Optimal Resource Allocation for Containing Epidemics on Time-Varying Networks

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***Abstract*— This paper studies the Susceptible-Infected-Susceptible (SIS) epidemic model on time-varying interaction graphs in contrast to the majority of other works which only consider static graphs. After presenting the mean-field model and characterizing its stability properties, we formulate and solve an optimal resource allocation problem. More specifically, we first assume that a cost can be paid to reduce the amount of interactions certain nodes can have with others (e.g., by imposing travel restrictions between certain cities). Then, given a budget, we are interested in optimally allocating the budget to best combat the undesired epidemic. We show how this problem can be equivalently formulated as a geometric program and solved in polynomial time. Simulations illustrate our results.**

I. INTRODUCTION

The study of spreading processes on complex networks has recently gained a massive surge of interest. With the wide range of applications including the spreading of a computer virus, how a product is adopted by a marketplace, or how an idea or belief is propagated through a social network, it is no surprise that a plethora of different models and studies have been devoted to this. However, an overwhelming majority of the stochastic models considered assume a fixed interaction topology. Unfortunately, this may not be a fair assumption depending on the time-scale of a spreading process. For instance, in the context of diseases, the network of contacts in a human population is constantly changing. Hence, a time-varying network model might be more appropriate, albeit more challenging to analyze. We are only aware of a few works analyzing these types of time-varying models, which seems to be a promising new branch of epidemics research.

Literature review

One of the oldest and most commonly studied spreading models is the Susceptible-Infected-Susceptible (SIS) model [1]. Early works such as the one above often consider simplistic assumptions such as all individuals in a population being equally likely to interact with everyone else in the population [2]. One of the first works to consider a continuous-time SIS model over arbitrary graphs using mean field theory is [3], which provides conditions on when the disease-free state of the system is globally asymptotically stable.

In addition to the simple SIS model, a myriad of different models have also been proposed and studied in the literature; see [4] for a recent survey. In [5], [6], the authors add various states to model how humans might adapt their behavior when given knowledge about the possibility of an emerging epidemic. The work [7] considers the possible effect of human behavior changes for the three state Susceptible-Alert-Infected-Susceptible (SAIS) model. In [8], a four-state generalized Susceptible-Exposed-Infected-Vigilant (G-SEIV) model is proposed and studied.

However, a large drawback is that *all* of the works above consider a fixed known topology. In the context of epidemics, this may be a very crude assumption since the people a given person interacts with is time-varying in the real world [9]. In fact, we can find several works mentioning the nontrivial effect that the dynamics of the network has on the behavior of spreading processes. For example, the authors in [10] observe from extensive simulations that the spreading speed of epidemic processes over a time-varying network can be significantly slower than in its aggregated network. We also find recent studies pointing out the key role played by the addition and removal of links [11], as well as the distribution of contact durations between nodes [12] in spreading dynamics.

Apart from these empirical studies, we also find several theoretical results about spreading processes over time-varying networks. The authors in [13] derived the value of the epidemic threshold in a type of time-varying networks, where randomly chosen two pairs of nodes swap their neighbors at random time instants. A wide and flexible class of TV network model, called *edge-Markovian graphs*, was proposed in [14] and analyzed in [15]. In this model, edges appear and disappear independently of each other according to Markov processes. Taylor et al. derived in [16] the value of the epidemic threshold in edge-Markovian graphs and proposed control strategies to contain an epidemic outbreak, assuming homogeneous spreading and recovery rates. Finally, epidemic processes over time-varying networks whose topology adaptively changes by responding to the prevalence of epidemics are studied in [17], [18].

Statement of contributions

In this paper we have presented an SIS epidemic model for a class of time-varying networks, contrary to the overwhelming majority of works that study only a static network. After presenting the model, a sufficient condition for almost sure global asymptotic stability of the disease-free equilibrium is presented. Leveraging this result, we propose an optimal resource allocation problem in which a cost can be paid to reduce the stochastic interactions between nodes (e.g., enforcing travel restrictions between certain cities). More formally, given a fixed budget, we are interested in maximizing the chance of eradicating the epidemic. We show how this problem can be equivalently formulated as a geometric

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program which can be solved efficiently using off-the-shelf solvers. Simulations illustrate our results.

