Preface

When I first started writing this article at the beginning of 2008, the full magnitude of the current financial crisis was not yet apparent. However, nothing that has happened since then has rendered the following information false; indeed, the primary lesson we have learned from the crisis—the need to understand risk—makes the information in this article even more valuable.

1 Introduction

I am writing this article because I wish I’d had it when I started trying to navigate the world of investing. As far as I can tell, there are roughly three kinds of investment books out there: 1) popular books, 2) semi-technical books written for MBA programs (e.g. Sharpe and Marcus)[1],[2], and 3) highly technical books on portfolio theory (e.g. Huang and Litzenberger)[3].

None of these books was sufficient for me on its own: the popular books didn’t have enough useful advice, the MBA books had a lot of useful information, but assumed you didn’t know calculus (though it would have helped) and were about a thousand pages long. The Huang and Litzenberger filled in the mathematical gaps, but had way more in it than I needed and was, well, a math textbook. And when’s the last time you read one of those for fun? (Don’t answer that.)

1.1 The aims of this article

What I’ve tried to do is to distill down the basics of these various references to give you, the recent graduate, what you need to make informed investment decisions. My point is to help you understand the models people use to design investment products, not to teach you how to design them yourself. If you want to know more, consult the books I reference; I certainly haven’t covered everything.

I’m writing this for technical majors, so I assume you’re comfortable with multivariable calculus, basic linear algebra and maybe a little bit of differential equations. If you’re not a technical person don’t be scared - I’ve written the article to be readable to anyone, but added math in the sections marked “advanced” so that the interested reader can go deeper without reading a textbook. We’ll need some statistics (not much, I promise), but I’ll go over all that quickly in the next section. First, let’s figure out what investing can do.
1.2 The aims of investing

In keeping with the aims of this article, I’m not going to get into day trading, selling on margin, options, or a bunch of more involved investment topics. The sort of question I’m concerned about is “How much should I save for retirement and where should I put it?” A really good example of a question I can’t and won’t answer is “How can I make a lot of money?”

Another issue I won’t cover is how to pick individual stocks. Partially this is because I’m totally unqualified to do this, but also because the set of skills required to make successful stock picks are almost totally unrelated to those that are required to put together a relatively successful overall portfolio. The best analogy I can think of is that of the water in the ocean: picking stocks is like trying to predict where any individual water molecule will be in a month, while picking a portfolio is like trying to predict the general currents in the sea. The first problem is really hard and the answers will, by necessity, be very imprecise, but the second problem, that of the aggregate of a bunch of individual molecules, is much easier, since the individual idiosyncrasies of particular molecules average out in the mix.

Furthermore, there are lots of books about how to pick individual stocks out there, and I don’t know that any of them would really benefit from much more math, at least at a level that doesn’t mean you have to spend a large part of the day thinking about your stocks. Of course, there are complex mathematical models that try to predict which individual stocks are good bets, but to use them would take an enormous amount of time and resources. In other words, you need to be doing this professionally for it to be worth your time, in which case you should be reading something else.

1.3 A word about the past vs. the future

In every investing document you’ll always see a sentence like “past performance is no indication of future success.” It’s absolutely true. All the models I show you here are based on knowledge of future quantities, like expected rates of return and variances, but all we have are the past data.

We can make some intelligent guesses as to what the future will look like, but there’s no way to know for certain. Numbers that look totally reasonable today may turn out to have been totally ludicrous tomorrow. Therein lies a great deal of the risk in investing. The good news is that with some intelligent choices we can at least minimize our risk and achieve some substantial rewards – that’s what this article is about. Just remember this important caveat and exercise some common sense; it’ll help keep you from getting burned.

1.4 Math

Finally, if anything is too much for you or you don’t feel like going through all the math right now, feel free to skip the equations and just skim through sections labelled “advanced”; I’ve tried to write such that you won’t miss anything important. That being said, I include a mathematical refresher in the appendix.

