Γ_j^k for all $j \in \mathcal{N}_i^-$;
2. Update $\Gamma_i^k = \left(\bigcup_{j \in \mathcal{N}_i^-} \Gamma_j^k \right) \setminus \mathcal{L}_i$;
3. Transmit Γ_i^k;
4. Set $\mathcal{L}_i = \mathcal{L}_i \cup \Gamma_i^k$;
endFor
5. Set $z_i[0] = |\mathcal{L}_i|$;
For $k = 1, \ldots, \hat{n}$
6. $z_i[k+1] = \min\limits_{j \in \mathcal{N}_i^-} z_j[k]$,
endFor
7. If $z_i[\hat{n}] == |\mathcal{L}_i|$ and $i == \arg\min \mathcal{L}_i$, then i is a leader.

Proposition 2. Further, let $\mathcal{N}_j^T$, with $j = 1, \ldots, \beta$, correspond to the β non-top linked SCCs of the communication graph and $\mathcal{I}(\mathcal{N}_j^T)$ the ids of the agents' states in $\mathcal{N}_j^T$. Then, we have $z_i[\hat{n}] == |\mathcal{L}_i|$ where i corresponds to the ids of the agents in the same non-top linked SCC, i.e., $i \in \mathcal{I}(\mathcal{N}_j^T)$. Therefore, any collection of agents $\mathcal{J}$ such that $\{i_1, \ldots i_\beta\} \subset \mathcal{J}$, with $i_1 \in \mathcal{I}(\mathcal{N}_1^T), \ldots, i_\beta \in \mathcal{I}(\mathcal{N}_\beta^T)$, leads to a structurally controllable $(\bar{W}(\mathcal{D}), \mathbb{I}_n^{\mathcal{J}})$ by Corollary 1, since all follower agents' states have self-loops. In addition, Step 7 choses $i_1, \ldots i_\beta$ to be the smallest agent id (which are unique) in the corresponding non-top linked SCC, which implies that exactly one agent is considered from each non-top linked SCC, and, by Corollary 1, setting $\mathcal{J}^*$ to be the collection of the leaders' states ids as computed by Algorithm 1, the results follow.

The complexity of Algorithm 1 is $\mathcal{O}(\hat{n}^2)$ since it requires two for-loops bounded by $\hat{n}$, and where the operations involved in each step have either linear complexity in n (i.e., the total number of agents in the network) or logarithm complexity in n as it is the case of the min-operator; the results follows by noticing that $\hat{n} \geq n$. ■

Once the leaders have been identified by the indices in $\mathcal{J}^*$, the communication protocol (i.e., the weights of $\mathcal{W}(\mathcal{D})$) has to be determined at the agent level such that $(W(\mathcal{D}), \mathbb{I}_n^{\mathcal{J}^*})$ is controllable. Notice that a controllable pair $(W(\mathcal{D}), \mathbb{I}_n^{\mathcal{J}^*})$ is guaranteed to exist, by definition of structural controllability of the pair $(\bar{W}(\mathcal{D}), \bar{\mathbb{I}}_n^{\mathcal{J}^*})$. In Algorithm 2, given a collection of leaders from Algorithm 1, we present a solution where agents actively set to zero some of the free parameters in $\bar{W}(\mathcal{D})$, i.e., the $\star$-entries, such that controllability is ensured as presented in the next result.

ALGORITHM 2: Determining the weights in $W(\mathcal{D})$, given $\mathcal{J}^*$ as in Algorithm 1

Let $l_i[k]$ denote the distance (measured in terms of the length of the shortest path) from a leader agent with id in $\mathcal{J}^*$ to agent i at instance of time k.
Set $l_i[1] = 0$ for the leader agents $i \in \mathcal{J}^*$ and $l_i[1] = \tilde{n} + 1$ otherwise, where $\tilde{n}$ an upper-bound on the length of the longest shortest path between agents in the network.
For $k = 1, \ldots, \tilde{n}$
1. Receive the token of the agents denoted by $l_j[k]$ for $j \in \mathcal{N}_i^-$ and take $l_i^* = \min_{j \in \mathcal{N}_i^-} \{l_i[k] : j \in \mathcal{N}_i^-\}$.
2. If $l_i^* < l_i[k]$, then $l_i[k] = l_i^*$ and $a_i = \min\left\{\arg\min\{l_j[k] : j \in \mathcal{N}_i^-\}\right\}$ denotes an agent's id that is closest to a leader.
3. Transmit $l_i[k]$;
endFor
4. Agent i sets $[W(\mathcal{D})]_{i,i} = \alpha_i$, where $\alpha_i = \frac{id}{p_i} \in (-1, 0) \cup (0, 1)$ with $p_i \in \mathbb{N}$. In addition, the follower $i \in \{1, \ldots, n\} \setminus \mathcal{J}^*$ sets $[W(\mathcal{D})]_{i,a_i} = 1 - \alpha_i$, and $[W(\mathcal{D})]_{i,j} = 0$, with $j \notin \{i, a_i\}$.

