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Network Design for Controllability Metrics

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***Abstract*— In this paper, we address the constrained design of continuous-time linear dynamics to improve system control performance, which can be measured as a function of the controllability Gramian. In contrast with the problem of deployment of actuation capabilities to achieve a specified control performance, we seek to change the dynamics of linear systems while considering the deployed actuation mechanisms. Specifically, we consider spectral properties of the 'infinite' controllability Gramian as control performance metrics, and apply constrained (i.e., bounded) perturbations in the system's parameters while respecting its structure. We show that two different (yet related) re-design problems for control enhancement can be cast as bilinear or linear matrix equality problems. Lastly, we propose different strategies to obtain the solution of these problems, and assess their performance in the context of multi-agent networks in the leader-follower setup.**

I. Introduction

Due to the fast growth of components involved in networked dynamical systems, it is crucial to better understand how local dynamical interactions impact the overall system's dynamics and its properties. In particular, the notion of controllability plays a key role, since it assesses the capability of steering the system's state towards a desirable goal. Therefore, the last decade has witnessed a renewed interest in determining which actuation capabilities need to be deployed, e.g., which states need to be actuated, to ensure controllability of dynamical system while achieving some controllability performance. Often, this performance is assessed as a function of the controllability Gramian of the system, which implicitly depends on the system's dynamics and its actuation capabilities [1]–[11].

Nonetheless, in some scenarios it is not possible to change the actuation capabilities of the networked dynamical system. Therefore, one alternative consists of perturbing the system's autonomous dynamics to improve control performance. Although this might not always be possible due to physical constraints, we can envision scenarios where this is feasible. For example, in power systems, we can change the inductance of the transmission lines [12]; in multi-agents networks, the dynamics resulting from the interaction between agents can usually be represented by linear update rules that can be designed according to an objective [13].

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In the present work, we seek to re-weight network interdependencies to ensure higher control performance of the overall network with respect to its actuation capabilities. For instance, in multi-agent networks, each agent receives the state of neighboring agents and weights it with its state. In addition, in leader-follower configurations, there is a subset of the agents which are equipped with actuation capabilities, corresponding to the incorporation of an external signal that regulates the overall network, i.e., the leaders. Thus, the goal in this case is to re-design the weights that agents use, and thus, specify their dynamics such that controllability properties are maximized with respect to the actuation capabilities of the followers.

In the past years, the focus has been on determining the actuation capabilities required to improve control performance [1]–[11]. Also, there is a considerable amount of research on understanding how the network topology properties impact the control performance [14]–[20]. Recently, in [21], [22], the authors explore the minimum energy required by the inputs to transfer the state from the origin to a desired state, in the context of discrete-time bilinear networks. In fact, a reason why the authors focus on discrete-time is that, in continuous-time bilinear networks, there is no direct relationship between energy and Gramian-like metrics, and the controllability Gramian requires an integrability condition that imposes bounded actuation. In contrast with that study, we consider continuous-time linear networks, i.e., we do not address bilinear networks, since the network dynamics in our case does not depend on the input. To the best of our knowledge, the closest work to the one proposed in this paper is [23], [24], where the authors propose the notion of observability radius, which measures how much the entries of the dynamics can be perturbed such that the system becomes unobservable.

The rest of the paper is outlined as follows. In Section II, we formalize two problems related to different controllability metrics. In Section III, we provide a detailed description of the proposed methods to address them. In Section IV, we report our computational experiments and corresponding results. In Section V, we conclude an enumerate some possibilities of future work.

II. Problem Formulation

Consider the (possibly) large-scale network dynamics described by

$$\dot{x}(t) = A(\mathcal{G})x(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, and $u(t) \in \mathbb{R}^p$ is the input signal. The dynamics $A(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is induced by a directed *interdependency graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ given by a set of nodes $\mathcal{V} = \{1, \ldots, n\}$ and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, where $[A(\mathcal{G})]_{ij} \in \mathbb{R}$ if the edge $(j,i) \in \mathcal{E}$, and $[A(\mathcal{G})]_{ij} = 0$ if $(j,i) \in \mathcal{E}^c$. Also, the input matrix $B \in \mathbb{R}^{n \times m}$ is such that $[B]_{ik} \neq 0$ if the external input signal k is available to state i, and $[B]_{ik} = 0$ otherwise.