We need some basic statistics to understand how to make the best possible investments. In my experience technically-
minded students are more likely to be comfortable with calculus and related concepts than statistics (I certainly was), so I explain them quickly in the appendix. Most of what we’ll do doesn’t require much of what I review, but if you feel yourself wanting some clarification later, go ahead and skim the appendix. If you’re comfortable with random variables, means and variances of multiple variables go ahead and keep reading.

## 2 How much to invest?

Now, let’s consider the first part of investing: deciding when and how much to invest. The simple answer is that it depends on your goals for your investments. There are two important lessons that will become obvious through this section: start saving and investing early and invest as much as you feel comfortable with.

I feel obligated to point out that for all this emphasis on “investing”, we can’t afford to forget general saving. Keeping money in a savings account or possibly a CD is an important part of maintaining financial security; most people advise keeping three months of salary in one of these relatively liquid (that is, easily accessible) forms. It may seem like a lot but it’s a reasonable amount to keep on hand for some sort of rainy day, whether you lose your job or have some unforeseen expense. So to start, be sure to

**Keep about three months of salary on hand in a savings account or CD.**

As a simple example of why you’d want to do this, imagine you lose your job during a recession when the stock market is also way down. (The sharp reader will have noticed that these two events are likely to be correlated so they’re more likely to happen together.) You need money to pay your living expenses, but your stock isn’t worth nearly as much as it was, so you have to sell lots of it. If you’d had cash you could have waited for the stock prices to come back.

Cash loses value with inflation, but slowly, and you always know how much it’s worth. With that in mind, let’s think about defining goals for once you’ve accumulated that liquid reserve.

### 2.1 Defining goals for life

How much you invest and how you invest will vary greatly depending on what you want to have at the end of your investment period. Thus, the first thing to do is to set some investment goals, such as

- Save for retirement,
- Cover the down payment on a house in five years, or
- Pay for next summer’s vacation.

The two crucial elements are

- How much you need at the end, and
- How much time you have to get there.

These two things, along with the rate of return you can expect to get (more on this in below) will tell you how much you need to invest to get to your goal. The time horizon, however, has another important aspect: it can tell you how much risk you can afford to take on. This is crucial, because assets with higher average returns tend to carry correspondingly higher risk. As an example,
consider the difference between saving for retirement and paying for next summer’s vacation. If you’re currently 30 years from retirement, a portfolio that yields, on average, 15% but loses money one year in four would be better than a CD that yields a guaranteed 3%. For your vacation, the opposite might be true.

This sort of situation isn’t so unusual, but consider why you’d choose one investment over the other. In particular, consider two investments with corresponding standard deviations $\sigma_1$ and $\sigma_2$, and furthermore suppose that $\sigma_2 > \sigma_1$. If both investments pay the same return on average, you’d be an idiot to invest in the second one, which carries higher risk but no higher reward. (Or maybe you like gambling?) Thus, in a well-functioning market, the average rate of return on investment 2, $r_2$ will be pushed up so that it’s higher than the rate on investment 1, $r_1$. Thus, higher risks must be compensated by higher returns. This is generally true throughout the investment world, and the difference in the rates of return due to added risk is known as the risk premium.

A general note on employer-sponsored retirement plans: Often these plans have some sort of matching contribution component to them. For example, my current employer plan matches 80% of every dollar that I save through the program, up to 6% of my salary. So if my salary is $40,000 per year and I contribute 5%, or $2,000, my employer will pay in an additional $1,600 to my retirement account for me, subject to something called vesting. First, let me point out that this is a bunch of free money!

So what’s the catch? Okay, let’s get back to this vesting issue. Basically, “vesting” means “owning”, in that you generally don’t “own” the money the company has given you unless you work there for a certain amount of time. For example, in my current plan, I am immediately 100% vested in any contribution I make (meaning I totally own it, so the day I walk away I keep that money and any gains or losses it made). However, I’m initially 0% vested in the matching plan money, increasing by 20% per year until I’ve worked there five years, at which point I am 100% vested and I own all the money in the account.