A. Notation

For a positive integer n, define the set $[n] = \{1, \ldots, n\}$. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted by $\|x\|$. Let I_n denote the $n \times n$ identity matrix. A square matrix is said to be *Metzler* if its off-diagonal entries are nonnegative. The *spectral abscissa* of a square matrix A, denoted by $\eta(A)$, is defined as the maximum real part of its eigenvalues. We say that A is *Hurwitz stable* if $\eta(A) < 0$. Also, we define the *matrix measure* [19] of A by $\mu(A) = \eta(A + A^\top)/2$. For a square random matrix X, its expectation is denoted by $E[X]$ and its variance is defined by $\mathrm{Var}(X) = E[(X - E[X])^2]$. The diagonal matrix with diagonal elements $a_1, \ldots, a_n$ is denoted by $\mathrm{diag}(a_1, \ldots, a_n)$.

For a real random variable X, define

$$\overline{\mathrm{Var}}(X) = (\operatorname{ess\,sup} X - E[X])(E[X] - \operatorname{ess\,inf} X). \tag{1}$$

It is known that [20]

$$\mathrm{Var}(X) \leq \overline{\mathrm{Var}}(X) \tag{2}$$

II. Problem Statement

In this section, we state the problems studied in this paper. Consider a network of nodes $1, \ldots, n$ connected over a time-varying graph $\mathcal{G}(t)$ with the weighted adjacency matrix $W(t) = [w_{ij}(t)]_{i,j}$. We study the SIS disease spread over this time-varying network. Using mean-field approximation, we model the evolution of the infection probability over the time-varying graph by the differential equations

$$\dot{p}_i(t) = (1 - p_i(t))\beta_i \sum_{j=1}^{n} w_{ij}(t) p_j(t) - \delta_i p_i(t),$$

for $i = 1, \ldots, n$, where $p_i(t)$ is the probability that node i is infected at time t, β_i is the infection rate to i from a neighbor of i, and δ_i is the recovery rate of node i. By comparison principle [21], we can focus on the linearized equation

$$\dot{p}_i(t) = \sum_{j=1}^{n} w_{ij}(t) p_j(t) - \delta_i p_i(t). \tag{3}$$

Let $D = \mathrm{diag}(\delta_1, \ldots, \delta_n)$. Then, equations (3) yield

$$\Sigma : \dot{p} = (W(t) - D)p.$$

We model the weighted time-varying graph $\mathcal{G}(t)$ through its adjacency matrix $W(t)$ as follows. Let $i, j \in \{1, \ldots, n\}$ satisfy $i < j$. Let $\underline{w}_{ij}$ and $\bar{w}_{ij}$ be positive numbers such that $\underline{w}_{ij} \leq \bar{w}_{ij}$ and also let N_i be a positive integer. Define $\Delta w_{ij} = \bar{w}_{ij} - \underline{w}_{ij}$ and $h_i = \Delta w_{ij}/N_{ij}$. Then we define w_{ij} as the Markov process with the state space $\{\underline{w}_{ij}, \underline{w}_{ij} + h_{ij}, \ldots, \underline{w}_{ij} + (k_{ij} - 1)h_{ij}, \bar{w}_{ij}\}$ and the transition probability

$$\begin{aligned} &P(w_{ij}(t + \Delta t) = \underline{w}_{ij} + \ell h_{ij} \mid w_{ij}(t) = \underline{w}_{ij} + k h_{ij}) \\ &\quad = \begin{cases} p_{ij}\Delta t + o(\Delta t) & \text{if } \ell = k+1 \text{ or } \ell = k = N_{ij}, \\ q_{ij}\Delta t + o(\Delta t) & \text{if } \ell = k-1 \text{ or } \ell = k = 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where p_{ij} and q_{ij} are nonnegative numbers.

