FEASIBLE FORMATIONS OF MULTI-AGENT SYSTEMS

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Abstract. Formations of multi-agent systems, such as mobile robots, satellites. and aircraft, require individual agents to satisfy their kinematic equations while constantly maintaining inter-agent constraints. In this paper, we develop a systematic framework for studying formation feasibility of multi-agent systems. In particular, we consider *undirected formations* for centralized formations, and *directed formations* for decentralized formations. In each case, we determine differential geometric conditions that guarantee formation feasibility given the individual agent kinematics. Our framework also enables us to extract a smaller control system that describes the group kinematics while maintaining all formation constraints.

1. Introduction

Advances in communication and computation have enabled the distributed control of multi-agent systems. This philosophy has resulted in next generation automated highway systems [27], coordination of aircraft in future air traffic management systems [26], as well as formation flying aircraft, satellites, and multiple mobile robots [2, 3, 10, 9]. The control of multi-agent systems is greatly simplified when the agent's mission can be executed by means of a *formation*. In several applications, maintaining a formation is even fundamental as in multiple aircraft where the formation is used to explore aerodynamic effects [18, 6] or in robotic exploration of large areas with restricted sensor capabilities [8].

The various approaches to *formation control* of a group of agents can roughly be divided into three categories: Behavior-based, Leader-Follower, and Rigid-Body type formations. Behavior based approaches [2, 17, 15] start by designing simple and intuitive behaviors or motion primitives for each individual agent. Then, by a weighted sum of this simple primitives more complex motion patterns are generated through the interaction of several agents. These motion patterns are usually called the group behavior that is said to emerge from the individual ones. Although this approach is characterized by being difficult to analyze in a rigorous and formal way, there have been some attempts to formally define and model behavior-based control schemes [11] and some of these simple schemes have already been proven to be stable and convergent [14]. In leader-follower

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approaches [28, 10, 16, 25] one or more agents are designated as leaders and are responsible for guiding the formation. The remaining agents are required to follow the leader with a predefined offset. Finally, in Rigid-Body type formations [24, 4, 19, 20, 12] the distance between the agents configurations (usually their positions) is required to be constant during all formation motions.

Many fundamental questions remain unanswered in these recent area of formation control. The control of a formation requires individual agents to satisfy their kinematics while constantly satisfying inter-agent constraints. In typical leader-follower formations, the leader has the responsibility of guiding the group, while the followers have the responsibility of maintaining the inter-agent formation. Distributing the group control tasks to individual agents must be compatible with the control and sensing capabilities of the individual agents. As the inter-agent dependencies get more complicated, a systematic framework for controlling formations is vital.

In this paper, we propose a framework to determine formation feasibility of multi-agent systems. Formations are modeled using *formation graphs* which are graphs whose nodes capture the individual agent kinematics, and whose edges represent inter-agent constraints that must be satisfied. A similar modeling framework has been proposed in [10], and in [25] graph theoretical methods are used to analyze formation stability properties. Similar problems arise in the study of formation rigidity properties [20, 12]. In [13] graphs are also used in the context of formation control, but the emphasis is on the communication flow and not on formation constraints. We assume kinematic models for each agent described by drift free control systems. This class of systems is rich enough to capture holonomic, nonholonomic, or underactuated agents. Two distinct types of formations are considered : *undirected formations* and *directed formations.*

In undirected formations each agent is equally responsible for maintaining the formation. For each edge constraining two agents of the formation graph, both agents cooperate in order to satisfy the constraint. Undirected formations therefore present a more centralized approach to the formation control problem as communication between agents is, in general, necessary. In directed formations, for each edge constraining two agents, only one of the agents (the follower) is responsible for maintaining the constraint. Directed formations, therefore, represent a more decentralized solution to the formation control problem. Not only the required information flow is restricted to the pairs of agents linked by an edge as also the synthesis of feedback control laws enforcing the constraints is also simpler.

In this paper, we focus on the *feasibility* problem: *Given the kinematics of several agents along with the inter-agent constraints, determine whether there exist agent trajectories that maintain the constrains.* For both directed and undirected formations we obtain differential-geometric conditions that determine formation feasibility. A related problem is to determine formation rigidity and in [12] it is shown how rigidity can be determined through the rigidity matrix which is the matrix representation of the codistribution Ω introduced in Section 4. However, we focus on formation feasibility for a larger class of formations including, but not restricted to rigid formations. When a formation is feasible, the *formation control abstraction* problem is then considered: *Given a feasible formation, extract a smaller control system that maintains formations along its trajectories.* The extracted control system allows to control the formation as a single entity, therefore being well suited for higher levels of control. A different type of formation abstraction was considered in [5] were only center of mass and variance information is retained about the group of agents. In the case of undirected formations, the centralized nature of the problem allows us to determine feasibility using a single mathematical object. A unified approach that captures both the agent kinematics as well as the formation constraints is offered by differential forms and exterior differential systems [21]. A preliminary version of the results presented in this paper appeared in [22, 23].

The structure of this paper is as follows: In Section 2 we introduce some mathematical notions to establish notation, and in Section 3 we define the notion of a formation graph. In Section 4, we consider the feasibility problem for undirected formations, whereas in Section 5, we consider it for directed formations. Finally, Section 6 describes many interesting directions of further research.

2. Mathematical Preliminaries

In this section we review some facts from differential geometry in order to establish consistent notation. The reader may wish to consult numerous books on these subject such as [1] and [7].

Let M be a differentiable manifold. We denote by T_xM its tangent space at $x \in M$ while the tangent bundle of M is denoted by $TM = \cup_{x\in M} T_xM$. Dually, the cotangent bundle is denoted by $T^*M = \cup_{x\in M} T_x^*M$, where T_x^*M is the cotangent space of M at x. Given a smooth map $f : M \to N$, we denote by Tf the tangent map of f, taking vectors $X \in T_xM$ to $T_xf(X) \in T_{f(x)}N$.

