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Decentralized Observability with Limited Communication between Sensors

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Abstract— **In this paper, we study the problem of jointly retrieving the state of a dynamical system, as well as the state of the sensors deployed to estimate it. We assume that the sensors possess a simple computational unit that is capable of performing simple operations, such as retaining the current state and model of the system in its memory.**

We assume the system to be observable (given all the measurements of the sensors), and we ask whether each sub-collection of sensors can retrieve the state of the underlying physical system, as well as the state of the remaining sensors. To this end, we consider communication between neighboring sensors, whose adjacency is captured by a communication graph. We then propose a linear update strategy that encodes the sensor measurements as states in an augmented state space, with which we provide the solution to the problem of retrieving the system and sensor states.

The present paper contains three main contributions. First, we provide necessary and sufficient conditions to ensure observability of the system and sensor states from any sensor. Second, we address the problem of adding communication between sensors when the necessary and sufficient conditions are not satisfied, and devise a strategy to this end. Third, we extend the former case to include different costs of communication between sensors. Finally, the concepts defined and the method proposed are used to assess the state of an example of approximate structural brain dynamics through linearized measurements.

I. INTRODUCTION

In the last decade, a significant effort was placed in developing strategies that enable the recovery of the system state, i.e., the problem of *estimating* the system's state. The applications of such mechanisms include the monitoring of the electric power grid, process control, swarms of robots, and social networks [1], [2], [3]. Furthermore, retrieval of the state of the system enables the assessment of the overall behavior of the plant, and allows us to design control strategies that enable the proper control of the system. Such control strategies can either steer the system to a specific target, as in a swarm of robots trying to keep a formation [4], or stabilize the system, as in the case of the electric power grid whose frequency should be kept within a given range [5].

The estimation strategies can be broadly classified into *centralized*, *decentralized* and *distributed*. In centralized schemes, it is assumed that all the data collected by the sensors deployed in the plant is available to a central entity that, together with the system's model, performs the estimation of the system's state. Observability of the system is commonly sought to ensure that it is possible to design stable estimators [6]. Whereas the system might be observable when all sensors deployed in the system are considered, the same does not necessarily hold true when only a sub-collection of the sensors is considered. As a consequence, we need to enhance the classical schemes with strategies that only consider a subset of sensors, as in the case of decentralized estimation [7]. Moreover, we need to add communication between different sensor locations to obtain additional information about the system state, as in distributed estimation [8]. For instance, the sensors can average either the estimates of the state obtained by different sensors [9] or the innovations [10], which usually leads to a smaller amount of information exchanged between sensors.

In this paper, we propose a *distributed-decentralized* scheme, i.e., a decentralized estimation in the sense previously defined, which resorts to communication in a similar fashion as distributed scenarios. Although it combines elements from both decentralized and distributed approaches, the distributed-decentralized approach is distinct from them in the following sense: it encodes the sensor measurements as states in an augmented state space that considers the physical system's dynamics and the dynamics induced by the communication between the sensors. As a consequence, the sensors only need to share their state instead of estimates of the system's state or innovations. Moreover, the measured output of each sensor includes the measurements performed over the system's states, as well as the states of the remaining sensors it communicates with.

This paper was developed in the context of large distributed systems with wireless sensors, where it is more convenient to add communication links rather than terminals to the system. Our method is also a one-time offline step, so that reaching consensus online in the context of distributed estimation is avoided, and the problem of not finishing computations before the following time step is bypassed.

Related Work

Some of the most recent developments in distributed and decentralized estimation approaches are overviewed, for instance, in [11], [12], [9], [13]. More specifically, the problem of designing communication networks to solve dynamic estimation problems has been previously addressed in [14], where sufficient conditions are provided in terms of the communication structure and classification of the different agents (sensors/state variables) in the system.

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In [15], [16], the topology of a static network of sensors is designed in order to minimize the transmission cost among sensors and from the sensors to a central authority, allowing centralized field reconstruction. More recently, [17] addresses the problem of determining the minimum communication topologies to ensure observability of a multi-agent's network, given a potential communication graph. In [18], some strategies based in consensus-like methods for distributed estimation are provided by resorting to structural systems theory. In addition, [19] addresses the stabilization of a wireless control network with strategies from structural systems theory applied to augmented state systems.

