# Bisimilar Control Affine Systems[1]

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**Abstract**

The notion of bisimulation plays a very important role in theoretical computer science where it provides several notions of equivalence between models of computation. These equivalences are in turn used to simplify analysis and synthesis for these models. In system theory, a similar notion is also of interest in order to develop modular analysis and design tools for purely continuous or hybrid control systems. In this paper, we introduce two notions of bisimulation for nonlinear systems. We present a differential-algebraic characterization of these notions and show that bisimilar systems of different dimensions are obtained by factoring out certain invariant distributions. Furthermore, we also show that all bisimilar systems of different dimension are of this form.

## 1 Introduction

In theoretical computer science the notion of bisimulation inspired the definition of various notions of equivalence between models of computation. Each of these equivalences identifies classes of systems with similar properties, so that proving a property for a certain system can be done on a smaller equivalent system, thereby simplifying the process.

Similar notions are also important in the context of hybrid systems, where the inherent complexity of the hybrid model render its analysis or design very difficult. Motivated by this, we were naturally led to understand the continuous counterpart of this notion. Previous steps towards this objective have been given in [15] where linear control systems are embedded in the class of transition systems for which the notion of bisimulation was originally introduced in [19] and also [11]. It is shown in [15] that different embeddings give rise to semantically different notions of bisimulation being characterized by different conditions. For nonlinear systems no such attempt has appeared in the literature so far, except in [5] where the notion of bisimulation is presented in a sufficiently abstract categorical context to unify discrete and continuous interpretations. Compared to that work, in this paper we seek not to unify, but to characterize the notion by easily checkable (algebraic) conditions.

A characterization of bisimulation for nonlinear systems is important for several reasons that go beyond its application in hybrid systems. In the series of papers [17, 18, 21], a methodology has been introduced to compute abstractions of linear and nonlinear control systems. These abstractions are clearly important for verification problems, but also for hierarchical synthesis. For example, in [16] hierarchical stabilization of linear systems is discussed in the framework of abstractions. The ability to perform hierarchical synthesis depends on finding low-level trajectories that implement or refine trajectories of the abstracted model. A sufficient condition is given by bisimilarity, and this fact constitutes another reason to provide algebraic tests for its characterization.

The notion of bisimulation is also very interesting from a system theoretic point of view as it provides an equivalence relation on the class of control systems. This can be regarded as another tool in the quest of classifying nonlinear control systems. Furthermore, this equivalence relation has the important property of rendering as equivalent, control systems of possibly different dimensions. This contrasts with other known equivalences such as diffeomorphisms [10], or feedback transformations [2, 7, 9]. Furthermore, the notion of bisimulation also has interesting connections with other well known notions in systems theory such as controlled invariance [6, 8, 12] and symmetries for nonlinear control systems [4, 13].

In this paper we introduce two notions of bisimulation for nonlinear control systems based on the original definitions in [11]. We then focus on control affine systems and relations between them defined by subimmersions, and provide algebraic characterizations for these notions. These characterizations turn out to be related

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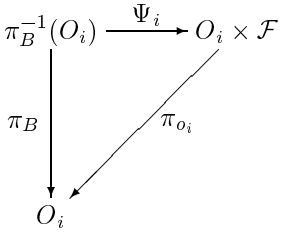
with the notion of $\Phi$-related control systems introduced in [17]. We then show that by factoring out certain invariant distributions one obtains bisimilar systems and that all bisimilar systems are obtained in this way. The distinguishing power of the two introduced notions is also discussed by showing that, locally, they are equivalent up to a feedback transformation. This is achieved by relating the introduced notions of bisimulation with controlled invariance.

For space reasons all the proofs have have been eliminated, however the interested reader may wish to consult them in [20].

## 2 Geometrical Preliminaries

Let $M$ be a differentiable manifold and $T_xM$ its tangent space at $x \in M$. In this paper, we will consider that all the manifolds are $C^\infty$, and that all related mathematical objects are also smooth. The tangent bundle of $M$ is denoted by $TM = \cup_{x\in M} T_xM$ and $\pi_M$ is the canonical projection map $\pi_M : TM \longrightarrow M$ taking a tangent vector $X(x) \in T_xM \subset TM$ to the base point $x \in M$. Now let $M$ and $N$ be manifolds and $\phi : M \longrightarrow N$ a map, we denote by $T_x\phi : T_xM \longrightarrow T_{\phi(x)}N$ the induced tangent map which maps tangent vectors $X$ at $T_xM$ to tangent vectors $T_x\phi \cdot X$ at $T_{\phi(x)}N$. If $\phi$ is such that $T_x\phi$ is of constant rank at $x \in M$ then we say that $\phi$ is a subimmersion at $x$. When $\phi$ is a subimmersion at every $x \in M$ we simply say that it is a subimmersion. When $\phi$ has an inverse which is also smooth we call $\phi$ a diffeomorphism.