It’s a decent-sized catch, particularly if you don’t stay at the company for a full year; in that case you don’t get to keep any of the matching contributions. However, even if you stay just one year and are only allowed to keep 20% of the 80% match, it’s still a guaranteed 16% return on your money, and I challenge you to find that anywhere else. So, if you have an employer matching plan, contribute at least up to the maximum amount the employer will match: otherwise you’re giving up free money.

2.2 Other things to consider

So far I’ve assumed several things: particularly that you’re employed working for someone else and that you’re primarily concerned with saving.

First, if you’ve got any form of debt (credit cards, student loans), think about paying off the debt before you invest. In any case, save up the three month cushion first. Credit cards usually carry the highest interest rates, if you contribute $1 of your money, they contribute 80 cents. After a year you walk away with 20% of their portion, or 16 cents, for a total of $1.16 for an initial $1 out of your pocket.
so think about paying them off first. Student loans have some tax advantages (usually you can deduct the interest from income), so it might be somewhat more tricky to decide what to do. Generally the rule of thumb is to compare the rate of return you’d get on investments versus the interest rate you’re paying on the debt.

Another thing to consider is whether to work for someone else or to start your own business, or invest in your future in some other way. Investing for life means more than saving for retirement: if there’s something you’re interested in and it has the potential to bring you happiness, consider adding it to your overall ‘portfolio’.

2.3 The miracle of compound interest

Compound interest is what makes investing over the long term so lucrative, and it’s really almost magic. We’ll go through a simple example here to show how time has value in much the same way money does.

Say you’re currently 21 and starting to think about saving for retirement. (I doubt many 21 year olds think about this, but this example will show you why they should.) You intend to retire at age 65, meaning you’ve got 44 years to save up. You consider two plans:

1. Invest $1,000 each year until you retire at 65, for a total of 44 years, or

2. Invest $2,000 each year starting at age 43 until you retire at 65, for a total of 22 years.

Notice that in either case, you’ll have invested $44,000 in contributions to your fund. Let’s see how these two plans compare at retirement. I’ll assume you invested relatively conservatively and earned a return of 6% every year on your money. At retirement, the first plan gives you a balance of $199,758.03, whereas the second plan gives you $86,784.58. Which one would you rather have? The contrast becomes even larger if you earn greater returns. Leaving everything else constant and changing the rate of return from 6% to 10% makes the ending balances $652,640.76 and $142,805.50, respectively. See Figure 1 for the evolution of the 6% case with time.

Obviously, the time you had the money invested had a large value; in the case of the 6% return, being able to put the money away earlier was worth nearly an extra $113,000 at the end. Thus, the first rule of investing is

Start early!

As you can see in the figure, both curves follow a basically exponential path; this is due to the compounding nature of the interest. If we had invested the $44,000 in one chunk, it would have grown as $e^{0.06t}$, where the 0.06 is the interest rate and $t$ is measured in years.

2.4 Inflation

Generally we see everything quoted in nominal numbers, but what we care about, presumably, are real numbers and rates: not how many pieces of paper do we have in our pocket but what can we buy with them?

The value of money tends to erode over time due to inflation, or a rise in the general price level. The Federal government tries to measure inflation through a number called the CPI, or Consumer Price In-
Figure 1: Evolution of portfolio value with time for 6% returns. The dashed line is for the plan starting in year 0 (at age 22), the triangles for the plan starting in year 22 (at age 44).
index, that measures the price of a “basket” of goods representing a set of common consumer purchases. One measure of the inflation rate is the rate of change of the CPI, otherwise known as headline inflation. Some economists prefer core inflation, another measure of inflation that excludes prices of food and energy (think gasoline and electricity), since these prices tend to be volatile and don’t necessarily represent the long-term trend of inflation.

As you can see in Figure 2, inflation has historically been high and highly volatile, though it has significantly moderated in the last 20 years or so. Current central bank policy throughout the developed world is to try to keep inflation between about 1 and 3 percent per year, and in the recent past they have generally been pretty good at attaining that goal.