The main contributions of this paper are threefold. First, we provide necessary and sufficient conditions to ensure distributed-decentralized observability. Second, we devise a method that guarantees observability when the necessary and sufficient conditions are not satisfied, by adding communication links between sensors. This implies providing a constructive algorithm that solves a maximum matching minimum cost problem. Third, we extend the former case to include different costs of communication between sensors.

The rest of the paper is organized as follows. In Section II, we provide the formal problem statement. Section III reviews graph theoretical concepts used in structural systems theory. Section IV presents the main technical results, and discusses the computational complexity of the strategies proposed. Furthermore, the case study in Section V illustrates the main results in the context of the brain dynamics. Conclusions and discussions on further research are presented in Section VI. Due to space limitations, all proofs and additional examples are omitted and can be found in [20].

II. PROBLEM STATEMENT

Consider a linear time-invariant system described by

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k], \quad k = 0, 1, \dots \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^{n\times 1}$ is the system's state, and $\mathbf{A} \in \mathbb{R}^{n\times n}$ the matrix that determines the autonomous dynamics of the system. In addition, consider m deployed sensors, whose measurements are described by

$$y_i[k] = \mathbf{c}_i\mathbf{x}[k], \quad i = 1, \dots, m, \tag{2}$$

where $y_i \in \mathbb{R}$ is the measured output, and $\mathbf{c}_i \in \mathbb{R}^{1\times n}$ the output vector that encodes the linear combination of the states measured by the sensor i.

In addition, we consider that (1)-(2) is observable, but possibly not when only some subset of sensors is considered, so that decentralized estimation is not guaranteed. Each sensor is equipped with a computational unit that is capable of performing elementary operations: it contains enough memory to retain the state estimates of the system and of the sensors, and it is capable of communicating with other sensors. These assumptions are common in distributed estimation.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the directed *communication graph* that encodes the interactions between sensors. Here $\mathcal{V} = \{1, \dots, m\}$ identifies the m sensors described in (2), and an edge $(i, j) \in \mathcal{E}$, which we refer to as a communication link, shows that sensor j transmits to sensor i. We consider that each sensor possesses a scalar state z_i, with $i = 1, \dots, m$, and its evolution over time is described as a linear combination of its previous state, the measured output of the system and the incoming states from neighboring sensors, i.e.,

$$z_i[k+1] = w_{ii}z_i[k] + y_i[k] + \sum_{j\in\mathbb{N}_i^-} w_{ij}z_j[k], \ i \in \mathcal{V}, \tag{3}$$

where $\mathbb{N}_i^- = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ are the indices of the in-neighbors of sensor i given by the communication graph $\mathcal{G}$.

Therefore, the overall dynamics described by (1)-(3) can be re-written as a linear augmented system:

$$\tilde{\mathbf{x}}[k+1] = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{0}_{n\times m} \\ \mathbf{C} & \mathbf{W}(\mathcal{G}) \end{bmatrix}}_{\tilde{\mathbf{A}}(\mathcal{G})} \tilde{\mathbf{x}}[k], \tag{4}$$

where $\tilde{\mathbf{x}} = [x_1 \ \dots \ x_n \ z_1 \dots z_m]^\intercal$ is the augmented system's state, $\mathbf{C} = [\mathbf{c}_1^\intercal \ \mathbf{c}_2^\intercal \ \dots \ \mathbf{c}_m^\intercal]^\intercal$ the measured output and $\mathbf{W}(\mathcal{G})$ the dynamics between sensors induced by the communication graph, i.e., $[\mathbf{W}(\mathcal{G})]_{ij} = w_{ij}$ when $(i, j) \in \mathcal{E}$ and zero otherwise. We consider that each sensor has access to its own state, thus, $\mathbf{W}(\mathcal{G})$ has a non-zero diagonal. Subsequently, the output measured by the sensors is given by (2) and the incoming states from the neighboring sensors, since the average rule in (3) is performed at the sensor level, i.e., in its computational unit. Thus, the measured output for the augmented system is given by

$$\tilde{\mathbf{y}}_i[k] = \underbrace{\begin{bmatrix} - \ \mathbf{c}_i \ - & \mathbf{0}_{1\times m} \\ \mathbf{0}_{|\mathbb{N}_i^-|\times n} & \mathbb{I}_m^{\mathbb{N}_i^-} \end{bmatrix}}_{\tilde{\mathbf{C}}_i} \tilde{\mathbf{x}}[k], \ i \in \mathcal{V}, \tag{5}$$

where $\mathbb{I}_m^{\mathcal{J}}$ is the sub-matrix containing the rows of the $m\times m$ identity matrix with indices in $\mathcal{J} \subset \{1, \dots, m\}$.