Roughly speaking, p_{ij} gives the rate at which the weight w_{ij} increases by h_{ij}, and q_{ij} the rate at which the rate decreases by h_{ij}. The positive number N_{ij} represents the "resolution" of the weight w_{ij}. We assume that the stochastic processes $\{w_{ij}\}_{i<j}$ are independent of each other. Also, since $\mathcal{G}(t)$ is assumed to be undirected at each time t, the above definition defines $W(t)$ by extending $w_{ji} = w_{ij}$ for all (i, j) such that $i < j$. The above defined model generalizes the edge-Markovian graph [22], where it is assumed that $N_{ij} = 1$, $\underline{w}_{ij} = 0$, $\bar{w}_{ij} = 1$, $p_{ij} = p$, and $q_{ij} = q$ for some constants $p_{ij}, q_{ij} > 0$.

We are concerned with the stability of the infection-free equilibrium $p \equiv 0$ of Σ, defined as follows:

Definition 2.1: We say that Σ is *almost surely stable* if

$$P\left(\lim_{t\to\infty} \|p(t)\| = 0\right) = 1,$$

for all initial states $p(0) = p_0$ and $w_{ij}(0) = w_{ij,0}$.

Now we can state the problems studied in this paper. The first problem is the stability analysis:

Problem 2.2: Determine if Σ is almost surely stable or not.

The other problem is the optimal resource allocation problem. We assume that the rates p_{ij} and q_{ij} can be designed with an accompanying cost $c_{ij}(p_{ij}, q_{ij})$. The total cost for realizing the rates (p_{ij}, q_{ij}) over the whole network is then given by

$$C = \sum_{i<j} c_{ij}(p_{ij}, q_{ij}).$$

Now we can state the following design problem:

Problem 2.3: Given a maximum budget $\bar{C}$, find the rates $\{p_{ij}\}_{i<j}$ and $\{q_{ij}\}_{i<j}$ such that the total cost C is less than or equal to $\bar{C}$ while Σ is almost surely stable.

III. Stability Analysis

We start with recalling the notion of Markov jump linear systems [23]. Let $A(t)$ be an $\mathbb{R}^{n\times n}$-valued Markov process with a finite state space $\{A_1, \ldots, A_N\}$. Then, we say that the stochastic differential equation

$$\dot{x} = A(t)x, \tag{4}$$

where $x(0) \in \mathbb{R}^n$ and $A(0)$ are arbitrary constants, is a *Markov jump linear system* [23]. We say that the Markov jump linear system (4) is *almost surely stable* if $P(\lim_{t\to\infty} \|x(t)\| = 0) = 1$ for all $x(0)$ and $A(0)$. We quote the following sufficient condition for the almost sure stability of Markov jump linear systems:

Lemma 3.1 ([24, Theorem 4.2]): Assume that $A(t)$ is symmetric for each $t \geq 0$ with probability one and the Markov process A is irreducible. Let A_π denote the random matrix following the stationary distribution of A. If $E[\mu(A_\pi)] < 0$, then the Markov jump linear system (4) is almost surely stable.

We also need to recall the following estimate of the maximum eigenvalue of the sum of random matrices:

Proposition 3.2 ([22], [25]): Let $X_1, \ldots, X_N$ be independent random $n \times n$ symmetric matrices. Let C and v be nonnegative constants such that $\|X_k - E[X_k]\| \leq C$ for every

$k \in [N]$ with probability one and $\|\sum_{k=1}^{N} \mathrm{Var}(X_k)\| \leq v^2$. Then the sum $X = \sum_{k=1}^{N} X_k$ satisfies

$$P(\eta(X) > \eta(E[X]) + s) \leq \kappa_{C,v^2}(s) \tag{5}$$

for every $s \geq 0$, where $\kappa_{C,v^2}(s)$ is defined by (6).

We present stability analysis in this section. For two positive constants C and v, define the decreasing function κ_{C,v^2} by

$$\kappa_{C,v^2}(s) = ne^{s/C}\left(\frac{Cs+v^2}{v^2}\right)^{-\frac{Cs+v^2}{C^2}}, \ s > 0 \tag{6}$$

where n is the number of the nodes in the network.