A mild level of inflation such as the target level for most central banks is not a big problem. Inflation only really becomes a problem when it is large or unexpected, such as what happened in the 1970s. Then savings deteriorate since they are unlikely to be earning a rate of return higher than the rate of inflation and general badness ensues. Hopefully periods like this won’t return any time soon, but if they do, there’s not a whole lot one can do to hedge against them, beyond buying things whose prices rise with the general price level, such as gold or commodities, e.g. wheat, corn or oil.

While we’re on the topic, it would be nice to see how to compute one’s real rate of return given the nominal rate of return and the inflation rate. This tells us how well our investments are doing in terms of purchasing power, or actual worth.

Suppose that today we invest $1 at a rate $r$. Furthermore suppose that we know the inflation rate and it is equal to $\pi$. Today that $1$ is worth exactly that, $\$0.1$. Note the subscript on the dollar sign: a dollar today isn’t worth the same as a dollar tomorrow! Then tomorrow, what do we have? Our investment is now worth $\$(1 + r)$, while things that cost $\$0.1$ yesterday cost $\$1(1 + \pi)$ today. Thus, our investment has purchasing power of $\frac{1 + r}{1 + \pi}$.

If $\pi$ is small, we can take the Taylor series of the rate of return and see that $1 + r_{\text{real}} \approx 1 + r - \pi$, so the real rate of return is $r_{\text{real}} \approx (r - \pi)$.

This approximation is generally very close to the true value.

2.5 Taxes

The goal of investing is to make money, and if you’re making money you’ll probably have to pay taxes on it. However, there are several ways the tax system can affect your choice of investment options.

Generally, you have to pay taxes on realized capital gains. That is, if you buy some stock and it goes up in value and then you sell it, you have to report the difference in price as income that you made and pay tax on it. There are some exceptions to this, but generally that’s the rule. Watch out for some mutual funds that have a high turnover ratio, since these can make you pay taxes even though you didn’t buy or sell anything yourself: basically, some funds that you buy as a single unit will buy and sell lots of stocks during the year, realizing capital gains. Even if you don’t sell your shares in this fund (meaning you don’t get any cash),
Figure 2: Headline inflation since 1914 (solid line) and core inflation since 1958 (line with circles). Headline inflation measures the change in prices of a general set of consumption goods, and core inflation is the same excluding food and energy costs, which tend to be volatile. Note the general moderation of size and volatility of inflation since about 1990. Source: Bureau of Labor Statistics.
you may have to pay taxes on the capital gains realized by the fund. This is annoying since you didn’t get any cash yourself, just a higher paper value for your fund shares.

Retirement accounts are another special case. There are basically two things you can do with regards to taxes on retirement accounts: pay them now or pay them later. There are two factors to consider when deciding when to pay taxes. One, since the income tax system is progressive you pay a higher proportion of your income in taxes the more money you make, and two, tax rates can change over time. So, if you expect to earn more money later, you’re better off paying the taxes at the low rate now and avoid paying them when you’ll have a higher income in retirement. Vice versa, if you’re making a lot of money now it might make more sense to invest the money before taxes now and pay taxes on it (and the capital gains) when you have a smaller income in retirement.

Plans like standard IRAs and 401(k)s are of the pay tax later type. You contribute pre-tax, meaning every dollar you contribute reduces the amount of your taxable income by that same amount. However, you have to pay taxes on the money and the capital gains you’ve accrued when you withdraw it during retirement. Roth IRAs and 401(k)s have the opposite structure: you pay income tax on the money now, but then the money grows tax-free (including the capital gains), so you pay no taxes when you make withdrawals, subject to some relatively minor caveats. There is a limit to how much you can contribute to a Roth IRA in any given year, though.

There’s one thing that I mentioned earlier that we need to keep in mind: that income tax rates can change over time. Income tax rates are historically pretty low right now and are pretty much sure to go up. Why? Look at the deficit and the national debt. Right now we’re borrowing money as a country to keep the government running, and one day we’ll have to pay it back. How will we get the money to pay it back? Through higher taxes, of course. So anything you can do now to reduce your tax liability later is probably a good thing, particularly if you’re young and have a relatively low salary.