In this setup, we aim to ensure that each sensor's computational unit is capable of retrieving the state of the augmented system. In other words, (4)-(5) (or, equivalently, $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$), is observable for $i = 1, \dots, m$. Subsequently, we aim to address the following three related problems.

Problem 1 Determine the necessary and sufficient conditions that $\mathbf{W}(\mathcal{G})$ must satisfy to ensure observability of $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$ for $i = 1, \dots, m$. ◦

Problem 2 If the necessary and sufficient conditions to attain observability of $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$ for $i = 1, \dots, m$ do not hold, then determine the minimum number of additional communication links which yield such conditions. ◦

In practice, the sensors might be deployed at varying distances from one another, making it more convenient to add some communication links instead of others. In other words, we can associate costs to the potential communication links from sensor j to sensor i, which we denote by γ_{ij}, which is zero if the communication link already exists. Thus, we consider the cost of setting a communication link between any two sensors and we pose the following problem:

Problem 3 If the necessary and sufficient conditions to attain observability of $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$ for $i = 1, \dots, m$ do not hold, then determine the minimum number of additional communication such that the cost is minimized and observability holds. ○

III. PRELIMINARIES AND TERMINOLOGY

The following standard terminology and notions from structural systems theory and graph theory can be found, for instance, in [21]. Structural systems deal with the sparsity (i.e., location of zeroes and, non-zeroes) patterns of matrices, rather than with the numerical values of the elements. Let $\bar{\mathbf{A}} \in \{0, *\}^{n \times n}$ be the matrix that represents the structural pattern of $\mathbf{A}$ with the following encoding: if $\bar{\mathbf{A}}_{ij} = 0$, then $\mathbf{A}_{ij} = 0$ and if $\bar{\mathbf{A}}_{ij} = *$, where $*$ is an arbitrary non-specified value $*$, then $\mathbf{A}_{ij}$ can take any value. Following the sparsity pattern, we associate structural matrices to every matrix in (1)-(2), (4)-(5), that will be employed further.

A pair $(\bar{\mathbf{A}}, \bar{\mathbf{C}})$ is structurally observable if and only if there exists an observable pair $(\mathbf{A}, \mathbf{C})$ with the same sparseness as $(\bar{\mathbf{A}}, \bar{\mathbf{C}})$. Moreover, given a structurally observable pair $(\bar{\mathbf{A}}, \bar{\mathbf{C}})$, then *almost all* pairs of real matrices $(\mathbf{A}, \mathbf{C})$ with the same structure as $(\bar{\mathbf{A}}, \bar{\mathbf{C}})$ are observable [22].

Let $\mathcal{D}(\bar{\mathbf{A}}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the digraph representation of $\bar{\mathbf{A}}$, to be referred to as the *state digraph*, where the vertex set $\mathcal{X}$ represents the set of state variables and $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : \bar{\mathbf{A}}_{ji} \neq 0\}$ denotes the set of edges. Similarly, we define the *state-output digraph* $\mathcal{D}(\bar{\mathbf{A}}, \bar{\mathbf{C}}) = (\mathcal{X} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}})$, where $\mathcal{Y}$ represents the set of output variables and $\mathcal{E}_{\mathcal{X},\mathcal{Y}} = \{(x_j, y_i) : \bar{\mathbf{C}}_{ji} \neq 0\}$. A digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ is *strongly connected* if there exists a directed path between any pair of vertices, i.e., a sequence of edges that starts and ends in those vertices. A sub-graph $\mathcal{D}_s = (\mathcal{V}_s, \mathcal{E}_s)$, with $\mathcal{V}_s \subset \mathcal{V}$ and $\mathcal{E}_s \subset \mathcal{E}$, is a *strongly connected component (SCC)* if between any two vertices in $\mathcal{V}_s$ there exists a directed path, and $\mathcal{D}_s$ is maximal among such subgraphs. Visualizing each SCC as a supernode, one may generate a *directed acyclic graph* (DAG), in which each node corresponds to a single SCC and a directed edge exists between two SCCs *if and only if* there exists a directed edge connecting vertices in the SCCs in the original digraph. In the DAG representation, if the SCC does not have an edge from any of its states to the states of another SCC, then it is referred to as *sink SCC*. Similarly, an SCC that does not have any incoming edge from other SCC is called a *source SCC*.