A fiber bundle is a tuple $(B, M, \pi_B, \mathcal{F}, \{O_i\}_{i\in I})$, where $B$, $M$ and $\mathcal{F}$ are manifolds called the *total space*, the *base space* and *standard fiber* respectively. The map $\pi_B : B \longrightarrow M$ is a surjective submersion and $\{O_i\}_{i\in I}$ is an open cover of $M$ such that for every $i \in I$ there exists a diffeomorphism $\Psi_i : \pi_B^{-1}(O_i) \longrightarrow O_i \times \mathcal{F}$ making the following diagram commutative:

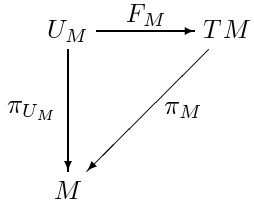



(2.1)

that is, satisfying $\pi_{o_i} \circ \Psi_i = \pi_B$, where $\pi_{o_i}$ is the projection from $O_i \times \mathcal{F}$ to $O_i$. The submanifold $\pi_B^{-1}(x)$ is called the fiber at $x \in M$ and is diffeomorphic to $\mathcal{F}$. Since a fiber bundle is locally a product, we can always find local coordinates, which we shall call trivializing coordinates, of the form $(x, b)$, where $x$ are coordinates for the base space and $b$ are coordinates for the local representative of the standard fiber.

**Definition 2.1** *A control system $\Sigma_M = (M \times V, F_M)$ consists of smooth manifolds $M$ called the state space, $V$ called the input space and a smooth map $F : M \times V \longrightarrow TM$ that assigns a vector $X \in T_xM$ to each pair $(x, v) \in M \times V$.*

Although the previous definition captures the usual notions of control systems, in certain situations it is more natural to model available inputs as being dependent on the state space. This dependence can be captured by replacing the product $M \times V$ by a fiber bundle. In this situation, we define a control system as $\Sigma_M = (U_M, F_M)$ consisting of a fiber bundle $\pi_{U_M} : U_M \longrightarrow M$ called the control bundle and a map $F_M : U_M \longrightarrow TM$ making the following diagram commutative:




(2.2)

that is, $\pi_M \circ F_M = \pi_{U_M}$, where $\pi_M : TM \longrightarrow M$ is the tangent bundle projection.

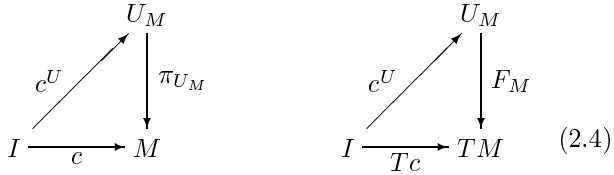
In trivializing coordinates $(x, v)$, the map $F : U_M \longrightarrow TM$ reduces to the familiar expression $\dot{x} = f(x, v)$ with $v \in \pi_{U_M}^{-1}(x)$. In the special case where the control bundle is trivial, that is, $U_M = M \times V$ we recover Definition 2.1.

Having defined control systems the concept of trajectories or solutions of a control system is naturally expressed as follows:

**Definition 2.2** *A smooth curve $c : I \longrightarrow M$, $I = ]\tau_1, \tau_2[ \subseteq \mathbb{R}_0^+$ is called a trajectory of control system $\Sigma_M = (U_M, F_M)$, if there exists a smooth curve $c^V : I \longrightarrow V$ such that:*

$$\frac{d}{dt}c(t) = F(c(t), c^V(t)) \qquad \forall\, t \in I \tag{2.3}$$

When we need to consider a fiber bundle $U_M$ instead of the product $M \times V$, we replace $c^V$ by $c^U : I \longrightarrow U_M$ and require commutativity of the following diagrams:




(2.4)

where we have identified $I$ with $TI$. These commutative diagrams are equivalent to the following equalities:

$$\begin{aligned} \pi_{U_M} \circ c^U &= c \\ Tc &= F_M(c^U) \end{aligned}$$

which express globally the equality (2.3).