It is often the case that we want to steer the overall state of the network to a target state by designing an input control law $u(t)$ for $t \in [0, t_f]$, where $t_0 = 0$ and t_f are the initial and final times, respectively. Further, let (without loss of generality) $x_0 = 0$ be the initial state and x_f be the desired state at the final time. If any $x_f \in \mathbb{R}^n$ is attainable, then the system (1) is *controllable*, which we refer to as $(A(\mathcal{G}), B)$ being controllable. Furthermore, it is known that the minimum control energy [25] to steer the system to x_f incurs a total energy given by

$$\int_0^{t_f} \|u(\tau)\|^2 d\tau = x_f^\intercal \left[W_c^{t_f}(\mathcal{G}) \right]^{-1} x_f,$$

where $W_c^{t_f}(\mathcal{G})$ is the controllability Gramian, which can be computed as a function of $(A(\mathcal{G}), B)$. In order to assert controllability, we can rely on the Lyapunov test for controllability [25]. Then, the 'infinite' controllability Gramian $W_c^\infty \equiv W_c^\infty(\mathcal{G}) = \int_0^\infty e^{A(\mathcal{G})\tau} B B^\intercal e^{A(\mathcal{G})^\intercal \tau} d\tau$ is positive definite if and only if $(A(\mathcal{G}), B)$ is controllable. Furthermore, W_c^∞ can be used as a numerically stable sub-optimal approximation of the minimum energy control law, which is often implemented in practical scenarios. Specifically, W_c^∞ can be computed as the unique solution to the following Lyapunov equation:

$$A(\mathcal{G}) W_c^\infty + W_c^\infty A(\mathcal{G})^\intercal + B B^\intercal = 0,$$

i.e., as a function of $(A(\mathcal{G}), B)$. Consequently, due to implementation considerations [26], we will rely on the 'infinite' Gramian to assess the energy consumption by the controllers in the long-run, not considering explicitly the transient behavior [27]. Because W_c^∞ is positive definite when $(A(\mathcal{G}), B)$ is controllable, it can be described as $W_c^\infty = V \operatorname{diag}(\lambda_1, \ldots, \lambda_n) V^\intercal$, $V = [v_1 | \ldots | v_n]$, where $\{(\lambda_i, v_i)\}_{i=1}^n$ are eigenvalue-eigenvector pairs associated with W_c^∞. Subsequently, we refer to v_i as the i-th eigenvector, i.e., an eigenvector associated with eigenvalue λ_i. Furthermore, we assume that $0 < \lambda_1 \leq \ldots \leq \lambda_n$, where $\lambda_{\min} = \lambda_1$ and $\lambda_{\max} = \lambda_n$ are referred to as the minimum and maximum eigenvalues, respectively. Therefore, it follows that the total energy incurred by the minimum energy control in a specific final state $x_f = c v_i$ is $c^2 \lambda_i^{-1}$, where $c \in \mathbb{R}$.

In the worst case, the most energy-consuming states are those in the direction of v_1, i.e., the eigenvector associated with the minimum eigenvalue of $W_c^\infty(\mathcal{G})$. As a consequence, to mitigate the limitations imposed in this case, we propose a scenario where we re-design the corresponding dynamics, while satisfying the interdependency graph constraints. Subsequently, equation (1) becomes as follows:

$$\dot{x}(t) = [A(\mathcal{G}) + \Delta(\mathcal{G})] x(t) + B u(t), \tag{2}$$

where $[\Delta(\mathcal{G})]_{ij} \in [\iota_{ij}, \mu_{ij}] \subset \mathbb{R}$ for $(j,i) \in \mathcal{E}$, and $[\Delta(\mathcal{G})]_{ij} = 0$ otherwise. Simply speaking, we perform a finite additive structural perturbation on the dynamics to ensure desirable control properties measured by spectral properties of the 'infinite' controllability Gramian.

Considering (2), in this paper we are interested in the following two problems:

$\mathcal{P}_1$ (*Worst-case controllability*) Given the interdependency graph $\mathcal{G}$ and $(A(\mathcal{G}), B)$ controllable, find $\Delta(\mathcal{G})$, with $[\Delta(\mathcal{G})]_{ij} \in [\iota_{ij}, \mu_{ij}] \subset \mathbb{R}$ for $(j,i) \in \mathcal{E}$ and $[\Delta(\mathcal{G})]_{ij} = 0$ otherwise, such that $(A(\mathcal{G}) + \Delta(\mathcal{G}), B)$ is controllable and

$$\begin{aligned} \max_{\Delta(\mathcal{G}), W_c^\infty \in \mathbb{S}_+} \quad & \lambda_{\min}(W_c^\infty) \\ \text{s.t.} \quad & (A(\mathcal{G}) + \Delta(\mathcal{G})) W_c^\infty + W_c^\infty (A(\mathcal{G}) + \Delta(\mathcal{G}))^\intercal \\ & \qquad + B B^\intercal = 0. \end{aligned}$$