A distribution is a smooth assignment of a subbundle of the tangent bundle, that is, at each point $x \in M$ a distribution Δ assigns a linear subspace Δ_x of T_xM. In this paper we will assume that all distributions and codistributions have constant rank which implies that, locally, there exist vector fields $X_1, X_2, \ldots, X_n$ such that $\mathrm{span}\{X_1(x), X_2(x), \ldots, X_n(x)\} = \Delta_x$. Furthermore, we define the dimension of Δ at x as the dimension of the basis generating Δ_x. Given a distribution Δ, there is a unique annihilating codistribution $\Delta^\perp$ defining Δ. This codistribution is defined as:

$$\Delta^\perp = \{\alpha \in T^*M \quad | \quad \alpha(X) = 0 \quad \forall X \in \Delta\} \tag{2.1}$$

Conversely, a codistribution $\Delta^\perp$ defines a unique distribution Δ given by the set of all vector fields $X \in TM$ such that $\Delta^\perp(X) = 0$, that is, $\alpha(X) = 0$ for every $\alpha \in \Delta^\perp$.

Consider, for example, a unicycle type robot. If we model its state space by the manifold $M = \mathbb{R}^2 \times S^1$, denoting a point in M by (x, y, θ) where x and y represent the position of the robot, and θ its orientation we can define its kinematics by:

$$\begin{aligned} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_2 \end{aligned} \tag{2.2}$$

Introducing the vector fields:

$$X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \tag{2.3}$$

we can rewrite (2.2) as:

$$\dot{x} = X_1 u_1 + X_2 u_2 \tag{2.4}$$

Expression (2.4) shows that the possible directions of motion allowed by (2.2) are captured by the distribution:

$$\Delta = \mathrm{span}\{X_1, X_2\} \tag{2.5}$$

or equivalently by its annihilating codistribution:

$$\Delta^\perp = \mathrm{span}\{-\sin\theta \mathbf{d}x + \cos\theta \mathbf{d}y\} \tag{2.6}$$

Given distributions Δ on M_1 and Δ' on M_2 we define their direct sum $\Delta \oplus \Delta'$ as the direct sum of the vector space Δ_{x_1} with the vector space Δ'_{x_2} for every $(x_1, x_2) \in M_1 \times M_2$ [1].

3. Formation Graphs

Consider n heterogeneous agents with states $x_i(t) \in M_i$, $i = 1, \ldots, n$ and kinematics defined by:

$$\dot{x}_i = \sum_{j=1}^{l_i} X_j^i(x) u_j^i \tag{3.1}$$

where u_j are the control inputs and X_j vector fields on M_i. With each control system we associate a distribution Δ_i defined by:

$$\Delta_i = \text{span}\{X_1^i, X_2^i, \ldots, X_{l_i}^i\} \tag{3.2}$$

These distributions are general enough to model nonholonomy and underactuation and provide an alternative, but equivalent description of each agent kinematics.

Making use of the above distributions we model the formation of a set of agents by *formation graphs* which completely describe individual agent kinematics and global inter-agent constrains.

Definition 3.1 (Formation Graph)**.** A formation graph $F = (V, E, C)$ consists of:

- A finite set V of n vertices, where n is the number of agents in the formation. Each vertex v_i is a distribution Δ_i modeling agent i kinematics, as described in (3.2).
- A binary relation $E \subseteq V \times V$ representing a bond or link between the agents.
- A family of constraints C indexed by the set E, $C = \{c_e\}_{e \in E}$. For each edge $e = (v_i, v_j)$, c_e is a vector of $\phi(e) \in \mathbb{N}$ smooth real valued functions $c_e^k : M_i \times M_j \to \mathbb{R}$, $k = 1, 2, \ldots, \phi(e)$ defining the formation constraints between agents i and j. The constraint is enforced when $c_e^k(x_i, x_j) = 0$.

Two different types of formation graphs will be considered: undirected formations where (V, E) will be an undirected graph and directed formations where (V, E) will be a directed graph. In undirected formations, for each edge $e = (v_i, v_j)$ both agents are equally responsible for maintaining the associated constraint c_e. Undirected formations are represented by the underlying undirected graph (V, E) as displayed in Figure 1 for a formation with two agents and an edge between them. In directed formations the constraint c_e associated

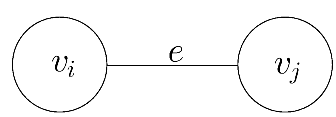

FIGURE 1. Undirected graph representing an undirected formation consisting of 2 agents and a constraint between them.

with the edge e must only be enforced by agent i. Directed formations are represented by the underlying directed graph as in Figure 2. At this point no further structure is assumed on the edge set E. Additional structure will be explicitly mentioned when needed throughout the paper.

In this paper, we focus on the formation feasibility problem, more precisely:

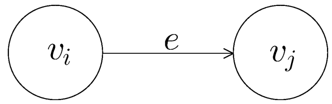

FIGURE 2. Directed graph representing a directed formation consisting of 2 agents and a constraint between them

Problem 3.2 (Feasibility)**.** *Given a formation graph $F = (V, E, C)$ determine whether there are solutions $x_i(t)$ of all agent kinematics (3.1) that maintain the constraints c_e for all $e \in E$ and $t \in \mathbb{R}_0^+$.*

We will solve Problem 3.2 for both undirected and directed formations. In case the formation is feasible, a new problem immediately emerges, the extraction of a formation control abstraction which characterizes the solution space of Problem 3.2 :

Problem 3.3 (Abstraction)**.** *Given a feasible formation graph $F = (V, E, C)$, extract a smaller control system that describes all feasible formation motions.*

Problem 3.3 will also be solved for the undirected case. In the directed case, the kinematics of the leaders is already an abstraction of the formation, since by defining the leaders motion we are also defining (up to some degrees of freedom) the motion of the followers.