The following theorem, the first main result of this paper, provides a condition for Σ to be almost surely stable:

Let π_{ij} denote the probability distribution on the set $\{\underline{w}_{ij}, \underline{w}_{ij}+h_{ij}, \dots, \bar{w}_{ij}\}$ given by

$$\pi_{ij}(\underline{w}_{ij} + mh_{ij}) = \frac{\rho_{ij}^m}{1+\rho_{ij}+\cdots+N_{ij}\rho_{ij}}. \tag{7}$$

We assume that all the probability distributions π_{ij} $(i < j)$ are independent.

Theorem 3.3: Assume that w_{ij} is irreducible for all i and j. Define $\mathcal{W} \in \mathbb{R}^{n\times n}$ by

$$\mathcal{W}_{ij} = \underline{w}_{ij} + h_{ij}\frac{\rho_{ij}+\cdots+N_{ij}\rho_{ij}^{N_{ij}}}{1+\rho_{ij}+\cdots+\rho_{ij}^{N_{ij}}}, \ 1 \leq m \leq M_{ij} \tag{8}$$

for all i and j. Define

$$\begin{aligned} C &= \max_{i<j}(\Delta w_{ij}), \\ v^2 &= \max_{1\leq i\leq n}\Bigg(\sum_{j<i}(\bar{w}_{ij} - E[\pi_{ij}])(E[\pi_{ij}] - \underline{w}_{ij}) + \\ &\qquad \sum_{j>i}(\bar{w}_{ij} - E[\pi_{ij}])(E[\pi_{ij}] - \underline{w}_{ij})\Bigg). \end{aligned} \tag{9}$$

Then, the disease-free equilibrium of Σ is almost surely stable if there exists $s > 0$ such that

$$\lambda_{\max}(\mathcal{W} - D) + s + \lambda_{\max}(W_{\max} - D)\kappa_{C,v^2}(s) < 0. \tag{10}$$

Proof: Since all the processes w_{ij} are assumed to be irreducible, the Markov process $\{W(t)\}_{t\geq 0}$ is irreducible. Therefore $\{W(t) - D\}_{t\geq 0}$ is also irreducible and therefore has a unique stationary distribution. Let M be the random matrix following the stationary distribution. Then, by Lemma 3.1, it is sufficient to show $E[\mu(M)] < 0$ under the assumptions stated in the theorem. Since $W(t) - D$ is similar to a symmetric matrix, so is M with probability one. Therefore we have $E[\mu(M)] = E[\lambda_{\max}(M)]$. Hence, we need to show $E[\lambda_{\max}(M)] < 0$.

By the definition of W, we can decompose the random matrix $W(t)$ into the sum of random variables as $W(t) = \sum_{i<j} w_{ij}(t)E_{ij}$. Therefore $W(t) - D = -D + \sum_{i<j} w_{ij}(t)E_{ij}$. It is immediate to see that Markov process $\{w_{ij}(t)\}_{t\geq 0}$ has the stationary distribution π_{ij} defined in (7). Therefore, M admits the expression

$$M = -D + \sum_{i<j}\pi_{ij}E_{ij} = -D + \sum_{i<j} X_{ij},$$

where $X_{ij} = \pi_{ij}E_{ij}$. We use this expression of M and Proposition 3.2 to evaluate $E[\lambda_{\max}(M)]$. Notice that

$$E[M] = \mathcal{W}. \tag{11}$$

Let C and v^2 be the constants defined as in (9). We temporarily assume that these constants satisfy

$$\|X_{ij} - E[X_{ij}]\| \leq C, \tag{12}$$

$$\|\sum_{i<j}\mathrm{Var}(X_{ij})\| \leq v^2, \tag{13}$$

which will be proved later. Let $(\Omega, \mathcal{M}, P)$ be the fundamental probability space, and define $\Omega_s = \{\omega \in \Omega : E[\lambda_{\max}(M)] > \lambda_{\max}(E[M]) + s\}$. Proposition 3.2 shows that $P(\Omega_s) < \kappa_{C,v^2}(s)$. Moreover, if $\omega \in \Omega_s$, we have