For any two vertex sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{V}$, we define the *bipartite graph* $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ associated with $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, to be a directed graph, whose vertex set is given by $\mathcal{S}_1 \cup \mathcal{S}_2$ and the edge set by $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2} = \{(s_1, s_2) \in \mathcal{E} : s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2 \}$. We refer to the bipartite graph $\mathcal{B}(\bar{\mathbf{A}}) \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ as the *state bipartite graph*, and to $\mathcal{B}(\bar{\mathbf{A}}, \bar{\mathbf{C}}) \equiv \mathcal{B}(\mathcal{X}, \mathcal{X} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}\cup\mathcal{Y}})$ as the *state-output bipartite graph*. Given $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, a matching M corresponds to a subset of edges in $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ that do not share vertices. In addition, a maximum matching M^* is defined as a matching M that has the largest number of edges among all possible matchings. Such a matching decomposes the digraph into a disjoint set of cycles and elementary paths. The term *left-unmatched vertices* (with respect to a maximum matching M^* associated to $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$) refers to those vertices in $\mathcal{S}_1$ that do not have an outcoming matching edge in M^*. For simplicity, we often just say *a set of left-unmatched vertices*, omitting the explicit reference to the maximum matching when the context is clear. If we need to emphasize that the set of left-unmatched vertices is associated with a specific maximum matching M of the state bipartite graph, then we make the dependency explicit by $\mathcal{U}_L(M)$. Given a vector of weights w associated to the edges in $\mathcal{E}$, we define a *weighted matching* as the matching with weights corresponding to its constituting edges. Moreover, a *minimum cost maximum matching (MCMM)* is a maximum matching M^* whose edges achieve the minimum cost among all maximum matchings. Finally, we notice that there may exist several sets of left-unmatched vertices, since the maximum matching M^* may not be unique.

Next, we state a known result regarding structural observability. The following theorem is an extension of the dual of Theorem 5 and Theorem 7 in [21] regarding the necessary and sufficient conditions for structural observability (see [20] for details).

Theorem 1: Let $\mathcal{D}(\bar{\mathbf{A}}, \bar{\mathbf{C}}) = (\mathcal{X} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}})$ denote the state-output digraph and $\mathcal{B}(\bar{\mathbf{A}}, \bar{\mathbf{C}}) \equiv \mathcal{B}(\mathcal{X}, \mathcal{X} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}\cup\mathcal{Y}})$ the state-output bipartite representation. The pair $(\bar{\mathbf{A}}, \bar{\mathbf{C}})$ is structurally observable if and only if the following two conditions hold:

(i) there is a path from every state vertex to an output vertex in $\mathcal{D}(\bar{\mathbf{A}}, \bar{\mathbf{C}})$; and
(ii) there exists a maximum matching M^* associated to $\mathcal{B}(\bar{\mathbf{A}}, \bar{\mathbf{C}})$ such that $\mathcal{U}_L(M^*) = \emptyset$. ◇

IV. MAIN RESULTS

In this section, we present the main results of the present paper. To give a solution to *Problem 1*, we propose to decouple the structure of the problem from its numeric parametrization. First, we rely on structural system theory to ensure the necessary conditions, and then we show that the necessary conditions on the structure are sufficient to ensure observability through parametrization of the solution. These results are described in Theorem 2 and Theorem 3.

Furthermore, we investigate the cases when the conditions for *Problem 1* do not hold, as stated in *Problem 2* and *Problem 3*. The computational complexity of *Problem 2* is given in Theorem 4. Because this is NP-hard, we provide an approximate polynomial constructive solution to *Problem 2*, by describing an algorithm that decides which edges have to be added to ensure distributed-decentralized structural observability. The proposed procedure is provided in Algorithm 1 and its correctness and computational complexity are asserted in Theorem 5. Subsequently, in *Problem 3*, we address the situation of different costs for the communication links between sensors, which is also NP-hard, and whose approximate solution is similar to that of *Problem 2*.

A. Solution to Problem 1

First, we ensure that *Problem 1* can be solved when structural observability is sought.