Theorem 2: Given a collection of leaders identified by $\mathcal{J}^*$ (as provided by Algorithm 1) and a communication protocol $W(\mathcal{D})$, constructed as in Algorithm 2, then $(W(\mathcal{D}), \mathbb{I}_n^{\mathcal{J}^*})$ is controllable. In addition, each agent runs Algorithm 2 with complexity $\mathcal{O}(N^2)$, with $N = \max\{\hat{n}, \tilde{n}\}$, where $\hat{n}$ is an upper-bound on the total number of agents in the network and $\tilde{n}$ an upper-bound on the length of the longest shortest path between agents in the network. ⋄

Proof: First, note that (by construction) Algorithm 2 produces a directed spanning forest with directed trees rooted in the leaders' states, and, additionally, the remaining state vertices associated with the followers have self-loops. To see that Algorithm 2 produces a directed spanning forest with directed trees rooted in the leaders' states, we recall the properties of the min-consensus algorithm and notice that agent i changes the value of $l_i[k]$ only after a leader is k hopes away form it; in addition, this data requires at most as many iterations as the length $\tilde{n}$ of the longest shortest path from the leaders to the agents in the network. Further, we notice that each follower uses a_i to select a single agent from which agent i receives data from, which implies that there exists only a single directed path from a leader to an agent i, which is a possible characterization of a directed tree rooted in the leaders. Moreover, because those leaders belong to non-top linked SCCs, it readily follows that there exists a directed path from at least one leader to every follower. Thus, the conclusion that Algorithm 2 produces a directed spanning forest with directed trees rooted in the leaders' states is obtained. Subsequently, $\mathcal{D}$ is a spanning tree where $\{\alpha_i\}_{i=1}^n$

with $\alpha_i \neq \alpha_j$ for $i \neq j$; hence, it is controllable [31], [32].

The complexity of Algorithm 2 is $\mathcal{O}(N^2)$ since it requires one for-loop bounded by $\tilde{n}$, where the operations involved in each step have linear complexity in $\hat{n}$, i.e., the upper-bound on the total number of agents in the network. ■

Furthermore, as a direct consequence of Theorem 1 and Theorem 2, we obtain one of the main results of this paper:

Theorem 3: Let $\mathcal{J}^*$ and $W^*(\mathcal{D})$ be the output of in Algorithm 1 and Algorithm 2, respectively. Then, $(W^*(\mathcal{D}), \mathcal{J}^*)$ is a solution to (5). ◇

Remark 1: In Step 4 in Algorithm 2, we choose to set $[W(\mathcal{D})]_{i,i} = \alpha_i$ with $\alpha_i \in (-1, 0) \cup (0, 1)$ for follower agent i, but from the proof of Theorem 2 it readily follows that $[W(\mathcal{D})]_{i,i} \in \mathbb{R} \setminus \{0\}$ and $[W(\mathcal{D})]_{i,a_i} \in \mathbb{R} \setminus \{0\}$ also ensures controllability of $(W(\mathcal{D}), \mathbb{I}_n^{\mathcal{J}^*})$. Nonetheless, the choice we made is closely related with the asymptotic stability of the communication protocol $W(\mathcal{D})$ described next. ◇

In Theorem 3, we obtained a controllable system, yet in order to define a control strategy, each leader would require knowledge on $W(\mathcal{D})$ or its structure. In practice this may not be the case, therefore, we relax the constraints, as we already pointed out in Section II, and instead of considering controllability we focus in determine $W(\mathcal{D})$ in a distributed manner such that the overall system is asymptotically stable towards leaders' states. Notice that in terms of leader-follower problems this makes sense in the case where agents try to keep relative position to their neighbors. In fact, Algorithm 1 and Algorithm 2 already provide such dynamics, as stated in the next result.