$$\lambda_{\max}(M) \leq \lambda_{\max}(BW_{\max} - D)$$

with probability one since $M \leq W_{\max} - D$ entrywise and also $\lambda_{\max}$ is monotonic over the set of Metzler matrices with respect to the entrywise ordering. Therefore, for the $s > 0$ satisfying (10), we have

$$\begin{aligned} E[\lambda_{\max}(M)] &\leq (\lambda_{\max}(E[M]) + s)P(\Omega\backslash\Omega_s) \\ &\qquad + \lambda_{\max}(W_{\max} - D)P(\Omega_s) \\ &\leq \lambda_{\max}(E[M]) + s + \lambda_{\max}(W_{\max} - D)\kappa(s) \\ &< 0, \end{aligned}$$

as desired.

Let us prove (12) and (13) in order to complete the proof. The first inequality (12) is obvious. Let us prove (13). A straightforward calculation shows that

$$\mathrm{Var}(X_{ij}) = \mathrm{Var}(\pi_{ij})(E_{ii} + E_{jj}).$$

Therefore,

$$\sum_{i<j}\mathrm{Var}(X_{ij}) = \sum_{i=1}^{n}\left(\sum_{j>i}\mathrm{Var}(\pi_{ij}) + \sum_{j<i}\mathrm{Var}(\pi_{ij})\right)E_{ii}$$

and hence

$$\begin{aligned} \left\|\sum_{i<j}\mathrm{Var}(X_{ij})\right\| &= \max_{1\leq i\leq n}\left(\sum_{j>i}\mathrm{Var}(\pi_{ij}) + \sum_{j<i}\mathrm{Var}(\pi_{ij})\right) \\ &\leq \max_{1\leq i\leq n}\left(\sum_{j>i}\overline{\mathrm{Var}}(\pi_{ij}) + \sum_{j<i}\overline{\mathrm{Var}}(\pi_{ij})\right) \\ &= v^2. \end{aligned}$$

This completes the proof of the theorem. ∎

Remark 3.4: An argument similar to the proof of Theorem 3.3 is used in [22] for proving a sufficient condition for stability of spreading processes on a time-varying network called aggregated-Markovian arc independent graphs, which include edge-Markovian graphs as their special case. The authors in [22] also confirm that the conservativeness arising from the argument is relatively small using numerical simulations.

Directed time-varying network

In the previous subsection, we have considered epidemic processes over a time-varying network whose topology is undirected. We can naturally extend the definition of the time-varying network to the directed case, as done in [22]. Below, we present a corresponding stability condition analogue to Theorem 3.3. We omit the proof of the condition as it is similar to the proof of [22, Theorem 3.4].

Theorem 3.5: Assume that w_{ij} is irreducible and define $\mathcal{W} \in \mathbb{R}^{n\times n}$ by (8) for all i and j. Define

$$C = \max_{i<j}(\Delta w_{ij}),$$
$$v^2 = \max_{1\le i\le n}\left(\sum_{j\neq i}(\bar{w}_{ij} - E[\pi_{ij}])(E[\pi_{ij}] - \underline{w}_{ij}) + \sum_{j\neq i}(\bar{w}_{ij} - E[\pi_{ij}])(E[\pi_{ij}] - \underline{w}_{ij})\right).$$

Then, the disease-free equilibrium of Σ is almost surely stable if there exists $s > 0$ such that

$$\mu(\mathcal{W} - D) + s + \mu(W_{\max} - D)\kappa_{C,v^2}(s) < 0,$$

where $\mu(\cdot)$ denotes the matrix measure [19] of a square matrix defined by $\mu(A) = \lambda_{\max}(A + A^\top)/2$.

IV. Optimal Resource Allocation

In this section we present the resource allocation we are interested in solving. Building upon the SIS epidemic model on a time-varying network $W(t)$, we now consider the case where $\bar{w}_{ij}$ are control parameters that can be chosen. For example, one might be able to decrease $\bar{w}_{ij}$ by limiting interactions between nodes i and j (e.g., limiting the amount of traffic between different cities).