Theorem 2: Let $\mathcal{D}(\tilde{\mathbf{A}}(\mathcal{G})) = (\mathcal{V} \equiv (\mathcal{X} \cup \mathcal{Z}), \mathcal{E}_{\mathcal{V},\mathcal{V}})$ be the state digraph, where $\mathcal{X}$ corresponds to the labels of state vertices and $\mathcal{Z}$ to the labels of sensors' states. Let $\mathcal{N}_i^- = \{v \in \mathcal{V} : (v, z_i) \in \mathcal{E}\}$ be the set of in-neighbors of a vertex z_i representing a sensor in $\mathcal{D}(\tilde{\mathbf{A}}(\mathcal{G}))$. The following two conditions are necessary and sufficient to ensure that $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$, for $i = 1, \ldots, m$, is generically observable:

(i) for every $z \in \mathcal{Z}$ there exists a directed path from any $v \in \mathcal{V}$;

(ii) for every $z \in \mathcal{Z}$ there exists a set of left-unmatched vertices $\mathcal{U}_L$, associated with a maximum matching of the bipartite representation of $\mathcal{D}(\tilde{\mathbf{A}}(\mathcal{G}))$, such that $\mathcal{U}_L \subset \mathcal{N}_i^-$ and $\mathcal{U}_L \cap \mathcal{X} = \emptyset$. $\diamond$

Structural observability holds for almost all numerical realizations of both $\tilde{\mathbf{A}}(\mathcal{G})$ and $\tilde{\mathbf{C}}_i$. Nonetheless, unlike [23], we want the parametrization of $(\mathbf{A}, \mathbf{C})$ to be fixed. Therefore, we show that with the observability of $(\mathbf{A}, \mathbf{C})$ and with a parametrization of $\mathbf{W}(\mathcal{G})$ alone, it is possible to attain the observability of $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$.

Theorem 3: If $(\mathbf{A}, \mathbf{C})$ is observable and $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$, for $i = 1, \ldots, m$, is structurally observable, then almost all realizations of $\mathbf{W}(\mathcal{G})$ ensure that $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$ is observable for $i = 1, \ldots, m$. $\diamond$

Remark 1: The proof of Theorem 3 provides a criterion for choosing the parametrization of $\mathbf{W}(\mathcal{G})$ in a deterministic fashion, such that $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$ is observable for $i = 1, \ldots, m$.

B. Solution to Problem 2

Next, we leverage the conditions prescribed for the solution to *Problem 1* to obtain a solution to *Problem 2*. For simplicity, we provide solutions to *Problem 2* when one of the conditions of Theorem 2 holds, hence, only the remaining condition needs to be ensured. More specifically, we assume that $\mathcal{G}$ is strongly connected (the other case is treated in [20]). Observe that this is a mild condition since many results in literature like [18] assume that the communication graph is strongly connected from its construction. Consequently, we only need to "bring" the left-unmatched vertices to a neighborhood of the sensors' states, as required by condition (ii) in Theorem 2. Notice that there might exist several possible sets of left-unmatched vertices associated with maximum matchings. In particular, some of these left-unmatched vertices might be either the dynamical system's state vertices or sensors' state vertices. Since we have assumed that the original system is observable, we are guaranteed that there exists a maximum matching such that only $\mathcal{U}_L \subset \mathcal{Z}$. However, in order to assess if the condition $\mathcal{U}_L \subset \mathcal{N}_i^-$ holds for a sensor i, we have to explore the matchings that produce no left-unmatched system's state vertices.

Theorem 4: The problem of determining the minimum number of communication links that ensure the observability of the pair $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$, for every $i = 1, \ldots m$ is NP-hard. $\diamond$

Remark 2: In practice, the total number of left-unmatched vertices is often 0, 1 or 2 [24]. In the latter case, it is required to find two disjoint paths from the left-unmatched vertices of the state-output digraph to the in-neighboring vertices of the sensor's state, where one can often be obtained through the state-output digraph and the other through the communication digraph, if this is assumed strongly connected. $\diamond$

As a result of Remark 2, we are able to propose computationally tractable solutions to *Problem 2*. We need to produce a strategy that penalizes the left-unmatched vertices that are the system's state vertices, and favors those states in the neighborhood of a given sensor's state. To this effect, we employ the MCMM routine, which can be found in [25], for example. We define $\mathcal{G}^*$ as the union between the communication graph $\mathcal{G}$ and the set of the communication links that have to be added to guarantee observability of the state-output digraph. The procedure is detailed in Algorithm 1.