Theorem 4: Let $W^*(\mathcal{D})$ be constructed as in Algorithm 2, given the indices $\mathcal{J}^*$ of leader agents constructed using Algorithm 1. Then, $W(\mathcal{D})$ is asymptotically stable. In addition, if $u_j[k] = u^*$ in (4) is a constant signal provided to the leaders $j \in \mathcal{J}^*$, then we have $x[k] \to \mathbf{1}u^*$ when $k \to \infty$, where $\mathbf{1}$ is the vector of ones with appropriate dimensions. ◇

Proof: By construction, Algorithm 2 produces a directed spanning forest $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ rooted in the leaders' states, where state vertices associated with the followers have self-loops. Consequently, $W(\mathcal{D})$ is a triangular matrix (up to permutation of the columns and rows) where the diagonal entries are $\{\alpha_i\}_{i=1,\ldots,n}$. These entries are the eigenvalues of $W(\mathcal{D})$ that are in the range $(-1, 1)$, which implies that $W(\mathcal{D})$ is asymptotically stable. Further, by expanding the $x_i[k]$ in its series it is easy to see that these converge, since both α_i and $(1-\alpha_i)$ are stricly less than 1 in absolute value; in fact, one obtains that $x_i[k] \to x_j[k]$ where $(j, i) \in \mathcal{E}$ and $x_l[k] = u^*$ for the leader agent l, where $l \in \mathcal{J}^*$, and the result follows. ■

Remark 2: The parameters α_i ($i \in \{1, \ldots, n\} \setminus \mathcal{J}^*$) as prescribed in Algorithm 2, accordingly to Theorem 4, ensure that the followers will converge to the leaders' state. Nonetheless, in practice, the choice of α_i should depend on a specific goal of the agents and/or physical constrains imposed on the agents dynamics. For example, when the agents aim to keep a fixed distance/position with respect to the neighboring agents, or energy constraints that depend on the choice of α_i. ◇

V. AN ILLUSTRATIVE EXAMPLE

In this section, we illustrate the main results obtained in this paper. Consider the communication graph depicted in Figure 1. By executing Algorithm 1, and considering that all agents have upper-bound on the total number of agents in the network given by $\hat{n} = 10$, we obtain that $\mathcal{L}_i = \{1, 2, 3, 4\}$ with $i \in \{1, 2, 3, 4\}$; $\mathcal{L}_j = \mathcal{L}_i \cup \{6, 7\}$ with $j = 6, 7$; and $\mathcal{L}_5 = \{5\}$. Therefore, we have $z[0] = [4\ 4\ 4\ 4\ 1\ 7\ 7]$, and $z[\hat{n}] = [z_1[\hat{n}] \ldots z_7[\hat{n}]] = [4\ 4\ 4\ 4\ 1\ 1\ 1]$; therefore, one of the agents $\{1, 2, 3, 4\}$ can be a potential leader, but only the one with lowest index declares itself as a leader, i.e., agent 1 becomes a leader. Similarly, it follows that agent 5 is also a leader, which implies that $\mathcal{J}^* = \{1, 5\}$ is the output of Algorithm 1.

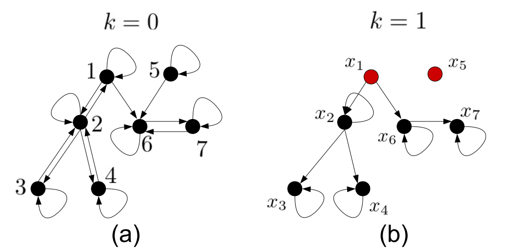

Fig. 1. Evolution of the communication graph, where the edges depict the communication being performed in the network. In particular, there exist self-loops in all state vertices (not depicted), except the leaders' states depicted by red vertices, and the solid edges correspond to the the transmissions being considered by the agents into their dynamics. In a) we have the original communication graph, and in b) the communication protocol $W(\mathcal{D})$ used in the remaining of the computations.