## 2.6 Math for all this

I wouldn’t be properly taking advantage of your technical background if I didn’t go into a little bit more detail about the math behind these ideas, particularly how to figure out how much money you’ll have in your portfolio like in the example in §2.3.

I should mention that this analysis is generally highly inexact and is best used to give a rough idea of how to construct a portfolio. As a simple example, it’s very difficult to estimate the expected future rate of return on a portfolio or any individual stock or other asset. So take it seriously, but as a tool to give you guidelines, not precise advice.

### 2.6.1 Z-transform (advanced)

While the example is easy enough to compute in a brute-force way with any spreadsheet program, sometimes it’s nice to have a simple closed-form solution. The Z-transform gives you a quick way to find such a closed-form solution.

The example above reduces to the following equation:

\[ A_{t+1} = A_t(1 + r) + C, \quad t = 1, 2, \ldots, 43, \quad (1) \]
where $A_i$ is the balance of your account at the end of period $i$, $r$ is the rate of return, and $C$ is the amount of your yearly contribution. If you ever took a class on signal processing or discrete math, you’ll remember that this looks like a good candidate for the Z-transform. Let’s review how to use it here.

First, let’s rewrite Equation 1 in a form that’s closer to something from signal processing:

$$y[t + 1] = (1 + r)y[t] + x[t],$$  \hfill (2)

where $x[t]$ is the input to the system, or the annual contribution in our case. Again using signal processing terminology, our annual contribution is $C$, or $Cu[t]$, where $u[t]$ is the unit step function that takes the value 0 if $t < 0$, and 1 otherwise.

Then we can solve the equation by taking the Z-transform of both sides:

$$zY(z) = (1 + r)Y(z) + X(z)$$
$$\Rightarrow (z - (1 + r))Y(z) = X(z),$$

where $X(z)$ is the Z-transform of $x[t]$, etc. Now substitute in the input to the system (i.e. the annual contribution), $Cu[t]$, to get

$$(z - (1 + r))Y(z) = C \frac{1}{1 - z^{-1}}$$
$$\Rightarrow Y(z) = C \frac{1}{1 - z^{-1} z - (1 + r)}.$$

It’s hard to take the inverse Z-transform of that above equation, so we take the partial fraction decomposition\(^\text{3}\) to split the right hand side into a sum of several terms with first-order denominators:

$$Y(z) = \frac{C}{r} \left( \frac{(r + 1)z^{-1} - z^{-1}}{1 - (1 + r)z^{-1} - z^{-1}} \right),$$

which has the relatively easy inverse Z-transform of

$$y[t] = \frac{C}{r} \left( (1 + r)^t - 1 \right) u[t - 1].$$

Note that we start where $t - 1 = 0$, or $t = 1$, so we have exactly the same setup as before.

This sort of analysis can be made arbitrarily complicated, for example you could have your contribution grow over time, etc. Again, a spreadsheet can do all of this in a brute-force, clumsy sort of way, but the closed-form solution is much more elegant and possibly faster for some computations.

### 2.6.2 Portfolio simulation

Having shown you an elegant way to solve for the value of your portfolio over time when the portfolio process is simple, let’s now examine a way to deal with the example of §2.3 when the portfolio process becomes much more complicated (read: realistic). The scenario I detailed in §2.3 is useful for getting a basic idea of what’s going on and how much to invest over time, but if you want to examine a less restrictive model, it’s tough to do with the sorts of Z-transforms I detailed in the previous section.

Why would you want to do this? Well, consider the fact that rates of return aren’t generally guaranteed, but are random variables that tend to follow some distribution through time. So instead of our constant $r = 0.06$, we might well want to let $\tilde{r} \sim N(0.06, 0.10)$, that is, let $\tilde{r}$ be a random

\(^{2}\)See a website like http://en.wikipedia.org/wiki/Z-transform to look up the Z-transform pairs and to refresh your understanding of the Z-transform, if necessary.\(^{3}\)Feel free to look this up, too. You did it in calculus but probably haven’t since then.
variable following a normal distribution with
mean 6% and standard deviation 10%. Or
perhaps you just have some history of re-
turns on your portfolio that you can use as
the distribution of $\tilde{r}$.