It is worth noticing that in the aforementioned problem, the objective does not impose constraints on the different controllability modes, i.e., the improvement of performance might be achieved at the expense of a change in the eigen-structure of the 'infinite' controllability Gramian. Subsequently, this might impact the performance of the network dynamics while heading for a specific state configuration. In other words, it might be desirable to improve the efficiency in the control towards a specific linear combination of states instead of others; thus, establishing a *controllability profile*. Therefore, the second problem that we address is as follows:

$\mathcal{P}_2$ (*Controllability Profile*) Given the interdependency graph $\mathcal{G}$ and $(A(\mathcal{G}), B)$ controllable, as well as a controllability profile $\{(\lambda_i, v_i)\}_{i=1}^n$ describing the eigenvalue-eigenvector pairs of a desirable positive definite 'infinite' controllability Gramian $W_c^\infty = \sum_{i=1}^n \lambda_i v_i v_i^\intercal$, find $\Delta(G)$, with $[\Delta(\mathcal{G})]_{ij} \in [\iota_{ij}, \mu_{ij}] \subset \mathbb{R}$ for $(j,i) \in \mathcal{E}$ and $[\Delta(\mathcal{G})]_{ij} = 0$ otherwise, such that

$$(A(\mathcal{G}) + \Delta(\mathcal{G})) W_c^\infty + W_c^\infty (A(\mathcal{G}) + \Delta(\mathcal{G}))^\intercal + B B^\intercal = 0.$$

III. Constrained Design of Network Dynamics for Control Enhancement

In this section, we address problems $\mathcal{P}_1$ and $\mathcal{P}_2$ by proposing computational methods for their solution. First, we notice that problem $\mathcal{P}_1$ presents a source of non-convexity due to the bilinear products occurring in the Lyapunov equation, which can be described as a bilinear matrix equality (BME). Unfortunately, this class of problems is, in general, NP-hard [28]. Nonetheless, we show that this particular equality is equivalently attained when a related rank optimization problem is solvable. Consequently, problem $\mathcal{P}_1$ can be addressed by sequentially solving associated convex optimization relaxations, as it is stated in Theorem 1. Next, we show that by constraining the control profile as in problem $\mathcal{P}_2$, the computation of its solution is simplified and can be solved as a linear matrix equality, which makes it suitable for large-scale design applications. However, it is worth noticing that, in defining such a fixed control profile, the space of feasible structural perturbations in $\mathcal{P}_2$ becomes comparatively more restricted than the one in $\mathcal{P}_1$, where the variables W_c^∞ and $\Delta(\mathcal{G})$ are jointly optimized.

First, we let $A \equiv A(\mathcal{G})$ and notice that the structural perturbation $\Delta \equiv \Delta(\mathcal{G})$ and the 'infinite' controllability Gramian $W_c^\infty \in \mathbb{S}_+^n$ are related by the Lyapunov equation

$$(A+\Delta)W_c^\infty + W_c^\infty(A+\Delta)^\intercal + BB^T = 0, \qquad (3)$$

which involves a sum of a bilinear terms in Δ and W_c^∞. In particular, let the matrices $M \in \mathbb{R}^{n\times 2n}$, $N \in \mathbb{R}^{2n\times n}$, and $Q \in \mathbb{R}^{n\times n}$ be such that

$$M \equiv M(\Delta, W_c^\infty) := \begin{bmatrix} A+\Delta & W_c^\infty \end{bmatrix},$$
$$N \equiv N(\Delta, W_c^\infty) := \begin{bmatrix} W_c^\infty & A+\Delta \end{bmatrix}^\intercal,$$

and $Q := -BB^\intercal$. Then, we have that (3) can be rewritten as the BME

$$MN = Q, \qquad (4)$$

which is also satisfied when

$$\text{rank}(Q - MN) = 0. \qquad (5)$$

Next, following a similar strategy to [29], we consider the structured matrix $Z \in \mathbb{R}^{3n\times 3n}$, defined as

$$Z = \begin{bmatrix} Q + XY + MY + XN & M + X \\ N + Y & I_{2n} \end{bmatrix}, \qquad (6)$$

which is parameterized by the matrices $X \in \mathbb{R}^{n\times 2n}$ and $Y \in \mathbb{R}^{2n\times n}$. This matrix will allow us to restate the constraint imposed by (5) as a rank minimization problem. In particular, we note that when $X = -M$ and $Y = -N$ we have

$$Z = \begin{bmatrix} Q - MN & 0 \\ 0 & I_{2n} \end{bmatrix}; \qquad (7)$$

hence, in this case $\text{rank}(Z) = 2n + \text{rank}(Q - MN)$. Therefore, the minimum value achieved by latter is attained when $\text{rank}(Q - MN) = 0$, which, in turn, implies (4). Furthermore, the matrix Z presents an affine dependency on Δ and W_c^∞ (through the matrices M and N), which we emphasize by noting that Z can be decomposed as

$$Z(\Delta, W_c^\infty; X, Y) = Z_0(X,Y) + Z_1(\Delta; X, Y) + Z_2(W_c^\infty; X, Y),$$

having X and Y as the parameters in (6).