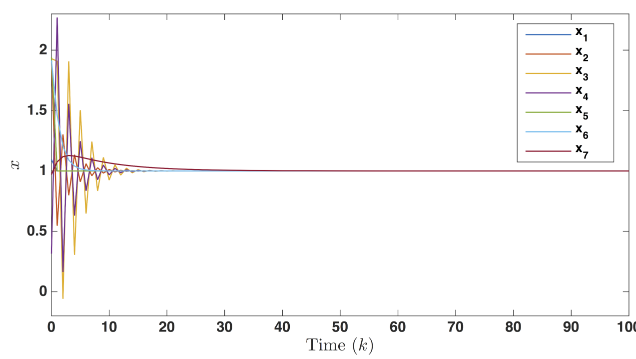

Fig. 2. Evolution of the agents' states over time.

After the leaders are chosen using Algorithm 1, Algorithm 2 is performed, leading to the communication graph depicted in Figure 1-b) at $k = 1$. The ids of the agents 1-7 are [2 3 5 7 11 13 17], respectively; therefore, considering all these ids divided by 18 to ensure that $\alpha_i \in (0, 1)$, we obtain $\alpha = [0.1111\ \ 0.1667\ \ 0.2778\ \ 0.3889\ \ 0.6111\ \ 0.7222\ \ 0.9444]$.

The evolution of the system's states is presented in Figure 2, where the initial conditions where chosen uniformly from $(0, 2)$ and the reference value shared to the leaders is $u^* = 1$, around which we can see that the agents's states have converged to. Finally, we notice that, as observed in

Remark 2, the choice of α_i has implications in the evolution of the dynamics and should be tuned to satisfy additional constraints. In particular, in Figure 2, we notice that agents 5-7 have "smooth" dynamics since $0.6 \leq |\alpha_i| < 1$, with $i = 5 - 7$. On contrary, agents 1-3 have an abrupt change in the dynamics to speed-up the convergence to the remaining agents' states.

VI. CONCLUSIONS AND FURTHER RESEARCH

In this paper, we provided a distributed solution to the leader selection problem where agents have scalar states, and the communication graph can be an arbitrary time-invariant digraph. In this problem, agents not only decide who has to become a leader, but also design a communication protocol, to ensure some control-theoretic specifications and/or performance guarantees. Whereas in the current paper we explored control strategies that aim to asymptotically converge towards a common (and constant) reference given to the leaders, it is left to explore how the proposed scheme can cope with time-varying (and potentially different) references provided to the leaders. Additionally, it would be of interest to explore how the current results can be extended to agents with arbitrary state space dimension and time-varying topologies. Finally, it would be interesting to explore practical applications, for instance, swarm formation, and understand how the present solution can be adapted to model biological scenarios.

REFERENCES

[1] Y. Chen, J. Lu, X. Yu, and D. Hill, "Multi-agent systems with dynamical topologies: Consensus and applications," *Circuits and Systems Magazine, IEEE*, vol. 13, no. 3, pp. 21–34, 2013.

[2] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination." *IEEE Trans. Industrial Informatics*, vol. 9, no. 1, pp. 427–438, 2013.

[3] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and Cooperation in Networked Multi-Agent Systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[4] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, june 2003.

[5] V. Tzoumas, M. A. Rahimian, G. J. Pappas, and A. Jadbabaie, "Minimal Actuator Placement with Bounds on Control Effort," *ArXiv e-prints*, Sep. 2014.

[6] S. Pequito, S. Kar, and A. Aguiar, "A structured systems approach for optimal actuator-sensor placement in linear time-invariant systems," in *Proceedings of American Control Conference*, June 2013, pp. 6108–6113.

[7] S. Pequito, S. Kar, and A. P. Aguiar, "A framework for structural input/output and control configuration selection of large-scale systems," *To Appear in IEEE Transactions on Automatic Control*. [Online]. Available: http://arxiv.org/pdf/1309.5868v1.pdf

[8] F. Jiang, L. Wang, G. Xie, Z. Ji, and Y. Jia, "On the controllability of multiple dynamic agents with fixed topology," in *American Control Conference*, June 2009, pp. 5665–5670.

[9] C. Commault and J.-M. Dion, "Input addition and leader selection for the controllability of graph-based systems." *Automatica*, vol. 49, no. 11, pp. 3322–3328, 2013.

[10] S. Pequito, S. Kar, and A. P. Aguiar, "On the Complexity of the Constrained Input Selection Problem for Structural Linear Systems," *ArXiv e-prints*, Mar. 2014.