4. UNDIRECTED FORMATIONS

4.1. **Feasibility.** In undirected formations each agent is equally responsible for maintaining constraints. Because of this property it will be useful to collect all agent kinematics and constraints on a single manifold:

$$M = \prod_{i=1}^{n} M_i \tag{4.1}$$

Given an element x of M the canonical projection on the ith agent:

$$\pi_i : M \to M_i \tag{4.2}$$

allow us to denote the state of the individual agents by $x_i = \pi_i(x)$. The formation kinematics is obtained by appending the individual kinematics through direct sum, that is:

$$\Delta = \oplus_{i=1}^{n} \Delta_i \tag{4.3}$$

This new control system Δ on M describes the kinematics of all agents, however it does not model any interaction between them. This interaction will be induced by the formation constraints that we now lift to the group manifold M. Each constraint c_e^k linking agent i to agent j induces a constraint $\mathcal{C}_e^k$ on M defined by:

$$\mathcal{C}_e^k(x) \quad = \quad c_e^k(\pi_i(x), \pi_j(x)) \tag{4.4}$$

All of these constraints can now be grouped in a single map from M to $\mathbb{R}^d$ with $d = \sum_{e \in E} \phi(e)$. This constraint map $\mathcal{C}$ is obtained by stacking all individual constraints as follows:

$$\mathcal{C} = [\mathcal{C}_1^1 \quad \mathcal{C}_1^2 \quad \ldots \quad \mathcal{C}_1^{\phi(1)} \quad \mathcal{C}_2 \quad \mathcal{C}_2^2 \quad \ldots \quad \mathcal{C}_2^{\phi(2)} \quad \ldots \quad \mathcal{C}_m^1 \quad \mathcal{C}_m^2 \quad \ldots \quad \mathcal{C}_m^{\phi(m)}]^T \tag{4.5}$$

where we have considered an enumeration $\{1, 2, \ldots, m\}$ of the edge set E. Without loss of generality[1] we assume that $\mathcal{C}$ has constant rank on a neighborhood of $\mathbf{0}$, consequently the set $\mathcal{C}^{-1}(\mathbf{0}) = \{x \in M \quad | \quad \mathcal{C}(x) = \mathbf{0}\}$ defines a submanifold N of M. This manifold N characterizes the interaction between the agents since the state variables of each agent are required to live on this submanifold. Formation feasibility requires that the constraints are satisfied along the formation trajectories, that is, that the submanifold N is invariant under Δ trajectories:

$$\frac{d}{dt}\mathcal{C}_e^k = \mathcal{L}_X \mathcal{C}_e^k = \mathbf{d}\mathcal{C}_e^k(X) = 0 \qquad\qquad \forall X \in \Delta, k \in \{1, 2, \ldots, \phi(e)\}, e \in E \tag{4.6}$$

To develop a single mathematical object that will allow us to check for feasibility we will adopt a differential forms approach instead of working directly with the vector fields. Denoting by $\mathbf{d}\mathcal{C}$ the codistribution:

$$\mathbf{d}\mathcal{C} = \mathrm{span}\{\mathbf{d}c_1^1, \mathbf{d}c_1^2, \ldots \mathbf{d}c_1^{\phi(1)}, \mathbf{d}c_2^1, \mathbf{d}c_2^2, \ldots, \mathbf{d}c_2^{\phi(2)}, \ldots, \mathbf{d}c_1^m, \mathbf{d}c_1^m, \ldots \mathbf{d}c_m^{\phi(m)}\} \tag{4.7}$$

we see that a vector field $X \in TM$ satisfies (4.6) iff $\alpha(X) = 0$ for every $\alpha \in \mathbf{d}\mathcal{C}$. This we shall denote by:

$$\mathbf{d}\mathcal{C}(X) = 0 \tag{4.8}$$

The kinematics of the formation agents can also be modeled by differential forms through the annihilating codistributions $\Delta_i^\perp$ of each distribution Δ_i. This lead us to define a single codistribution $\Delta^\perp$ modeling the kinematics of all formation agents as:

$$\Delta^\perp = \oplus_{i=1}^n \Delta_i^\perp \tag{4.9}$$

Vector fields $X \in TM$ satisfying $\Delta^\perp(X) = 0$ represent directions of motion respecting the individual agent kinematics, while vector fields X satisfying $\mathbf{d}\mathcal{C}(X) = 0$ represent directions of motion respecting the formation

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[1]Since we can use Sard's theorem [1] on the map obtained from $\mathcal{C}$ by removing all the linearly dependent constraints. This local rank assumption ensures that $\mathcal{C}$ is a subimmersion and therefore $\mathcal{C}^{-1}(\mathbf{0})$ is a submanifold of M [1]. Note that although the map $\mathcal{C}$ depends on the chosen enumeration, the submanifold it defines does not.

constraints. Therefore by merging both objects into:

$$\Omega = \mathbf{d}\mathcal{C} + \Delta^{\perp} \tag{4.10}$$

we can check for formation feasibility in a single equation:

$$\Omega_x(X) = 0 \quad \forall x \in N \tag{4.11}$$

Note that this equation only needs to hold for points belonging to N, since outside N the agents are no longer in formation. The previous discussion leads to the following solution of Problem 3.2:

Theorem 4.1. *An undirected formation is feasible iff equation (4.11) has solutions, equivalently iff*

$$\dim \Omega_x < \dim T^*M$$

for every $x \in N$.