Algorithm 1: Finding the communication graph that ensures observability of $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$, given strong connectivity of $\mathbf{W}(\mathcal{G})$.

Input: $\mathcal{D}(\tilde{\mathbf{A}}(\mathcal{G})) = (\mathcal{V} \equiv (\mathcal{X} \cup \mathcal{Z}), \mathcal{E}_{\mathcal{V},\mathcal{V}}), \tilde{\mathbf{C}}_i, i = 1, \ldots, m$;
Output: The modified communication graph $\mathcal{G}^*$ and $\mathbf{W}(\mathcal{G}^*)$;
for $i = 1, \ldots, m$ **do**

- **1.** Let $\mathcal{S} = \{s_j : j \in \{1, \ldots, m\} \setminus \mathbb{N}_i^-\}$ be the set of slack variables corresponding to virtual outputs of the sensors' state, with $\mathcal{E}_{\mathcal{Z},\mathcal{S}} = \{(z_j, s_j) : j \in \{1, \ldots, m\} \setminus \mathbb{N}_i^-\}, z_j \in \mathcal{Z}, s_j \in \mathcal{S}\}$;
- **2.** Let $\mathcal{B}(\tilde{\mathbf{A}}, \tilde{\mathbf{C}}_i, S) = (\mathcal{V}, \mathcal{V} \cup \mathcal{S}, \mathcal{E}_{\mathcal{V},\mathcal{V}} \cup \mathcal{E}_{\mathcal{V},\mathcal{S}})$ be the state-output bipartite graph of the system with slack variables;
- **3.** Let the weight function of the edges be $w : \mathcal{E} \to \mathbb{R}^+$, where $\mathcal{E}$ is the set of all edges. Assign the following costs w^i for sensor i: (i) $w^i(e) = 0$, $e \in \mathcal{E}_{\mathcal{V},\mathcal{V}}$, and (ii) $w^i(e) = 1$, $e \in \mathcal{E}_{\mathcal{Z},\mathcal{S}}$;
- **4.** Define $(\mathcal{B}(\tilde{\mathbf{A}}, \tilde{\mathbf{C}}_i, S), w)$ as the weighted state-output bipartite graph;
- **5.** Run the MCMM on $(\mathcal{B}(\tilde{\mathbf{A}}, \tilde{\mathbf{C}}_i, S), w)$ and obtain M^*;
- **6.** For all $j = 1, \ldots, m$ such that $\{(z_j, s_j) \in M^* \setminus \mathcal{E}_{\mathcal{Z},\mathcal{S}}\}$, add (z_i, z_j) to $\mathcal{G}$.

Let $\mathcal{G}^* = \mathcal{G}$;
Find $\mathbf{W}(\mathcal{G}^*)$ that yields the result in Theorem 3.

Intuitively, the introduction of the slack variables $\mathcal{S}$ in Algorithm 1 is related to adding the communication links that ensure $\mathcal{U}_L \subset \mathcal{N}_i^-$. Any slack variable $s_j \in \mathcal{U}_L(M^*)$ suggests adding a communication link from sensor z_j to sensor z_i.

Theorem 5: The procedure outlined in Algorithm 1 is correct, i.e., its execution ensures the observability of the augmented pair $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$, for any sub-collection of sensors considered. In addition, the complexity of the algorithm is $\mathcal{O}(m(n + 2m)^3)$. $\diamond$

In summary, this strategy finds the maximum number of left-unmatched vertices in the neighborhood of a sensor. Hence, one can simply add communication links from $s_j \in \mathcal{U}_L \not\subset \mathcal{N}_i^-$ to sensor i. Denote the number of such left-unmatched vertices with respect to sensor i as q_i. Subsequently, we can perform the same procedure for each sensor, taking into consideration the edges added previously, and we end up having a total of $q_1 + \ldots + q_m$ additional links. This number is less than or equal to the number of communication links that have to be added for each sensor, independently.

One can argue that this strategy is not necessarily optimal since the number of edges added is not minimal, and some of the edges added in the procedure for sensor i might be used in constructing a path in the MCMM for sensor $i+1$. This problem is NP-hard, as shown in Theorem 4, but, by Remark 2, the maximum number of left-unmatched vertices that can appear in practice is two, the difference between the number of edges added by Algorithm 1 and the minimum number of edges that need to be added is likely negligible.