Now, we notice that, by fixing X and Y, the rank minimization problem in $Z(\Delta, W_c^\infty; X, Y)$ can be further relaxed to a convex problem. Specifically, we consider the convex envelope of the rank function, which is given by the nuclear norm, denoted by $\|Z(\Delta, W_c^\infty; X, Y)\|_*$, and whose minimization can be formulated as an SDP [30]. This allows us to accomodate additional semidefinite and affine constraints arising from the objective and constraints appearing in problem $\mathcal{P}_1$. The affine constraints encapsulate the maximum allowed structural pertubation, whereas the objective in $\mathcal{P}_1$ can reformulated as an SDP constraint. More precisely, for $\mathcal{P}_1$ we have that

$$\max_{W_c^\infty \in \mathbb{S}_+^n} \ \lambda_{\min}(W_c^\infty) \quad \Leftrightarrow \quad \begin{array}{cc} \max\limits_{\delta\in\mathbb{R}, W_c^\infty \in \mathbb{S}_+^n} & \delta \\ \text{s.t.} & W_c^\infty - \delta I_n \succeq 0. \end{array}$$

Besides, a fixed $\delta = \bar{\lambda}$ is feasible for $\mathcal{P}_1$ if there exist Δ and W_c^∞ such that both (4) and the following set of constraints holds

$$W_c^\infty - \bar{\lambda} I_n \succeq 0 \qquad (8)$$
$$\iota_{ij} \le [\Delta]_{ij} \le \mu_{ij}, \qquad (j,i) \in \mathcal{E} \qquad (9)$$
$$[\Delta]_{ij} = 0, \qquad (j,i) \in \mathcal{E}^c. \qquad (10)$$

Subsequently, we propose to rely on solving consecutive convex programs such that we determine a sequence $\{\Delta^{(k)}, W_c^{\infty(k)}\}_k$ (or, equivalently, $\{M^{(k)}, N^{(k)}\}_k$) converging to a pair jointly satisfying constraints (4), (8), (9) and (10). Thus, we propose a sequence of convex optimization problems that are described as follows. Given a pair (X, Y) and a *target* $\bar{\lambda}$, find a solution to

$$\begin{array}{cc} \min\limits_{\Delta\in\mathbb{R}^{n\times n}, W_c^\infty \in \mathbb{S}_+^n} & \|Z(\Delta, W_c^\infty; X, Y)\|_* \\ \text{s.t.} & (8), (9), (10), \end{array} \qquad (\mathcal{C}_1(X, Y, \bar{\lambda}))$$

where the objective in $\mathcal{C}_1(X, Y, \bar{\lambda})$ seeks to enforce constraint (4). Also, following the above reasoning, the following result holds.

Theorem 1: The solution to $\mathcal{P}_1$ is given by the solution to $\mathcal{C}_1(X, Y, \bar{\lambda})$ for the maximum value $\bar{\lambda}$, as well as some X and Y, such that $MN = Q$. ⋄

Remark 1: Given that the 'infinite' controllability Gramian is positive definite, it follows that the eigenvalues are positive, real, and ordered. Therefore, we can iteratively increase the value $\bar{\lambda}$ until a maximum feasible is reached. ◦

In summary, we propose to solve feasibility problems associated with $\mathcal{P}_1$ for increasing values of $\bar{\lambda}$, by invoking the procedure described in Algorithm 1, consisting of solving consecutive convex relaxations $\mathcal{C}_1(X, Y, \bar{\lambda})$. Furthermore, we start by considering the initial points $X^{(1)} = -\begin{bmatrix} A & W_c^\infty \end{bmatrix}$, and $Y^{(1)} = -\begin{bmatrix} W_c^\infty & A \end{bmatrix}^\intercal$, corresponding to $\Delta = 0$, and W_c^∞ as a solution to $AW_c^\infty + W_c^\infty A^T = -BB^\intercal$. The numerical stopping condition is given by the relative residual of the bilinear inequality constraint, i.e., $\|M^{(k)}N^{(k)} - Q\|_* / \|Q\|_* < \epsilon \ll 1$.