[11] S. Pequito, S. Kar, and A. Aguiar, "Optimal cost actuator/sensor placement for large scale linear time-invariant systems: A structured systems approach," in *Proceedings of European Control Conference*, July 2013, pp. 815–820.

[12] S. Pequito, S. Kar, and A. P. Aguiar, "Minimum cost input-output and control configuration selection: A structural systems approach," in *Proceedings of IEEE 52nd Annual Conference on Decision and Control*, Dec 2013, pp. 4895–4900.

[13] S. Patterson and B. Bamieh, "Leader selection for optimal network coherence," in *49th IEEE Conference on Decision and Control (CDC)*, Dec 2010, pp. 2692–2697.

[14] F. Lin, M. Fardad, and M. Jovanovic, "Algorithms for leader selection in stochastically forced consensus networks," *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1789–1802, July 2014.

[15] A. Clark and R. Poovendran, "A submodular optimization framework for leader selection in linear multi-agent systems," in *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, Dec 2011, pp. 3614–3621.

[16] A. Clark, B. Alomair, L. Bushnell, and R. Poovendran, "Minimizing convergence error in multi-agent systems via leader selection: A supermodular optimization approach," *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1480–1494, June 2014.

[17] S. Patterson, N. McGlohon, and K. Dyagilev, "Efficient, Optimal k-Leader Selection for Coherent, One-Dimensional Formations," *ArXiv e-prints*, Dec. 2014.

[18] S. Patterson, "In-Network Leader Selection for Acyclic Graphs," *ArXiv e-prints*, Oct. 2014.

[19] I. Shames, A. Teixeira, H. Sandberg, and K. H. Johansson, "Distributed leader selection without direct inter-agent communication," in *2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems (NecSys'10)*, ser. IFAC Proceedings Volumes, 2010, pp. 221–226.

[20] A. Franchi, H. H. Blthoff, and P. Robuffo Giordano, "Distributed Online Leader Selection in the Bilateral Teleoperation of Multiple UAVs," in *IEEE Conference on Decision and Control*, Orlando, FL, December 2011, pp. 3559–3565.

[21] F. Pasqualetti, S. Zampieri, and F. Bullo, "Controllability metrics, limitations and algorithms for complex networks," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 1, pp. 40–52, March 2014.

[22] T. H. Summers, F. L. Cortesi, and J. Lygeros, "On Submodularity and Controllability in Complex Dynamical Networks," *ArXiv e-prints*, Apr. 2014.

[23] A. Olshevsky, "Minimal controllability problems," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 3, pp. 249–258, Sept 2014.

[24] S. Pequito, S. Kar, and A. P. Aguiar, "On the Complexity of the Constrained Input Selection Problem for Structural Linear Systems," *ArXiv e-prints*, Mar. 2014.

[25] G. Ramos, S. Pequito, S. Kar, A. P. Aguiar, and J. Ramos, "On the Exact Solution of the Minimal Controllability Problem," *ArXiv e-prints*, Jan. 2014.

[26] D. T. Gavel and D. Šiljak, "Decentralized adaptive control: structural conditions for stability," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 413–426, Apr 1989.

[27] P. Ioannou, "Decentralized adaptive control of interconnected systems," *IEEE Transactions on Automatic Control*, vol. 31, no. 4, pp. 291–298, Apr 1986.

[28] J.-M. Dion, C. Commault, and J. V. der Woude, "Generic properties and control of linear structured systems: a survey." *Automatica*, pp. 1125–1144, 2003.

[29] K. J. Reinschke, *Multivariable control: a graph theoretic approach*, ser. Lect. Notes in Control and Information Sciences. Springer-Verlag, 1988, vol. 108.

[30] B. M. Nejad, S. Attia, and J. Raisch, "Max-consensus in a max-plus algebraic setting: The case of fixed communication topologies," in *in XXII International Symposium on Information, Communication and Automation Technologies*, Sarajevo, Bosnia and Herzegovina.

[31] H. Tanner, G. Pappas, and V. Kumar, "Leader-to-formation stability," *Robotics and Automation, IEEE Transactions on*, vol. 20, no. 3, pp. 443–455, June 2004.

[32] J. P. Hespanha, *Linear Systems Theory*. Princeton, New Jersey: Princeton Press, Sep. 2009.