C. Solution to Problem 3

The binary cost strategy designed for the solution of *Problem 2* is expandable to variable costs. Suppose we are provided with a matrix of costs $\mathbf{\Gamma}$, with elements γ_{ij} denoting the costs of adding a communication link from sensor j to sensor i. Using these costs, we can adapt the previous strategy to the cost constrained problem of choosing the edges to be added such that the system is distributed-decentralized observable. The procedure implements the idea of including the costs γ_{ij} in the weights for the bipartite representation $(\mathcal{B}(\tilde{\mathbf{A}}, \tilde{\mathbf{C}}_i, S), w)$.

Theorem 6: Let $\{\gamma_{ij}\}_{i,j\in\{1,\ldots,m\}}$ be the communication cost, and consider the following weight structure: (i) $w^i(e) = 0$, $e \in \mathcal{E} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}}$, (ii) $w^i(e) = \gamma_{ij}$, $e = (z_j, s_j), z_j \in \mathcal{Z} \setminus \mathcal{N}_i^-, s_j \in \mathcal{S}$, for every $i, j = 1, \ldots, m$. Algorithm 1 correctly computes and selects the communication links that have to be added to ensure the observability of the pair $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$, when the cost of communication is imposed and the weight pattern w^i for $i = 1, \ldots, m$ is considered. $\diamond$

The proof is similar to the proof of Theorem 5. As before, the algorithm for finding the MCMM will select the maximum number of edges of cost zero. Following the design of the new weighting pattern, when the communication links already available are not enough to satisfy $\mathcal{U}_L \subset \mathcal{N}_i^-$, the algorithm will try to minimize the cost of the links and not their number. In addition, observations regarding the minimum number of communication links can be made, in a similar manner as in the end of Section IV-B.

V. ILLUSTRATIVE EXAMPLE

In this section, we use the concepts and methodology proposed in Section IV in the context of brain dynamics. We consider a linearized brain dynamics associated with data obtained from electroencephalography (EEG) [26], whose structure is induced by the brain structural connectivity obtained via magnetic resonance imaging [27]. Specifically, consider the brain partitioned in 34 different regions [28], and its connectivity captured by the digraph $\mathcal{D} = (\mathcal{V} = \{1, \ldots, 34\}, \mathcal{E})$, where each $v \in \mathcal{V}$ labels a different region and $\mathcal{E}$ captures the existence of white-matter tracks between different regions. The activity in the different regions and between these is fluctuating, being more pronounced during the execution of certain tasks. Therefore, following [26], the state evolution can be captured by considering the first-order autoregressive model:

$$\mathbf{x}[k+1] = \mathbf{A}(\mathcal{D})\mathbf{x}[k] + \epsilon[k], \quad k = 0, 1, \ldots,$$

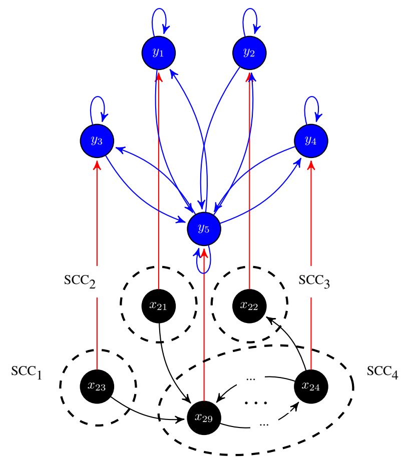

Fig. 1. DAG representation of the 34 regions of the brain, along with the sensors deployed for measurement of the system. The strongly connected component labeled as SCC$_4$ contains all the other state vertices not depicted.

where $\mathbf{x} \in \mathbb{R}^{34}$ is the state of the different regions, and $[\mathbf{A}_k(\mathcal{S})]_{i,j} = 0$ if $(j,i) \notin \mathcal{E}$, else, a scalar to be determined, and ϵ_k is the dynamics error. Nonetheless, we assume these to be zero-one as described in [28], where additionally, the diagonal entries were set to zero and only 30% of the off-diagonal entries were considered. The digraph $\mathcal{D}(\mathbf{A})$ is composed of four SCCs as shown in Figure 1, with two source SCCs containing the states numbered 21 and 23 and one sink SCC containing the state numbered 22. In addition, there exists a maximum matching of the state bipartite graph with unmatched-vertices $\mathcal{U}_L = \{21, 22\}$; hence, measuring these two variables suffices to ensure structural observability.