Algorithm 1 Feasibility sequence for $\mathcal{P}_1$

given $X^{(1)}, Y^{(1)}, \bar{\lambda}$
1: **while** $\|M^{(k)}N^{(k)} - Q\|_* / \|Q\|_* > \epsilon$ **do**
2: solve $\mathcal{C}_1(X^{(k)}, Y^{(k)}, \bar{\lambda})$
3: let $X^{(k+1)} = -M^{(k)}, Y^{(k+1)} = -N^{(k)}$
4: **end while**

We now consider problem $\mathcal{P}_2$, which is distinct from $\mathcal{P}_1$ in that we also prescribe a specific control profile. More precisely, given a control profile described in terms of eigenvalue-eigenvector pairs $\{(\bar{\lambda}_i, \bar{v}_i)\}_{i=1}^n$, we can construct a desirable positive definite 'infinite' controllability Gramian $\bar{W}_c^\infty = \sum_{i=1}^n \bar{\lambda}_i \bar{v}_i \bar{v}_i^\intercal$. Therefore, we seek to find constrained perturbations $\Delta(G)$, with $[\Delta(\mathcal{G})]_{ij} \in [\iota_{ij}, \mu_{ij}] \subset \mathbb{R}$ for $(j,i) \in \mathcal{E}$ and $[\Delta(\mathcal{G})]_{ij} = 0$ such that Lyapunov equation (3) is satisfied.

To solve $\mathcal{P}_2$, because $\bar{W}_c^\infty$ is fixed, we can reformulate the Lyapunov equation (3) as a linear matrix equation. To do so, we first rewrite (3) as

$$-(A\bar{W}_c^\infty + \bar{W}_c^\infty A^\intercal + BB^\intercal) = \Delta \bar{W}_c^\infty + \bar{W}_c^\infty \Delta^\intercal.$$

Applying the vectorization operator on both sides, we obtain

$$\begin{aligned} b_0 &:= -\text{vec}(A\bar{W}_c^\infty + \bar{W}_c^\infty A^\intercal + BB^\intercal) \\ &= \text{vec}(\Delta \bar{W}_c^\infty + \bar{W}_c^\infty \Delta^\intercal), \end{aligned}$$

and, using the identity $\text{vec}(LYR) = (R^\intercal \otimes L)\text{vec}(Y)$, we obtain

$$\begin{aligned} b_0 =& (\bar{W}_c^\infty \otimes I_n)\text{vec}(\Delta) + (I_n \otimes \bar{W}_c^\infty)\text{vec}(\Delta^\intercal) \\ =& \left[(\bar{W}_c^\infty \otimes I_n) + (I_n \otimes \bar{W}_c^\infty)T_{n,n}\right] \text{vec}(\Delta) \\ =:& M_0 \text{vec}(\Delta), \end{aligned}$$

where $T_{n,n}$ is an $n^2 \times n^2$ sparse orthogonal permutation matrix (also known as the *vectorized transpose matrix* [31]). Consequently, the problem of finding perturbations Δ fulfilling a desired controllability profile can be stated as the following feasibilty problem

$$\begin{aligned} &\text{find} && \Delta \in \mathbb{R}^{n \times n} && && (\mathcal{C}_2(\bar{W}_c^\infty)) \\ &\text{s.t.} && M_0 \text{vec}(\Delta) = b_0 && && \\ & && \iota_{ij} \le [\Delta]_{ij} \le \mu_{ij}, && (j,i) \in \mathcal{E} && \\ & && [\Delta]_{ij} = 0, && (j,i) \in \mathcal{E}^c. && (11) \end{aligned}$$

It is worth noticing that, contrarily to the solution proposed for $\mathcal{P}_1$, the solution to $\mathcal{P}_2$ is cast as a linear matrix equation that can be attained by using efficient off-the-shelf algorithms. Also, we notice that the matrix $M_0 \in \mathbb{R}^{n^2 \times n^2}$ is not full rank, which arises from the possible low-rankness of W_c^∞, as well as from the restriction in the degress of freedom imposed by its symmetry. In case W_c^∞ has full rank n, the rank of M_0 is $\frac{n}{2}(n+1) \le n^2$, whereas, in the general case, when $\text{rank}(W_c^\infty) = k$, we have that $\text{rank}(M_0) = kn - \frac{k}{2}(k-1)$, which is obtained by counting degrees of freedom. Therefore, we observe that the solution Δ to $\mathcal{C}_2(\bar{W}_c^\infty)$ for a fixed $\bar{W}_c^\infty$ is not necessarily unique.

IV. Enhancing Controllability in Multi-agent Networks

We now illustrate the applicability of the proposed framework in the context of multi-agent networks. Specifically, we first describe a multi-agent network and its dynamics, as well as the configuration of leaders in the network. Then, we address problems $\mathcal{P}_1$ and $\mathcal{P}_2$ in the context of the proposed multi-agent network.