Now we can simulate the portfolio by mak-
ing draws from the probability distribution
of the return. Say the distribution you have
is for annual returns and you want to simu-
late a portfolio for 44 years, as in the previ-
ous example. Then you draw a return 44
times from the distribution (with replace-
ment, if you’re thinking of this like a bunch
of lottery balls), and compute the 44-year
return with them.

At this point we’ve gotten one simulated
return on the portfolio, but this is just one
realization from the bigger probability distri-
bution of 44-year returns. (Realize that this
is different from the distribution of annual
returns!) In order to get an idea of this over-
all return distribution, redo the draws and
make another 44-year return. Redo this in-
dependently many times and you get a sim-
ulation of the distribution of the 44-year re-
turns. This process gives you an idea of the
returns you can expect and their variability.

As an aside, a recently published article in
the New York Times used this sort of tech-
nique (called bootstrapping if you want to
learn about it in more detail) to simulate
the history of baseball in order to find the
probability of Joe DiMaggio’s 56-game hit-
ting streak. If you like baseball or are inter-
ested in seeing what you can get from this
sort of simulation procedure, check out the
article.4

3 What can we invest in?
Okay, so we’ve spent a bunch of time going
over the mathematics of investing: now let’s
see what we can actually invest in. There
are basically two types of assets that most
people invest in: stocks and bonds. By “as-
et”, I mean anything you can buy as an
investment. Of course there are other types
of assets: commodities (like wheat or gold,
which can be bought in a convenient way
through futures contracts), or, importantly,
a relatively safe savings account or certificate
of deposit (CD), which you’ll want to use to
stash your three or so months of salary in
reserve.

I should stress that the information in this
section is necessarily brief and that this is a
place where the MBA-level investments text-
books excel. Look to them for more detailed
information on available assets and the de-
tails of how they function.

3.1 Savings and CD accounts
The obvious and simple place to keep your
money (beyond stashing it in your mattress)
is a savings or CD account with your bank.
Typically, savings accounts pay some rela-
atively low interest rate that adjusts slowly
(maybe once or twice a year), and are re-
ally simple and everyone understands them.
CDs are like savings accounts but you have
to keep the money in the bank for a fixed
amount of time (usually several months, but
as short as a week and as long as five or so
years). In exchange for this limited flexibil-
ity, the bank usually pays more in interest
on CDs, but not always.

In particular, many ‘online only’ banks
have sprung up recently (e.g. ING Direct,

4http://www.nytimes.com/2008/03/30/opinion/
30strogatz.html
3.2 Money market

A money market account is basically a savings account that pays a variable rate of interest by investing in short-term debt. What is this? Short-term debt is composed of bills having a maturity of about six months or less. These can be issued by the US Treasury (T-Bills), or corporations seeking financing for small debts over short time frames (called Commercial Paper). Since the maturities of these bills are short, there’s little risk of the issuers defaulting.

Thus, the money market account will pay whatever rate is prevailing in the marketplace at any given point in time (hence the term money market). The main risk in this sort of account is that the rate of return is not fixed, as in a CD, but rather floats with the market. This floating rate can be quite variable, as we can see in Figure 3, where I’ve plotted the 3-month Treasury bill rate, which is typical of the rate on a money market account. Finally, there are various types of money market accounts and funds, some of which are FDIC insured and others which are not.

3.3 Bonds

Bonds are similar to the bills that compose money market funds, but are of longer maturity. For example, US Treasury bonds of 15 or 30 year maturity or a corporation’s 20 year bond issued to pay for some new manufacturing plant. As they are longer-term than money market-type assets, they are somewhat more sensitive to changes in interest rates (what happens if you’ve bought a 30 year bond paying 5% interest and next year the interest rate goes up to 6%?), they are consequently more risky.