A solution of equation $\Omega_x(X) = 0$ with $x \in N$ specifies the motion of each individual agent. When more than one independent solution exists, a change in the direction of a single agent may require that all other agents also change their actions to maintain formation. This shows that, in general, solutions for undirected formations are centralized and require more inter-agent communication for their implementation.

4.2. **Example.** As an example of the methodology developed so far we consider an undirected formation consisting of three mobile robots of the unicycle type as displayed in Figure 3. The kinematics of each agent

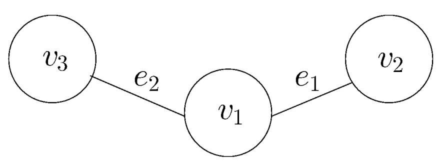

FIGURE 3. Undirected 3 agents formation.

is given by codistributions of the form (2.6). To completely specify the formation graph we need to define the constraints between the agents. Denoting by e_1 the edge between agent 1 and 2 we define the associated constraint as:

$$c_{e_1} = \begin{bmatrix} x_1 - x_2 - \delta_x \\ y_1 - y_2 - \delta_y \\ \theta_1 - \theta_2 \end{bmatrix} \tag{4.12}$$

where δ_x and δ_y are positive constants. The edge between agents 1 and 3 is denoted by e_2 and the associated constraint given by:

$$c_{e_2} = \left[\tfrac{1}{2}(x_1 - x_3)^2 + \tfrac{1}{2}(y_1 - y_3)^2 - \tfrac{1}{2}(\theta_1 - \theta_3)^2 - \delta\right] \tag{4.13}$$

with δ a positive constant. The constraint between agents 1 and 2 requires them to perform the same trajectories with an offset between their position coordinates given by δ_x and δ_y. It is intuitive that it is always possible to do so. However the constraint between agents 1 and 3 states that the distance between their positions should always equal $\sqrt{2\delta + \frac{1}{2}(\theta_1 - \theta_3)^2}$. This is clearly a non-intuitive constraint and no a priori answer can be given regarding feasibility. We will now study feasibility of this formation according to the methods developed so far. First we compute $\Delta^\perp$ which is given by:

$$\Delta^\perp = \mathrm{span}\{-\sin\theta_1 \mathrm{d}x_1 + \cos\theta_1 \mathrm{d}y_1, -\sin\theta_2 \mathrm{d}x_2 + \cos\theta_2 \mathrm{d}y_2, -\sin\theta_3 \mathrm{d}x_3 + \cos\theta_3 \mathrm{d}y_3\} \tag{4.14}$$

Since $\mathcal{C}$ is given by:

$$\mathcal{C} = \begin{bmatrix} x_1 - x_2 - \delta_x \\ y_1 - y_2 - \delta_y \\ \theta_1 - \theta_2 \\ \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}(y_1 - y_2)^2 - \frac{1}{2}(\theta_1 - \theta_3)^2 - \delta \end{bmatrix} \tag{4.15}$$

the codistribution $\mathbf{d}\mathcal{C}$ will be given by:

$$\begin{aligned} \mathbf{d}\mathcal{C} = \mathrm{span}\{ \quad & \mathrm{d}x_1 - \mathrm{d}x_2, \mathrm{d}y_1 - \mathrm{d}y_2, \mathrm{d}\theta_1 - \mathrm{d}\theta_2, \\ & (x_1 - x_3)\mathrm{d}x_1 + (y_1 - y_3)\mathrm{d}y_1 + (\theta_3 - \theta_1)\mathrm{d}\theta_1 + (x_3 - x_1)\mathrm{d}x_3 + (y_3 - y_1)\mathrm{d}y_3 + (\theta_1 - \theta_3)\mathrm{d}\theta_3\} \end{aligned} \tag{4.16}$$

Combining $\mathbf{d}\mathcal{C}$ and $\Delta^\perp$ into Ω one easily verifies that $\dim\Omega$ is at maximum 7 since a basis for Ω is formed by the 4 forms spanning $\mathbf{d}\mathcal{C}$ and more 3 forms spanning $\Delta^\perp$. This means that the formation is indeed feasible since $\dim\Omega \leq 7 < 9 = \dim T^*M$. We can therefore conclude by Theorem 4.1, that there are trajectories for each agent satisfying the formation constraints as well as its kinematics. In the next section we will see how one can control the individual agents while maintaining the formation and gain some insight into the group trajectories.

4.3. **Group Abstraction.** Whenever more than one independent solutions exist, the solution space of equation $\Omega(X) = 0$ can be used to extract a smaller control system that will preserve the formation along its trajectories. This new control system, $\Delta_G = \{X \in TM \ : \ \omega(X) = 0 \quad \forall\omega \in \Omega\}$ is an abstraction that hides away low-level control necessary to maintain the formation, and can be used in higher levels of control. If we now denote by $\{K_1, K_2, \ldots, K_k\}$ a basis for the kernel of Ω, we can write the solution of Problem 3.3 as Δ_G given by:

$$\Delta_G = \sum_{j=1}^{k} K_j v_j \tag{4.17}$$

The centralized nature of the problem is also reflected on the control abstraction. When one or more of the control inputs v_i are used, inter-agent cooperation is necessary to implement the new direction of motion since each vector K_j specifies the motion for all formation agents.