Formally, we let $\underline{w}_{ij}$ and $\bar{w}_{ij}$ be the natural bounds on the edge weight $w_{ij}(t)$. Instead, we consider new bounds $\underline{W}_{ij} = \underline{w}_{ij}$ and $\bar{W}_{ij} \in [\underline{w}_{ij}, \bar{w}_{ij}]$, where the new upper limits $\bar{W}_{ij}$ are control parameters. The cost of setting the new upper bounds is given by

$$C = \sum_{i<j} c_{ij}(\bar{W}_{ij}),$$

where $c_{ij}(\underline{w}_{ij}) = 0$ and $c_{ij}(\cdot)$ is nondecreasing for all $i, j \in \{1,\dots,n\}$. This means that there is no cost incurred when setting $\bar{W}_{ij} = \bar{w}_{ij}$ to the natural upper bound, and the cost to reduce the bound $\bar{W}_{ij}$ is nondecreasing.

We are now interested in optimization problems to allocate resources leveraging the results of Section III. We formalize the optimal budget allocation problem next.

Problem 4.1 (Optimal budget allocation): Given a fixed budget $B > 0$, find the optimal allocation that minimizes $\lambda_{\max}(\mathcal{W} - D)$. This can mathematically formulated mathematically by:

$$\begin{array}{ll} \min\limits_{\bar{W}} & \lambda_{\max}(\mathcal{W} - D) \\ \text{s.t.} & \sum_{i,j=1}^n c_{ij}(\bar{W}_{ij}) \le B, \\ & \underline{w}_{ij} \le \bar{W}_{ij} \le \bar{w}_{ij}, \end{array}$$

for all $i, j \in \{1,\dots,n\}$.

Theorem 4.2: Problem 4.1 can be solved exactly by the auxiliary geometric program:

$$\begin{array}{ll} \min\limits_{\bar{W},u,\lambda} & \lambda \\ \text{s.t.} & \frac{\sum_{j=1}^n(\bar{W}_{ij}-\delta)u_j - \delta u_i}{(\phi+\lambda)u_i} \le 1, \\ & \sum_{i,j=1}^n c_{ij}(\bar{W}_{ij}) \le B, \\ & \underline{w}_{ij} \le \bar{W}_{ij} \le \bar{w}_{ij}, \end{array}$$

where $\phi > \delta$ and $\lambda_{\max}(\mathcal{W} - D) = \lambda^* - \phi$.

Proof:

In our derivations, it will be useful to resort to the following result from [26, Chapter 4]. For a review of geometric programming in general, see [26], [27].

Proposition 4.3: Consider an $N \times N$ nonnegative, irreducible matrix $M(\mathbf{x})$ with entries being either 0 or posynomials with domain $\mathbf{x} \in \mathcal{S}$ where $\mathcal{S} = \cap_{i=1}^m \{\mathbf{x} \in \mathbb{R}^k_{>0} \mid f_i(\mathbf{x}) \le 1\}$ for some posynomials f_i. Then, minimizing the largest real part of the eigenvalues of $M(\mathbf{x})$, denoted by $\lambda_1(M(\mathbf{x}))$, over $\mathbf{x} \in \mathcal{S}$ is equivalent to solving the following GP:

$$\begin{array}{lll} \underset{\lambda,\{u_i\}_{i=1}^N,\mathbf{x}}{\text{minimize}} & \lambda & \\ \text{such that} & \frac{\sum_{j=1}^N M_{ij}(\mathbf{x})u_j}{\lambda u_i} \le 1, & i \in \{1,\dots,N\}, \\ & f_i(\mathbf{x}) \le 1, & i \in \{1,\dots,m\}. \end{array} \tag{14}$$

Leveraging this result, it is easy to see in this case that the auxiliary program is simply optimizing over the eigenvalues of $(\mathcal{W} - D + \mathbf{I}\phi)$ for which it is clear that $\lambda(\mathcal{W} - D + \mathbf{I}\phi) = \lambda(\mathcal{W} - D) + \phi$. ■

We finally remark that, using Theorem 3.5, we can easily extend the framework presented in this section to the case of directed networks. The details are omitted.