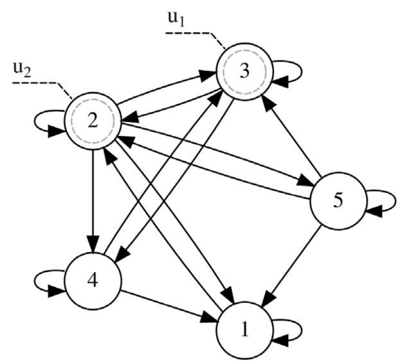

Fig. 1. Multi-agent network considered in problems $\mathcal{P}_1$ and $\mathcal{P}_2$, with agents 2 and 3 selected as leaders.

Multi-agent network

We consider a multi-agent network composed by $n = 5$ agents, whose communication capabilities are described by the interdependency graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which is depicted in Figure 1. Additionally, for each $(j,i) \in \mathcal{E}$, we generate a weight w_{ij} according to a standard uniform distribution $w_{ij} \sim [0,1]$, which we associate with the element $[\tilde{A}]_{ji}$ of an initial random system dynamic matrix $\tilde{A}(\mathcal{G}) \in \mathbb{R}^{n \times n}$. Correspondingly, for $(j,i) \in \mathcal{E}^c$, we set $[\tilde{A}]_{ji} = 0$. An asymptotically stable matrix $A(\mathcal{G}) \in \mathbb{R}^{n \times n}$ is then generated by taking $\tilde{A}$ and applying a shift in the real part of its eigenvalues $\xi_i(A), i = 1 \ldots, n$, $\Re(\xi_1(A)) \le \ldots \le \Re(\xi_n(A))$. More specifically, we let $\delta_n = \Re(\xi_n(\tilde{A}))$, and set $A = \tilde{A} - (\delta_0 + \delta_n)I_n$, such that $\Re(\xi_n(A)) = -\delta_0$. We choose $\delta_0 = 1.00 \times 10^{-3}$, which produces the following matrix

$$A = \begin{bmatrix} -1.393 & 0.559 & 0 & 0 & 0 \\ 0.732 & -0.781 & 0.581 & 0.071 & 0.374 \\ 0 & 0.034 & -0.987 & 0.658 & 0 \\ 0.575 & 0 & 0.976 & -1.393 & 0 \\ 0.442 & 0.778 & 0.569 & 0 & -1.372 \end{bmatrix}.$$

Besides, we consider agents 2 and 3 to be leaders, i.e., the input matrix can be described as follows

$$B = \left[\begin{smallmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{smallmatrix}\right]^\intercal.$$

Further, given the pair (A, B), the eigendecomposition of the Gramian matrix $W_c^\infty(A, B)$ (obtained as the solution of the Lyapunov equation (3)), whose elements are $\{(\lambda_i, v_i)\}_{i=1}^n$, are as follows

λ_1	λ_2	λ_3	λ_4	λ_5
$\mathbf{3.08 \times 10^{-5}}$	0.031	0.131	0.635	$\mathbf{633.666}$

v_1	v_2	v_3	v_4	v_5
0.798	0.442	−0.260	−0.146	0.282
−0.009	−0.150	0.637	−0.276	0.704
−0.001	0.322	0.250	0.893	0.192
0.307	−0.797	−0.320	0.323	0.251
−0.519	0.209	−0.601	−0.030	0.570

We note specifically the values $\lambda_1 = 3.08 \times 10^{-5}$ and $\lambda_n = 633.666$.

Next, we find the structural perturbation $\Delta(\mathcal{G})$ required to attain desirable goals to enhance the controllability of the initial multi-agent network $(A(\mathcal{G}), B)$, as captured by problems $\mathcal{P}_1$ and $\mathcal{P}_2$.

Enhancement of worst-case controllability

Given the spectral characteristics of the Gramian matrix associated with the initial multi-agent network (A, B), we would like to find a constrained structural perturbation $\Delta_{\mathcal{P}_1}(\mathcal{G})$ such that the minimum eigenvalue of the Gramiam matrix associated with perturbed system $A(\mathcal{G}) + \Delta_{\mathcal{P}_1}(\mathcal{G})$ satisfies a minimum target value $\bar{\lambda} = 0.100$. Recall that the sparsity of the structural perturbation $\Delta_{\mathcal{P}_1}$ is restricted to the interdependency graph $\mathcal{G}$, i.e. $[\Delta_{\mathcal{P}_1}]_{ij} = 0$, $(j, i) \in \mathcal{E}^c$. Additionally, the upper and lower limits for the structural perturbation are uniformly set to $\iota_{ij} = -1.00$ and $\mu_{ij} = 1.00$, for $i, j = 1, \ldots, n$.

To determine the structural perturbation necessary to attain the controllability target value for $\bar{\lambda}$, we resort to Algorithm 1, associated with a sequence of convex problems $\mathcal{C}_1$. The stopping criterion is met when the relative residual of the bilinear equality $\|M^{(k)}N^{(k)} - Q\|_*/\|Q\|_* < \epsilon$ with $\epsilon = 1.00 \times 10^{-6}$.