Example:(Cont.) Continuing with the previous example we will extract an abstraction representing the formation as a whole. Straightforward computations provide the following basis for the kernel of Ω:

$$\begin{aligned}
K_1 &= (\theta_3 - \theta_1)\cos\theta_1 \frac{\partial}{\partial x_1} + (\theta_3 - \theta_1)\sin\theta_1 \frac{\partial}{\partial y_1} \\
&+ (\theta_3 - \theta_1)\cos\theta_1 \frac{\partial}{\partial x_2} + (\theta_3 - \theta_1)\cos\theta_1 \frac{\partial}{\partial y_2} \\
&+ \Big((x_1 - x_3)\cos\theta_1 + (y_1 - y_3)\sin\theta_1\Big)\frac{\partial}{\partial \theta_3} \\
K_2 &= \Big((x_1 - x_3)\cos\theta_3 + (y_1 - y_3)\sin\theta_3\Big)\cos\theta_1 \frac{\partial}{\partial x_1} \\
&+ \Big((x_1 - x_3)\cos\theta_3 + (y_1 - y_3)\sin\theta_3\Big)\sin\theta_1 \frac{\partial}{\partial y_1} \\
&+ \Big((x_1 - x_3)\cos\theta_3 + (y_1 - y_3)\sin\theta_3\Big)\cos\theta_1 \frac{\partial}{\partial x_2} \\
&+ \Big((x_1 - x_3)\cos\theta_3 + (y_1 - y_3)\sin\theta_3\Big)\sin\theta_1 \frac{\partial}{\partial y_2} \\
&+ \Big((x_1 - x_3)\cos\theta_1 + (y_1 - y_3)\sin\theta_1\Big)\cos\theta_3 \frac{\partial}{\partial x_3} \\
&+ \Big((x_1 - x_3)\cos\theta_1 + (y_1 - y_3)\sin\theta_1\Big)\sin\theta_3 \frac{\partial}{\partial y_3} \\
K_3 &= (\theta_1 - \theta_3)\cos\theta_1 \frac{\partial}{\partial x_1} + (\theta_1 - \theta_3)\sin\theta_1 \frac{\partial}{\partial y_1} \\
&+ \Big((x_1 - x_3)\cos\theta_1 + (y_1 - y_3)\sin\theta_1\Big)\frac{\partial}{\partial \theta_1} \\
&+ (\theta_1 - \theta_3)\cos\theta_1 \frac{\partial}{\partial x_2} + (\theta_1 - \theta_3)\sin\theta_1 \frac{\partial}{\partial y_2} \\
&+ \Big((x_1 - x_3)\cos\theta_1 + (y_1 - y_3)\sin\theta_1\Big)\frac{\partial}{\partial \theta_2}
\end{aligned}$$

These vector fields define the abstraction through control system:

$$\Delta_G = K_1 v_1 + K_2 v_2 + K_3 v_3 \tag{4.18}$$

To gain some insight on the abstraction control system and the formation trajectories we display in Figure 4 the formation evolution when the open loop control $v_1 = 1$, $v_2 = 0$ and $v_3 = 0$ is used. Agent 1 is represented

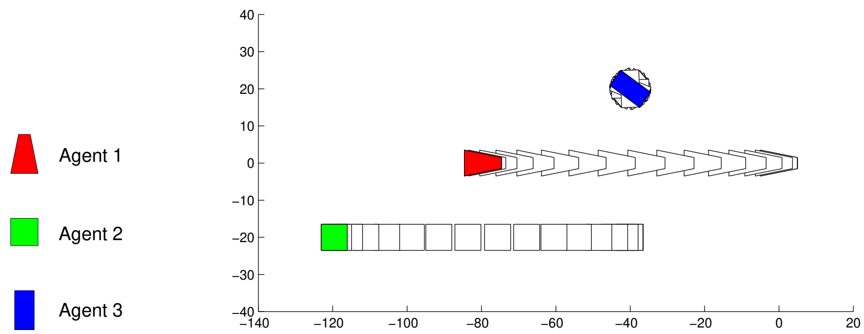

FIGURE 4. Formation flow along vector field K_1.

by a trapezoid, agent 2 by a square and agent 3 by rectangle. The formation evolution is characterized by agent 3 rotating around the same point while agent 1 and 2 perform straight line motions. When the formation flows along vector field K_2 corresponding to the open loop control $v_1 = 0$, $v_2 = 1$ and $v_3 = 0$ all the agents in the formation move along parallel trajectories as displayed in Figure 5. This was achieved since their initial orientations where identical, when this is not the case more complex motions characterize the flow along K_2. However it is always possible to achieve identical orientations by flowing along K_1 or K_3. The formation flow

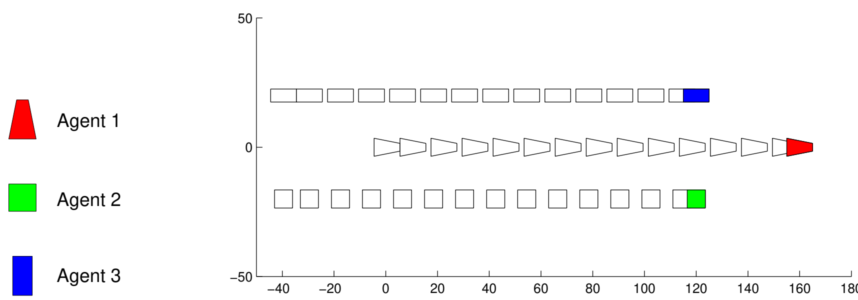

FIGURE 5. Formation flow along vector field K_2.

along basis vector K_3 is somewhat dual to K_1. Instead of agent 1 rotating around itself to achieve different configuration errors regarding agent 1, agent 3 is now stopped and the remaining agents revolve around it as suggested in Figure 6. To generate more complex motions for the formation other open or closed loops control laws can be used with the group abstraction (4.18).