V. Numerical Simulations

Here we demonstrate the correctness of the algorithm on a 20 node time-varying directed contact graph $\mathcal{W}$. For simplicity, we initially set $\delta_i = 1.2$ constant for all nodes, and lower and upper bounds on all edge weights as $\underline{w}_{ij} = 0.1$ and $\overline{w}_{ij} = 1$ for all links in $\mathcal{W}$. The cost for reducing the upper-bound is then given by $c_{ij}(\overline{W}_{ij}) = \frac{1}{\overline{W}_{ij}} - \frac{1}{\overline{w}_{ij}}$ for all links. Figure 1 shows the achieved $\lambda_{\max}(\mathcal{W} - D)$ given a certain budget.

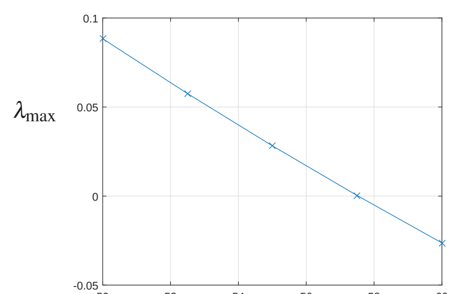

Fig. 1: Achieved $\lambda_{\max}$ for varying levels of budget.

Figure 2 shows the trajectories of the average of all states $\bar{p} = \sum_{i=1}^N p_i$ for varying levels of $\lambda_{\max}(\mathcal{W} - D)$. Interestingly, these simulations (and further simulations not shown here) seem to hint at the fact that $\lambda_{\max}(\mathcal{W}-D) \leq 0$ is both a necessary and sufficient condition for almost sure convergence to the disease-free equilibrium, but this has yet to be shown.

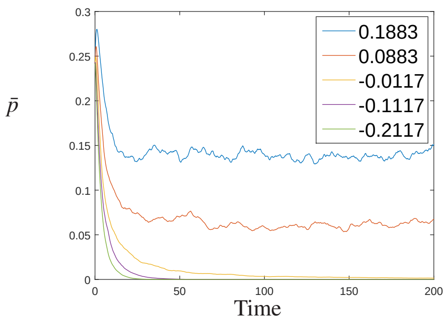

Fig. 2: Trajectories of mean-field approximation for varying $\lambda_{\max}$.

In Figure 3 we compare the evolution of the mean-field approximation (3) to the exact evolution of the original stochastic model. The dashed lines represents the average trajectory of all 20 nodes and the solid line represents the average trajectory of all 20 nodes over 500 simulations for the exact spreading model for $\lambda_{\max}(\mathcal{W}-D) = -0.1117$.

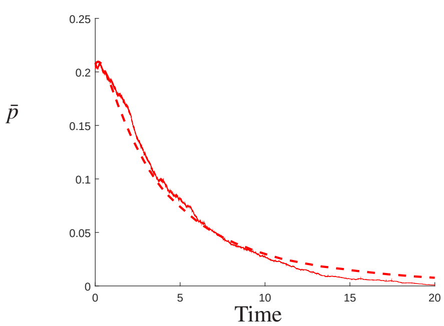

Fig. 3: Mean-field approximation (dashed) vs exact trajectories (average of 500 simulations, solid) for $\lambda_{\max} = -0.1117$.

VI. Conclusions

In this work we have presented the Susceptible-Infected-Susceptible (SIS) epidemic spreading model for a class of time-varying networks. After analyzing the stability properties of the mean-field dynamics, we leverage this result to formulate a geometric program that can efficiently solve an optimal resource allocation problem to best mitigate the effects of an undesired epidemic. For future work we hope to mathematically show that the condition $\lambda_{\max}(\mathcal{W}-D) \leq 0$ is a necessary and sufficient condition for almost sure convergence to the disease-free equilibrium.

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