In Figure 2, we display the result of the execution of Algorithm 1 in terms of the relative residual of the bilinear equality, along with the sequence of values obtained for $\log_{10}(\lambda_1^{(k)}/\bar{\lambda})$, where $\lambda_1^{(k)}$ is the minimum eigenvalue of $W_c^\infty(A + \Delta_{\mathcal{P}_1}(\mathcal{G}), B)$ at each iteration k of Algorithm 1. The resulting structural pertubation $\Delta_{\mathcal{P}_1}$ is as follows

$$\Delta_{\mathcal{P}_1} = \begin{bmatrix} \mathbf{1.00} & -0.400 & 0 & 0 & 0 \\ \mathbf{-1.00} & 0.435 & 0.172 & 0.165 & -0.174 \\ 0 & 0 & 0.301 & -0.184 & 0 \\ -0.447 & 0 & 0.040 & 0.469 & 0 \\ \mathbf{-1.00} & 0 & -0.544 & 0 & 0.675 \end{bmatrix}.$$

We emphasize the fullfilment of the sparsity constraints, as well as the active contraints $[\Delta_{\mathcal{P}_1}]_{ij} = \iota_{ij}$ and $[\Delta_{\mathcal{P}_1}]_{ij} = \mu_{ij}$ highlighted in bold. Lastly, the resulting eigendecomposition of W_c^∞, calculated from $(A + \Delta_{\mathcal{P}_1}, B)$ is as follows

λ_1	λ_2	λ_3	λ_4	λ_5
0.100	0.100	0.308	1.191	633.666

v_1	v_2	v_3	v_4	v_5
−0.427	−0.475	0.660	0.278	0.282
−0.191	−0.276	−0.626	0.006	0.704
0.407	−0.365	0.245	−0.777	0.192
−0.554	0.619	0.175	−0.464	0.251
0.555	0.426	0.286	0.322	0.570

where it can be seen that the target value $\lambda_1 = \bar{\lambda} = 0.100$ has been reached. Besides, we remark that the minimum eigenvalue increased by a factor of 3.24×10^3. Finally, as mentioned in Remark 1, one can iteratively increase the value $\bar{\lambda}$ to obtain the solution to $\mathcal{P}_1$.

Controllability profile design

We now seek to find constrained perturbations $\Delta_{\mathcal{P}_2}$ achieving three different predefined controllability profiles associated with the previous subsections, which we index by (a), (b) and (c).

Controllability profile (a), described by $\{(\bar{\lambda}_i, \bar{v}_i)\}_{i=1}^n$, is defined in terms of the eigenvalues and eigenvectors $\{(\underline{\lambda}_i, \underline{v}_i)\}_{i=1}^n$ of the Gramian of the initial multi-agent network $W_c^\infty(A, B)$, along with a minumum target

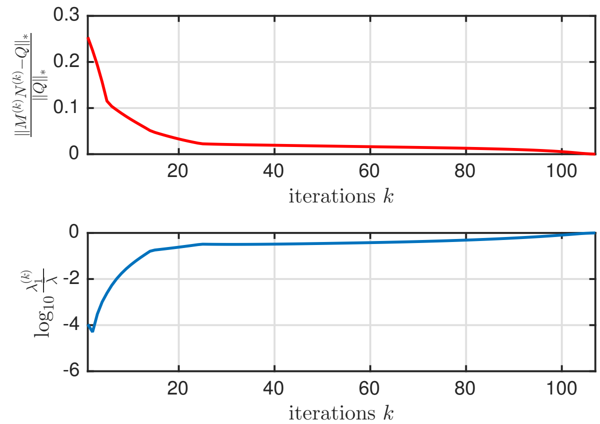

Fig. 2. Sequence of values of the relative residual of the bilinear equality constraint $\|M^{(k)}N^{(k)} - Q\|_*/\|Q\|_*$ (top) and and $\log_{10}(\lambda_1^{(k)}/\bar{\lambda})$ (bottom), obtained from Algorithm 1 for problem $\mathcal{P}_1$.