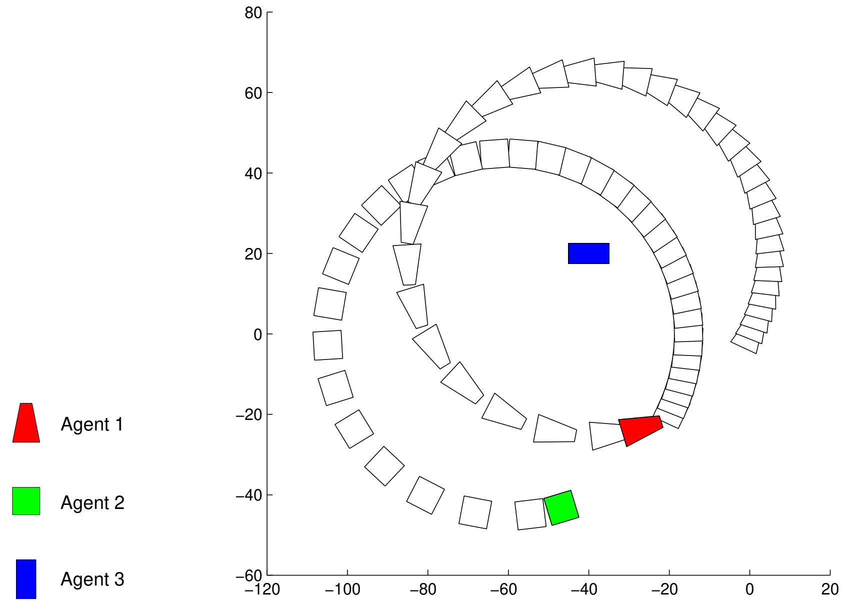

FIGURE 6. Formation flow along vector field K_3.

5. Directed Formations

Another important class of formations can be modeled by directed graphs. A directed graph assigns responsibilities to the formation members in an asymmetric way. For each edge $e = (v_i, v_j)$ agent i is responsible for maintaining the constraints c_e, while agent j is not affected by the edge constraint. Once agent j determines its motion, agent i should always be capable of *locally* computing a control strategy enforcing the formation constraint in order to render the formation feasible. From an implementation point of view, directed formations simplify the synthesis of the low level control laws responsible for maintaining the agents in formation. These control laws require only local information since each agent control actions depend only on its state and the state of the agents linked to it by outgoing arrows (its neighbors). The information flow is also simplified (unidirectional) since each agent determines its motion without the need of coordination/cooperation with other agents.

5.1. **Directed feasibility for two agents.** To illustrate the fundamental idea we start by considering a formation with two agents as displayed in Figure 2. Our goal is to determine if it is possible for agent i to be the single agent responsible for constraint c_e. To formalize this problem, let Δ_G be the set of vector fields in $TM = TM_i \times TM_j$ representing directions of motion satisfying the kinematics of each agent, as well as the formation constraint c_e, as discussed in Section 4. The question we are trying to solve can now be put in the following terms: is the distribution Δ_G such that its projection on TM_j is Δ_j? If this is the case, then agent i can be assigned the responsibility of the constraint, as for any direction of motion $X_j \in \Delta_j$ for agent j, there exists a direction of motion $[X_i \;\; X_j]^T \in \Delta_G$ for the group behavior respecting the formation constraints imposed by c_e. Instead of computing explicitly the projection of Δ_G we will provide an equivalent characterization directly in terms of the basis of one-forms spanning Δ_G.

To state our main result in a concise way, we need some additional notation. Given a one-form $\omega \in \Omega$, which we may write locally as $\omega = a(x_i, x_j)\mathbf{d}x_i + b(x_i, x_j)\mathbf{d}x_j$ for smooth real valued maps a and b, we denote by $\omega|_i$ the form $a(x_i, x_j)\mathbf{d}x_i$ and similarly by $\omega|_j$ the form $b(x_i, x_j)\mathbf{d}x_j$. Decentralization can now be characterized by:

Theorem 5.1. *Let F be a formation graph modeling a formation with two agents (i and j) and a single constraint c_e associated with the undirected edge $e = (v_i, v_j)$. The constraint c_e can be realized as a directed edge from v_i to v_j (agent i has the responsibility of enforcing the constraint) iff the following condition holds for any $\omega \in \Omega$, smooth real valued maps r_l and one-forms $\alpha_l \in \Omega$, $\alpha_l \neq \omega$, $l = 1, 2, \ldots, k$:*

$$\textit{if } \omega|_i = \sum_{l=1}^{k} r_l \alpha_l|_i \textit{ then } \omega - \sum_{l=1}^{k} r_l \alpha_l \in \Delta_j^{\perp} \tag{5.1}$$

Proof: Assume, for the sake of contradiction, that condition (5.1) holds and that there exists a $X_j \in \Delta_j$ such that for any $X \in T\pi_j^{-1}(X_j)$ we have $\Omega(X) \neq 0$. We start by fixing a basis $\{\omega_1, \ldots, \omega_n\}$ for Ω and expressing X as $X = [X_i \;\; X_j]^T$. We now see that $\omega_l(X) = \omega_l|_i(X_i) + \omega_l|_j(X_j) \neq 0$ implies that $\omega_l|_i(X_i) \neq -\omega_l|_j(X_j)$. To take into account every form $\omega_l \in \Omega$ we define the vector Y by:

$$Y = \begin{bmatrix} \omega_1|_j(X_j) \\ \omega_2|_j(X_j) \\ \vdots \\ \omega_n|_j(X_j) \end{bmatrix} \tag{5.2}$$

and rewrite $\Omega(X) \neq 0$ as:

$$L(X_i) = \begin{bmatrix} \omega_1|_i(X_i) \\ \omega_2|_i(X_i) \\ \vdots \\ \omega_n|_i(X_i) \end{bmatrix} \neq -Y \tag{5.3}$$

Regarding L as a pointwise linear operator we conclude that L is not surjective as $-Y$ does not belong to its range implying that not all of the forms $\omega_1|_i, \omega_2|_i, \ldots, \omega_n|_j$ are linearly independent. As such, we can