A (left) action of a Lie group $G$ on a manifold $M$ is a map $\theta : G \times M \longrightarrow M$ such that $\theta(e, x) = x$ and $\theta(g_1 g_2, x) = \theta(g_1, \theta(g_2, x))$, where $e$ is the group identity and $g_1, g_2 \in G$ (see [1]). Given a point $x \in M$ we can define the orbit of $\theta$ through $x$ to be the following subset of $M$:

$$\{x' \in M \ : \ x' = \theta(x, g) \text{ for some } g \in G\}$$

An action is said to be free when $\theta(g, x) = x \Rightarrow g = e$ and proper when the map $\overline{\theta}(g, x) = (x, \theta(g, x))$ is proper. When $\theta : G \times M \longrightarrow M$ is a free and proper action, then $M/G$, the space of orbits of $\theta$ is a smooth manifold and the projection $\pi : M \longrightarrow M/G$ taking each point in $M$ to its orbit is a smooth surjective submersion [1]. Furthermore by fixing any $g \in G$ we obtain $\theta(g, -) = \theta_g : M \longrightarrow M$ a diffeomorphism of $M$.

## 3 Bisimulation Relations

The notion of bisimulation is originally credited to [19] and [11], and since then many authors have made important contributions to its development. In the context of continuous control systems, bisimulations have been discussed for the first time in [15] for linear control systems. We start by recalling the concept of transition system and bisimulation as presented in [11].

**Definition 3.1** *A transition system is a tuple* $T = (S, L, \longrightarrow)$ *consisting of:*

- *A set of states* $S$*;*
- *A set of labels* $L$*;*
- *A transition relation* $\longrightarrow \subseteq S \times L \times S$*.*

We use the graphical representation $q_1 \xrightarrow{l} q_2$ to denote $(q_1, l, q_2) \in \longrightarrow$. Intuitively, one can regard a transition system as a nondeterministic control system. Given a state $s \in S$, one interprets the set of labels $l \in L$ such that $s \xrightarrow{l} s'$ for some $s' \in S$, as the set of control inputs available at state $s$. Choosing one of those inputs will make the transition system evolve to the new state or states $s'$ satisfying $s \xrightarrow{l} s'$. The nondeterminism is captured by the fact that different triples $(s, l, s')$ and $(s, l, s'')$ may belong to $\longrightarrow$. This is the analogy that we shall make use to provide a continuous counterpart of the notion of bisimulation that we now recall.

**Definition 3.2** *Let* $T_1 = (S_1, L, \longrightarrow_1)$ *and* $T_2 = (S_2, L, \longrightarrow_2)$ *be transition systems. A relation* $H \subseteq S_1 \times S_2$ *is said a bisimulation relation between* $T_1$ *and* $T_2$ *if* $(s_1, s_2) \in H$ *implies for all* $l \in L$*:*

- *if* $s_1 \xrightarrow{l}_1 s_1'$ *then there exists a* $s_2' \in S_2$ *such that* $s_2 \xrightarrow{l}_2 s_2'$ *and* $(s_1', s_2') \in H$*.*
- *if* $s_2 \xrightarrow{l}_1 s_2'$ *then there exists a* $s_1' \in S_1$ *such that* $s_1 \xrightarrow{l}_2 s_1'$ *and* $(s_1', s_2') \in H$*.*

To import this notion into the continuous context we face the difficulty of not being able to express the continuous dynamics in terms of the "atomic" jumps $s_1 \xrightarrow{l} s_1'$. We shall, therefore, replace the atomic jumps for *any* evolution, that is, we will ask a control system to match the evolution of another control system for *every* instant of time. Furthermore, as trajectories must be obtained by using the same input trajectory (the same input symbols), the input space cannot depend on the state space. We shall, therefore assume, that the control bundle is a product $U_M = M \times V$, being $V$ the input space. Naturally, this leads to the following notion of bisimulation for control systems:

**Definition 3.3** *Let* $\Sigma_M = (U_M, F_M)$ *and* $\Sigma_N = (U_N, F_N)$ *be control systems such that* $U_M = M \times V$ *and* $U_N = N \times V$*. A relation* $H \subseteq M \times N$ *is said to be a bisimulation relation between* $\Sigma_M$ *and* $\Sigma_N$ *if* $(x, y) \in H$ *implies:*