eigenvalue λ_0. More precisely, the target profile (a) is defined as follows: we set $\bar{\lambda}_i = \underline{\lambda}_i$ if $\underline{\lambda}_i \geq \lambda_0$, and $\bar{\lambda}_i = \lambda_0$ otherwise. Correspondingly, we set all target eigenvectors equal to those from the initial multi-agent network, i.e., $\bar{v}_i = \underline{v}_i$, $i = 1, \ldots, n$. Given this profile $\{(\bar{\lambda}_i, \bar{v}_i)\}_{i=1}^n$, we solve problem $\mathcal{C}_2(\bar{W}_c^\infty)$ to find a structural perturbartion $\Delta_{\mathcal{P}_{2a}}$. In doing so, we observe that when the upper and lower bounds on the structural pertubations are enforced (i.e., $\iota_{ij} = -1.00$ and $\mu_{ij} = 1.00$), problem $\mathcal{P}_{2a}$ results infeasible. As a verification, if these constraints are relaxed, i.e., we let $\iota_{ij} = -10.00$ and $\mu_{ij} = 10.00$, then the relaxed problem $\hat{\mathcal{P}}_{2a}$ becomes feasible, producing the structural perturbarions $\Delta_{\hat{\mathcal{P}}_{2a}}$ as follows

$$\Delta_{\hat{\mathcal{P}}_{2a}} = \begin{bmatrix} 0.807 & -0.324 & 0 & 0 & 0 \\ -1.053 & -0.192 & -0.538 & 0.788 & 0.593 \\ 0 & -0.117 & -0.155 & 0.446 & 0 \\ -0.678 & 0 & -0.764 & 1.346 & 0 \\ -1.629 & -0.105 & -0.242 & 0 & 1.018 \end{bmatrix}.$$

Next, in profile (b), we keep the same set of target eigenvalues as considered in profile (a), but generate a random set of orthogonal eigenvectors. The target eigenvectors $\{\bar{v}_i\}_{i=1}^n$ are obtained as the columns of an orthogonal matrix $Q = [\bar{v}_1 | \ldots | \bar{v}_n]$ obtained from performing a QR decomposition on a random matrix $X \in \mathbb{R}^{n \times n}$, where each entry of X is drawn from a standard uniform distribution, i.e., $[X]_{ij} \sim [0, 1]$. As a result, similarly to what was obtained for profile (a), when the upper and lower bounds on the structural pertubations are enforced (i.e., $\iota_{ij} = -1.00$ and $\mu_{ij} = 1.00$), we observed that $\mathcal{P}_{2b}$ resulted infeasible. In addition, when these constraints are relaxed, i.e, we let $\iota_{ij} = -10.00$ and $\mu_{ij} = 10.00$, the relaxed problem $\hat{\mathcal{P}}_{2b}$ becomes feasible, producing the structural perturbarions $\Delta_{\hat{\mathcal{P}}_{2b}}$ as follows

$$\Delta_{\hat{\mathcal{P}}_{2b}} = \begin{bmatrix} 1.937 & -0.392 & 0 & 0 & 0 \\ -2.549 & -3.271 & -0.780 & -1.285 & 1.116 \\ 0 & -0.499 & -2.475 & 0.742 & 0 \\ 1.152 & 0 & -1.912 & 1.424 & 0 \\ -3.251 & -1.519 & -1.150 & 0 & 1.070 \end{bmatrix}.$$

Finally, in profile (c), we feed the controllability profile resulting from the solution of $\mathcal{P}_1$, i.e., $\{(\lambda_i, v_i)\}_{i=1}^n$ from $W_c^\infty(A + \Delta_{\mathcal{P}_1})$. As expected, solving $\mathcal{C}_2(\bar{W}_c^\infty)$ produces a feasible structural perturbation $\Delta_{\mathcal{P}_{2c}}$, as given by

$$\Delta_{\hat{\mathcal{P}}_{2c}} = \begin{bmatrix} \mathbf{0.99} & -0.395 & 0 & 0 & 0 \\ -0.684 & 0.713 & 0 & 0.332 & -0.694 \\ 0 & 0.048 & 0.469 & -0.495 & 0 \\ -0.411 & 0 & 0.188 & 0.316 & 0 \\ -0.956 & 0.236 & -0.630 & 0 & 0.396 \end{bmatrix}.$$

It is worth mentioning that the obtained solution $\Delta_{\mathcal{P}_{2c}}$ differs from the structural perturbation $\Delta_{\mathcal{P}_1}$, which is expected due to the nonuniqueness of solutions.

V. Conclusions and Further Research

In this paper, we addressed the constrained design of continuous-time linear system dynamics to improve their control performance, measured as a function of the associated controllability Gramian. Specifically, we considered spectral properties of the 'infinite' controllability Gramian, and showed that worst-case performance problem can be cast as an optimization problem with bilinear matrix equality constraints. In contrast, the problem of achieving a specific control profile can be formulated as a linear matrix equality. Because the former problem is computationally intractable, we proposed to address it by resorting to a sequence of convex programs encoding parameterized feasibility problems. We then validated our approach in the context of multi-agent networks. Future research will focus on posing and addressing similar constrained design problems in the discrete-time domain, and on exploring distributed implementation of these algorithms in the context of decentralized networked systems.

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