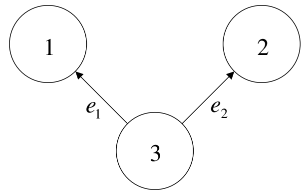

FIGURE 7. Single follower with two leaders.

write one of these forms, $\omega_1|_i$ for example, as $\omega_1|_i = \sum_{l=2}^n r_l \omega_l|_i$. It then follows that $\omega_1 - \sum_{l=2}^n r_l \omega_l$ cannot belong to $\Delta_j^\perp$ as this implies that ω_1 is linearly dependent of the remaining forms, violating the fact that $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is a basis for Ω. We thus conclude that $\omega_1|_i = \sum_{l=2}^n r_l \omega_l|_i$ and $\omega_1 - \sum_{l=2}^n r_l \omega_l \notin \Delta_j^\perp$ which contradicts (5.1) and proves sufficiency.

Necessity is also proved by contradiction by assuming that $T\pi_j(\Delta_G) = \Delta_j$, $\omega_1|_i = \sum_{l=2}^n r_l \alpha_l|_i$ and $\omega_1 - \sum_{l=2}^n r_l \omega_l = \beta \notin \Delta_j^\perp$. Since

$$\begin{aligned} \dim(T\pi_j(\Delta_G)) &\leq \dim(TM_j) - \dim(\Delta_j^\perp) - \dim(\mathrm{span}\{\beta\}) \\ &= \dim(TM_j) - \dim(\Delta_j^\perp) - 1 < \dim(\Delta_j) \end{aligned} \tag{5.4}$$

it follows that $T\pi_j(\Delta_G) \neq \Delta_j$ which contradicts $T\pi_j(\Delta_G) = \Delta_j$ and concludes the proof. □

We have thus solved Problem 3.2 for the simple case of a formation with two agents a single constraint. In the next section we describe how the developed results can also be applied to formations of several agents with multiple constraints. We also note that condition 5.1 is simple to check since it amounts to determine linear independence of one-forms.

5.2. **From two agents to acyclic formations.** We now briefly describe, through examples, how Theorem 5.1 can be applied to acyclic formations containing more then two agents.

We start by analyzing the formation displayed in Figure 7. Restricting our attention to edge e_1 we will determine if agent 3 is capable of fulfilling that constraint. We assume that all the agents have kinematics defined by codistributions of the form (2.6). The constraint associated to edge e_1 is given by $c_{e_1} = x_1 - x_3 - \delta$, where δ is a positive scalar. The codistribution Ω defining the space of solutions is given by:

$$\begin{aligned}
\Omega &= \Delta_1^\perp \oplus \Delta_3^\perp + \mathbf{d}c_{e_1} \\
\Omega &= \operatorname{span}\{\omega_1, \omega_2, \omega_3\} \\
\omega_1 &= \sin\theta_1\, \mathbf{d}x_1 - \cos\theta_1\, \mathbf{d}y_1 \\
\omega_2 &= \sin\theta_3\, \mathbf{d}x_3 - \cos\theta_3\, \mathbf{d}y_3 \\
\omega_3 &= \mathbf{d}x_1 - \mathbf{d}x_3
\end{aligned}$$

and we can see by inspection, that $\omega_1|_3 = 0\,\mathbf{d}x_3 + 0\,\mathbf{d}y_3 + 0\,\mathbf{d}\theta_3$, $\omega_2|_3 = \sin\theta_3\,\mathbf{d}x_3 - \cos\theta_3\,\mathbf{d}y_3 + 0\,\mathbf{d}\theta_3$ and $\omega_3|_3 = -\mathbf{d}x_3 + 0\,\mathbf{d}y_3 + 0\,\mathbf{d}\theta_3$ are linearly independent. Theorem 5.1 now ensures that agent 3 is indeed capable of fulfilling this constraint.

We now determine if it is possible for agent 3 to fulfill the constrains associated with *both* edges e_1 and e_2. The constraint associated with edge e_2 is defined by $c_{e_2} = y_2 - y_3 - \delta$. Group agent 1 and 2 into a new agent with kinematics given by:

$$\begin{aligned}
\Delta_{12}^\perp &= \Delta_1^\perp \oplus \Delta_2^\perp \\
&= \operatorname{span}\{\sin\theta_1\,\mathbf{d}x_1 - \cos\theta_1 \mathbf{d}y_1, \sin\theta_2\,\mathbf{d}x_2 - \cos\theta_2 \mathbf{d}y_2\}
\end{aligned} \tag{5.5}$$

allows to regard this formation as one with only two agents, agent 3 and the group agent linked by a single edge associated with the constraint:

$$c_e = \begin{bmatrix} x_1 - x_3 - \delta \\ y_2 - y_3 - \delta \end{bmatrix}$$

Computing Ω one obtains:

$$\begin{aligned}
\Omega &= \operatorname{span}\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} \\
\omega_1 &= \sin\theta_1\, \mathbf{d}x_1 - \cos\theta_1\, \mathbf{d}y_1 \\
\omega_2 &= \sin\theta_2\, \mathbf{d}x_2 - \cos\theta_2\, \mathbf{d}y_2 \\
\omega_3 &= \sin\theta_3\, \mathbf{d}x_3 - \cos\theta_3\, \mathbf{d}y_3 \\
\omega_4 &= \mathbf{d}x_1 - \mathbf{d}x_3 \\
\omega_5 &= \mathbf{d}y_2 - \mathbf{d}y_3
\end{aligned} \tag{5.6}$$

Applying Theorem 5.1 one determines that $\omega_3|_3 = \cos\theta_3\omega_5|_3 - \sin\theta_3\omega_4|_3$ and $\omega_3 - \cos\theta_3\omega_5 + \sin\theta_3\omega_4 = -\cos\theta_3\mathbf{d}y_2 + \sin\theta_3\mathbf{d}x_1$ which cannot be written as a linear combination of ω_1 and ω_2 spanning $\Delta_{12}^\perp$. This

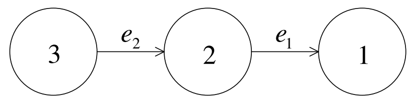

FIGURE 8. Platoon type formation.

implies, by Theorem 5.1, that edges e_1 and e_2 cannot be decentralized simultaneously. This agrees with our intuition, since if agents 1 and 2 move arbitrarily, then agent 3 cannot enforce both constraints.