For the simplicity of the model (and reproducibility of the results), we assume that an EEG sensor captures the behavior of a single region in a linearized manner, i.e., $y_k = \mathbb{I}_{34}^{\mathcal{J}} x_k$, where $\mathcal{J} = \{21, 22, 23, 24, 29\}$ correspond to sensors deployed in the following scalp locations {AF3, AF4, T7, T8, Pz} (see details of these locations in [29]), and consistent with the locations of the EMOTIV Insight [30]. In addition, the system is observable – in particular, we notice that considering that other regions were simultaneously measured by the EEG sensors would not compromise the observability under the present dynamics. Subsequently, considering the design topology of this device, we assume the following communication graph (through wiring) holds:

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Because the system is observable, it follows that it is also structurally observable, and, in particular, there are distinct sensors measuring the locations $\{21, 22\}$. Furthermore, notice that there are direct paths from every vertex in the digraph to every sensor, which implies that condition (i) of Theorem 2 is satisfied. However, condition (ii) is not fulfilled, and, therefore, we need to address *Problem 3*, where we consider a unit of cost c and a communication cost structure $\mathbf{\Gamma} = [\gamma_{ij}]_{i,j\in\{1,\ldots,m\}}$, in accordance to the distance between the sensors, given by:

$$\mathbf{\Gamma} = \begin{bmatrix} 0 & 2c & 3c & 4c & 4c \\ 2c & 0 & 4c & 3c & 4c \\ 3c & 4c & 0 & 5c & 3c \\ 4c & 3c & 5c & 0 & 3c \\ 4c & 4c & 3c & 3c & 0 \end{bmatrix}.$$

Running Algorithm 1 with the given costs, as in Theorem 6, we obtain that there is only one link from sensor 2 to sensor 4 that has to be added, while incurring the minimum cost. The resulting communication scheme that guarantees observability of the system with respect to each sensor is $\mathcal{G}^*$, with a total cost of $3c$, and a possible communication protocol $\mathbf{W}(\mathcal{G}^*)$ with random values of the parameters, given as following:

$$\mathcal{G}^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{W}(\mathcal{G}^*) = \begin{bmatrix} 0.77 & 0 & 0 & 0 & 0.48 \\ 0 & 0.43 & 0 & 0 & 0.44 \\ 0 & 0 & 0.30 & 0 & 0.50 \\ 0 & 0.51 & 0 & 1.00 & 0.81 \\ 1.00 & 0.79 & 0.64 & 1.00 & 0.37 \end{bmatrix}.$$

In conclusion, the system $(\tilde{\mathbf{A}}(\mathcal{G}), \tilde{\mathbf{C}}_i)$ as in (4)-(5) is observable for all $i = 1, \ldots, m$.

VI. CONCLUSIONS AND FURTHER RESEARCH

In this paper, we addressed the problem of exact distributed-decentralized retrieval of the states of an LTI system, from any subset of sensors of the given system. We considered that the sensors have storing and communicating capabilities, and that they are inter-linked according to a communication graph. Our approach involved associating states to the sensors and constructing an augmented system, for which we provided the necessary and sufficient conditions to ensure observability from any sensor. Furthermore, we addressed the problem of re-designing the communication graph to ensure distributed-decentralized observability when the previous conditions are not readily fulfilled and proved that it is NP-hard. We devised a suboptimal solution that can be attained in polynomial time such that the observability requirements are satisfied. Moreover, we proposed an extension to the previous strategy that takes into consideration the variable costs of adding communication links between the sensors. We explored the trade-offs between the different aspects of the control-communication-computation paradigm for the present setup that employed more communication and a lighter computational load.

Future research includes identifying subclasses of the problem in which the solution described in the present paper is optimal and examining suboptimality guarantees. Moreover, we notice that, in some scenarios, adding communication capabilities may be more prohibitive than adding more memory to the sensors. We will further investigate the implications of having multi-dimensional sensors' states, and address the possibility of relying on the same communication graph and providing different communication schemes such that a distributed-decentralized scheme is feasible.

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