1. *for any state trajectory* $c_M : I \longrightarrow M$ *of* $\Sigma_M$ *with* $c_M(0) = x$ *determined by input trajectory* $c^V : I \longrightarrow V$ *there exists a state trajectory* $c_N : I \longrightarrow N$ *of* $\Sigma_N$ *with* $c_N(0) = y$ *determined by input trajectory* $c^V : I \longrightarrow V$ *such that* $(c_M(t), c_N(t)) \in H$ *for every* $t \in I$*.*
2. *for any state trajectory* $c_N : I \longrightarrow N$ *of* $\Sigma_N$ *with* $c_N(0) = y$ *determined by input trajectory* $c^V : I \longrightarrow V$ *there exists a state trajectory* $c_M : I \longrightarrow M$ *of* $\Sigma_M$ *with* $c_M(0) = x$ *determined by input trajectory* $c^V : I \longrightarrow V$ *such that* $(c_M(t), c_N(t)) \in H$ *for every* $t \in I$*.*

As we shall see soon, this notion of bisimulation will be quite restrictive. This will motivate more relaxed notions of bisimulation, and in particular, we shall consider an input abstract version. This new notion relaxes the requirement that both systems have the same input trajectories and furthermore can be easily expressed without the assumption of trivial control bundles, being therefore, better suited for global analysis of control systems.

**Definition 3.4** *Let* $\Sigma_M = (U_M, F_M)$ *and* $\Sigma_N = (U_N, F_N)$ *be control systems. A relation* $H \subseteq M \times N$ *is said to be an input abstract bisimulation relation between* $\Sigma_M$ *and* $\Sigma_N$ *if* $(x, y) \in H$ *implies:*

1. *for any state trajectory* $c_M : I \longrightarrow M$ *of* $\Sigma_M$ *with* $c_M(0) = x$ *there exists a state trajectory* $c_N : I \longrightarrow N$ *of* $\Sigma_N$ *with* $c_N(0) = y$ *such that* $(c_M(t), c_N(t)) \in H$ *for every* $t \in I$.

2. *for any state trajectory* $c_N : I \longrightarrow N$ *of* $\Sigma_N$ *with* $c_N(0) = y$ *there exists a state trajectory* $c_M : I \longrightarrow M$ *of* $\Sigma_M$ *with* $c_M(0) = x$ *such that* $(c_M(t), c_N(t)) \in H$ *for every* $t \in I$.

We shall say that two control systems are (input abstract) bisimilar when there exists a (input abstract) bisimulation between then.

The above introduced notions of bisimulation are also important from a systems perspective since they allow a new type of classification of control systems. Indeed, the notion of bisimulation defines an equivalence relation in the class of control systems:

**Proposition 3.5** *Bisimulation and input abstract bisimulation are equivalence relations on the class of control systems.*

These equivalence relations have the important characteristic of rendering equivalent, systems of possibly different dimension. It therefore makes sense to consider as representative of each equivalence class, the system of smallest dimension, leading to notions of minimality.

## 4 A Characterization of Bisimulation

We start by making some assumptions that will allow to provide simple characterizations of bisimilar control systems:

1. The control systems are assumed to be control affine, that is, locally (globally in the case of trivial control bundles) there are coordinates where the system map takes the form $F_M = f_M(x) + \sum_{i=1}^{k} g_M^i(x)v_i$.
2. The associated affine distribution $\mathcal{D}_M = f_M + \Delta_M = f_M + \text{span}\{g_M^1, g_M^2, \ldots, g_M^k\}$ is of constant rank.
3. The relation $H \subseteq M \times N$ is induced by a smooth map $h : M \longrightarrow N$, that is $(x, y) \in H$ iff $h(x) = h(y)$ where $h$ is a subimmersion, that is, $T_x h$ has constant rank for every $x \in M$.

The first two assumptions are not very restrictive since the results obtained for affine control systems can be lifted to fully nonlinear control systems by making use of the notion of extended control system [14]. The third assumption is more restrictive but its justified by the fact that in [18] a construction has been presented for the computation of quotients of control systems based on such a quotient map. It is therefore of extreme importance to be able to determine when such quotients are in fact bisimilar to the original one with respect to the quotient map.