In addition, bonds issued by private corporations carry default risk, the risk that the company could go bankrupt and thus not be able to pay back its debts. This risk doesn’t affect government bonds since the government can always print more money to pay back the nominal value of its debts, but the probability that any given company will default varies widely, from practically nil (say GE) to quite high (the so-called ‘junk bonds’). Consequently, corporate bonds pay a higher rate of interest than government bonds to compensate for their higher risk.

Despite these risk factors, bonds (particularly those of big corporations) are relatively safe investments, and generally offer higher returns than a money market account. These returns represent a risk premium over the relatively risk-free money market account (cf. §3.1).

3.4 Stocks

Stocks are generally riskier than bonds and so, due to the risk premium effect, tend to offer correspondingly higher returns. Stocks represent the ownership of a company, so
Figure 3: The rate on the 3-month US Treasury bill since 1935. Note how it spikes with inflation around 1980 and is quite variable. The curve itself tracks inflation pretty well: 3-month real rates don’t move around nearly as much as you’d guess from the picture. Source: Federal Reserve H.15.
Figure 4: Market interest rates on Baa-rated (that is, low investment-grade, or medium default risk) bonds (solid line), with 3-month T-Bill rates for comparison (dashed line with circles). Note how they are generally higher than the 3-month T-Bill rate, a reflection of the default premium. Source: Federal Reserve H.15.
owning one share of a stock means you own a part of the company proportional to the number of shares of stock outstanding. As an owner of the company, you are entitled to a portion of the firm’s profits, which are paid out in the form of dividends. Thus, stocks generate returns both through capital gains (changes in the price of the stock) and dividends (the profits paid to you as a shareholder).

Stocks can be categorized many different ways, but the two most important criteria are size (the value of all outstanding shares, which is the value of the company), and something I’ll call maturity, which is a measure of how quickly the company is expected to grow.

Company size, also known as market capitalization, is generally broken up into three categories: small (or small-cap), medium (or mid-cap) and large. Similarly, companies are generally categorized by maturity as growth (obviously for companies that are expected to grow) or value (for companies that are considered mature and are unlikely to grow quickly). It is an empirical fact that small-cap stocks tend to deliver higher returns than large-cap stocks, and that value stocks tend to deliver higher returns than growth stocks. Professor Ken French maintains data on return histories since 1927 [5]. It is also a fact that the higher returns come with a price of higher volatility and therefore risk: there really is no free lunch.

The small vs. large and growth vs. value breakdown is the basis for several important industry benchmarks. Morningstar (an investments research company) classifies stocks and mutual funds based on these two criteria, as do other firms. Furthermore, these are two of the main inputs into the Fama-French model to predict portfolio returns, which is increasingly being used to design investment instruments.

3.5 Other stuff

Savings accounts, stocks and bonds are the most common investment assets, but there are tons of others as well, such as commodities, futures and options. Most of the time you wouldn’t want to mess with these things, but it’s worthwhile to know what they are. Comodities are just that: goods that are produced in large, essentially identical units, like gold, gasoline, corn, or pork bellies. Commodities are traded in various markets; for example, most agricultural contracts are traded at the Chicago Board of Trade, and others are traded around the world.

Now, you wouldn’t want to buy 100,000 gallons of gasoline and have to hold on to it while waiting around for gas to go up in price in order to sell. Enter the future contract. These are also traded on large mercantile exchanges, and require the seller to provide a given quantity of some commodity at some future date for a given price. You buy this contract and effectively make a bet that the market price of that commodity will be higher than the price you paid on the futures contract on the day it expires.

Options contracts are similar to futures contracts, but, as their name implies, give the buyer the right, but not the obligation, to buy or sell some underlying commodity or stock at some future date. These can make lots of money fast, but they’re also really dangerous if you don’t know what you’re doing and are basically impossible to profit from unless you have a sizable amount of money to invest in them.
4 Diversification: an introduction

Now that we’ve thought a little bit about what we can invest in, let’s think about what we might want to invest in. In the spirit of the saying “don’t put all your eggs in one basket”, you probably want to invest in several things; multiple stocks, some bonds, etc. This spreading around of investments is known as diversification. Let’s formalize this intuition a little bit with an example.