We now consider the formation displayed in Figure 8 where we analyze the feasibility of the edge linking agent 3 with agent 2. Agent 1 and 3 kinematics are are modeled by a codistribution of the form (2.6), while agent 2 is assumed to be holonomic. This means that its kinematics can be described by:

$$\dot{x}_2 = X_1 u_1 + X_2 u_2 + X_3 u_3$$

for

$$X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} -\sin\theta_2 \\ \cos\theta_2 \\ 0 \end{bmatrix}$$

leading to the codistribution $\Delta_2^\perp = \{0\}$. The several formation constraints are defined by:

$$\begin{aligned} c_{e_1}^1 &= x_1 - x_2 - \delta_1^2 \\ c_{e_1}^2 &= y_1 - y_2 - \delta_2^2 \\ c_{e_2}^1 &= x_2 - x_3 - \delta_3^2 \\ c_{e_2}^2 &= y_2 - y_3 - \delta_4^2 \end{aligned} \tag{5.7}$$

grouping agent 1 with agent 2 we obtain a new agent defined by:

$$\begin{aligned} \Delta_{12}^\perp &= \Delta_1^\perp \oplus \Delta_2^\perp + \mathrm{span}\{\mathbf{d}c_{e_1}^1, \mathbf{d}c_{e_1}^2\} \\ &= \mathrm{span}\{\sin\theta_1\,\mathbf{d}x_1 - \cos\theta_1\mathbf{d}y_1, \mathbf{d}x_1 - \mathbf{d}x_2, \mathbf{d}y_1 - \mathbf{d}y_2\} \end{aligned} \tag{5.8}$$

To employ Theorem 5.1 we determine Ω:

$$
\begin{aligned}
\Omega &= \Delta_{12}^{\perp} \oplus \Delta_{3}^{\perp} + \mathrm{span}\{\mathbf{d}c_{e_2}^{1}, \mathbf{d}c_{e_2}^{2}\} \\
&= \mathrm{span}\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\} \\
\omega_1 &= \sin\theta_1\, \mathbf{d}x_1 - \cos\theta_1 \mathbf{d}y_1 \\
\omega_2 &= \mathbf{d}x_1 - \mathbf{d}x_2 \\
\omega_3 &= \mathbf{d}y_1 - \mathbf{d}y_2 \\
\omega_4 &= \sin\theta_3\, \mathbf{d}x_3 - \cos\theta_3 \mathbf{d}y_3 \\
\omega_5 &= \mathbf{d}x_2 - \mathbf{d}x_3 \\
\omega_6 &= \mathbf{d}y_2 - \mathbf{d}y_3
\end{aligned}
\tag{5.9}
$$

and quickly determines that $\omega_4|_3 = -\sin\theta_3\omega_5|_4 + \cos\theta_3\omega_6|_3$ and $\omega_4 + \sin\theta_3\omega_5 - \cos\theta_3\omega_6 = \sin\theta_3\mathbf{d}x_2 - \cos\theta_3\mathbf{d}y_2 \notin \Delta_{12}^{\perp}$. Theorem 5.1 now tell us that agent 3 will not respect the constraint imposed by edge e_2, except when $\theta_3 = \theta_1$ which leads to $\omega_4 + \sin\theta_3\omega_5 - \cos\theta_3\omega_6 = \sin\theta_1\mathbf{d}x_2 - \cos\theta_1\mathbf{d}y_2 = -\sin\theta_1\omega_2 + \cos\theta_1\omega_3 + \omega_1 \in \Delta_{12}^{\perp}$. We thus see that imposing the additional constraint $\theta_3 = \theta_1$ renders the formation feasible. This shows that the proposed methodology is useful not only for analysis as well as for synthesis by providing guidelines regarding the formations topology and the nature of the constraints.

6. Conclusions

In this paper we have proposed a framework to determine formation feasibility of multi-agent systems. Two different types of formations were considered: undirected formations were inter-agent cooperation is required to maintain formation and directed formations were control responsibilities are distributed between the agents. Differential geometric conditions were developed to determine formation feasibility for undirected formations. The abstraction problem was also addressed and solved by constructing a model of the formation as a whole, guaranteeing that the formation constraints are preserved along any of its trajectories. This model, representing the group kinematics, significantly reduces the complexity associated with the formation trajectory generation problem. For directed formations, the feasibility problem was also solved by providing necessary and sufficient conditions at the level of linear independence of differential forms. Besides its simplicity, these results are important in decentralized formations where no cooperation between agents is required, the informations flow is reduced, and the design of feedback control laws to enforce the formation constraints is also much simpler.

There are several interesting directions along the presented results. If a directed formation is not feasible one could consider the problem of extracting the largest feasible directed subformation. This would allow to maximally decentralize the control effort in the formation. Also one can consider a fixed communication topology and study feasibility under those communication constraints. Another promising direction is to study the interplay between "quality" of formation control and the formation topology. Recent results along these lines are reported in [25] when "quality" is taken as formation stability.

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