Before characterizing bisimulation we recall that given a control system $\Sigma_M$ and a subset $S$ of $M$, we say that $S$ is invariant for $\Sigma_M$ iff every trajectory of $\Sigma_M$ starting at a point $x \in S$ remains in $S$ for all time. It follows easily that $S$ must contain the all the points reachable by $\Sigma_M$ from $S$. This notion of invariance allows the characterization of bisimulation given in the next theorem:

**Theorem 4.1** *Let* $\Sigma_M = (U_M, F_M)$ *and* $\Sigma_N = (U_N, F_N)$ *be two control affine systems such that* $U_M = M \times V$ *and* $U_N = N \times V$, *and* $h : M \longrightarrow N$ *a subimmersion. Then* $\Sigma_M$ *is bisimilar to* $\Sigma_N$ *via* $h$ *iff for every* $x \in M$*:*

- $h(M)$ *is invariant for* $\Sigma_N$
- $T_x h(\mathcal{D}_M(x)) = \mathcal{D}_N \circ h(x)$
- *for every* $X \in \mathcal{D}_M$ *there exists a* $Y \in \mathcal{D}_N$ *such that* $T_x h \cdot X(x) = Y \circ h(x)$.

The above characterization shows how restrictive the notion of bisimulation is, since every vector field in $\mathcal{D}_M$ must be $h$-related to some vector field in $\mathcal{D}_N$. Relaxing this condition was the motivating factor behind the notion of input abstract bisimulation whose characterization is now presented.

**Theorem 4.2** *Let* $\Sigma_M$ *and* $\Sigma_N$ *be two control affine systems and* $h : M \longrightarrow N$ *a subimmersion. Then* $\Sigma_M$ *is input abstract bisimilar to* $\Sigma_N$ *via* $h$ *iff for every* $x \in M$*:*

- $h(M)$ *is invariant for* $\Sigma_N$
- $T_x h(\mathcal{D}_M(x)) = \mathcal{D}_N \circ h(x)$

We note that since the relation $H \subseteq M \times N$ is defined by a map $h : M \longrightarrow N$, the first condition on the definition of input abstract bisimulation can be refrased as: *for every trajectory* $c_M$ *of* $\Sigma_M$, $h(c_M)$ *must be a trajectory of* $\Sigma_N$. This was the basic definition of abstraction introduced in [17], so that it is natural that the characterization of (input abstract) bisimulation is a stronger version of the concept of $\Phi$-related control systems, which is the algebraic characterization of abstractions. It is also interesting to note that the

characterization of input abstract bisimilarity, given in Theorem 4.2 distinguishes these systems from general abstractions at the level of the structure of the control bundle as discussed in [21]. In fact, when a control system is bisimilar to its abstraction, no new inputs will appear on the abstraction, a phenomena that does not occur for general abstracted systems [17].

We now clarify how different can (input abstract) bisimilar control systems be if they have different dimensions. For this we will assume that $\dim(M) > \dim(N)$, and recall the notions of invariant and controlled invariant distributions:

**Definition 4.3** *Let $\Sigma_M$ be a control affine system and let $\mathcal{E}$ be a regular distribution on $M$. Distribution $\mathcal{E}$ is said to be invariant for $\Sigma_M$ when:*

$$[\mathcal{D}_M, \mathcal{E}] \subseteq \mathcal{E}$$

*Distribution $\mathcal{E}$ is said locally controlled invariant if there exist a local feedback transformation around each $x \in M$, such that $\mathcal{E}$ is invariant for the feedback transformed system.*

Locally controlled invariant distributions also admit the following characterization:

**Theorem 4.4 (Adapted from [3])** *Let $\Sigma_M$ be a control affine system and $\mathcal{E}$ a regular distribution on $M$. The distribution $\mathcal{E}$ is locally controlled invariant for $\Sigma_M$ iff:*

$$[\mathcal{D}_M, \mathcal{E}] \subseteq \mathcal{E} + \Delta_M$$

*where $\Delta_M = \operatorname{span}\{g_M^1, g_M^2, \ldots, g_M^k\}$.*

Equipped with the notions of invariant and controlled invariant distributions we can now understand the relationship between (input abstract) bisimilar systems of different dimensions.

**Theorem 4.5** *Let $\Sigma_M = (U_M, F_M)$ and $\Sigma_N = (U_N, F_N)$ be control affine systems such that $U_M = M \times V$, $U_N = N \times V$, $\dim(M) > \dim(N)$ and let $h : M \longrightarrow N$ be a surjective subimmersion. Then $\Sigma_M$ is input abstract bisimilar to $\Sigma_N$ via $h$ iff $\ker(Th)$ is invariant for $\Sigma_M$ and $\Sigma_N$ is defined by the affine distribution:*

$$\mathcal{D}_N \circ h(x) = \bigcup_{x' \in h^{-1} \circ h(x)} T_{x'} h(\mathcal{D}_M(x'))$$

For input abstract bisimulation we recover local controlled invariance:

**Theorem 4.6** *Let $\Sigma_M$ and $\Sigma_N$ be control affine systems such that $\dim(M) > \dim(N)$ and $h : M \longrightarrow N$ a surjective subimmersion. Then $\Sigma_M$ is input abstract bisimilar to $\Sigma_N$ via $h$ iff $\ker(Th)$ is locally controlled invariant for $\Sigma_M$ and $\Sigma_N$ is defined by the affine distribution:*

$$\mathcal{D}_N \circ h(x) = \bigcup_{x' \in h^{-1} \circ h(x)} T_{x'} h(\mathcal{D}_M(x'))$$

The previous characterization of (input abstract) bisimulation shows that although dimension is not constant on the equivalence classes of this equivalence, two control systems $\Sigma_M$ and $\Sigma_N$ of different dimensions are in the same equivalence class if and only if it is possible to obtain one from the other by factoring out (controlled) invariant distributions. As an immediate corollary of the previous results we have that factoring out symmetries also produces bisimilar systems:

**Corollary 4.7** *Let $\Sigma_M$ be an affine control system and $\theta : G \times M \longrightarrow M$ be a free and proper action of a Lie group $G$ such that for every $X \in \mathcal{D}_M$ we have $\theta_g^* X = X$ for every $g \in G$. Then $\Sigma_M$ is bisimilar via $\pi$ to $\Sigma_M/G$ defined by:*

$$\mathcal{D}_N \circ \pi(x) = \bigcup_{x' \in \pi^{-1} \circ \pi(x)} T_{x'} \pi(\mathcal{D}_M(x'))$$

For input abstract bisimilar systems it is still the case that factoring out symmetries implies input abstract bisimilarity but we allow a larger class of symmetries:

**Corollary 4.8** *Let $\Sigma_M$ be an affine control system and $\theta : G \times M \longrightarrow M$ be a free and proper action of a Lie group $G$ such that for every $X \in \mathcal{D}_M$ we have $\theta_g^* X \in \mathcal{D}_M$ for every $g \in G$. Then $\Sigma_M$ is input abstract bisimilar via $\pi$ to $\Sigma_M/G$ defined by:*

$$\mathcal{D}_N \circ \pi(x) = \bigcup_{x' \in \pi^{-1} \circ \pi(x)} T_{x'} \pi(\mathcal{D}_M(x'))$$

We have not explicitly discussed the quotient system $\Sigma_M/G$ control bundle geometry. We defer the reader to the reference [21] where these issues are addressed for general quotients and to [4, 13] where symmetries are modeled by group actions acting on the control bundle as well.

It is clear that the equivalence relation defined by bisimulation is strictly finer (in the sense that it distinguishes more control systems) then the equivalence relation defined by input abstract bisimulation. However, locally, every two input abstract bisimilar control systems are bisimilar up to a feedback transformation. This fact is a simple consequence of Theorem 4.4. This proves the following result:

**Proposition 4.9** *Let $\Sigma_M$ and $\Sigma_N$ be affine control systems input abstract bisimilar via $h : M \longrightarrow N$. Then, locally, there exists a feedback transformation for $\Sigma_M$ rendering it bisimilar to $\Sigma_N$ via $h$.*

Note that the previous result does not assert that $\Sigma_M$ is bisimilar to $\Sigma_N$ since the feedback transformation is not a bisimulation relation.

## 5 Conclusions

Motivated by notions of equivalence in computer science and hybrid systems, we have introduced the notion of (input abstract) bisimulation for nonlinear control systems. A differential algebraic characterization was given for the introduced notions capturing the notion of $\Phi$-related control systems of [17]. Although this notion constitutes an equivalence relation on the class of control systems which does not require the dimension of the systems to be an invariant, it was shown that bisimilar systems of different dimensions must be related in a very special way. In fact, one of the systems must be obtained from the other by factoring out (controlled) invariant distributions.

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