The canonical example of why you’d want to diversify is comparing returns on stock in an oil company and an airline. Both of these companies should obviously be affected by oil prices, but a rise in the price of oil should increase returns on the oil company’s stock and decrease returns on the airline, since jet fuel is a major expense for their business. If we plot the simulated returns on two stocks like these over the past 10 years, we get Figure 5.

As expected, one was down when the other was up, and vice versa. Of course this relationship doesn’t hold exactly, but in general it’s true. So if you bought $100 worth of our Company A or $100 worth of our Company B’s stock in 1998, today you’d have stock worth $159.47 or $243.66, respectively.

Now let’s consider a third possibility: that you invest $50 in each; that is, construct a portfolio with equal weight on each stock. In this case, you achieve returns that are equal to the average of the two returns, and at the end of the same 10 year period you’d have $201.56 for the same initial investment of $100. Returns of the average portfolio are obviously much less volatile than those on either individual stock, which is a good thing: you’ve lowered your risk.

In general, any two assets that are imperfectly correlated (i.e. \( \rho < 1 \), cf. the appendix) will yield diversification benefits by lowering risk while not lowering returns by a corresponding amount. Here, the two stocks had a \( \rho \) of 0.0092 over 1998-2008. Thus, they were correlated, but not very much.

5 Diversification: modern portfolio theory (advanced)\(^5\)

Now that we’ve seen why we might want to diversify our portfolio, let’s consider how to do so optimally. This is a non-trivial problem, and as far as I know, the full version you’d want to use in practice has no nice analytical solution and has to be solved numerically.

Before we begin, I should point out that because of the difficulty involved in estimating the inputs to this problem, particularly for individual stocks or bonds, no one in their right minds uses this procedure to pick a portfolio of individual stocks, but rather uses it to decide how to allocate money among various asset classes, usually those discussed in Section 4.

Note: Before we go any farther into this, I should say that there are a number of rules of thumb that people also use: for example, a retirement portfolio composed of stocks and bonds should have a weight on bonds that’s approximately half your age, with the remainder in stocks or other relatively risky

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\(^5\)In this section I borrow heavily from Huang and Litzenberger. This work was pioneered by Markowitz in the 1950s, for which he won the Nobel Prize in 1990.
Figure 5: Returns on two relatively uncorrelated companies, and an equally-weighted portfolio of both. Company A (circles) had an average annualized return of 6.30%, Company B (crosses) of 12.19% and the equally-weighted (solid line) portfolio 9.25%. Corresponding standard deviations were 4.91%, 10.88% and 5.99%, respectively.
assets. Thus if you’re currently 30, about 15% of your portfolio should be in bonds and 85% in stocks. If that’s enough for you, skip this section. Otherwise, read on.

5.1 The optimal portfolio problem

This is really just a math problem, so let’s start out with our assumptions. We assume we have $N$ assets, each with a rate of return $\bar{r}_i = E[\tilde{r}_i]$, $i = 1, ..., N$.

For example, the assets $i$ could be individual stocks, bonds, portfolios of stocks, or any combination of these. In theory we could use the model that follows on individual stocks, but in practice it’s difficult to accurately estimate the inputs in the case of individual stocks, so generally the assets we use are general asset classes, eg large stocks, US bonds, European stocks, etc.

Suppose that we want to analyze a portfolio of assets $i$, $i = 1, ..., N$, with rates of return as above. We invest a proportion of our wealth $w_i$ in asset $i$.

Then the expected rate of return on the portfolio is

$$\bar{r}_P = E[\bar{r}_P] = \sum_{i=1}^{N} w_i \bar{r}_i = w'r, \quad (3)$$

where $w$ and $r$ are vectors composed of the $w_i$ and $r_i$, respectively. Since the $w_i$ are proportions of our invested wealth, they must sum to 1, so we have

$$\sum_{i} w_i = w'1 = 1, \quad (4)$$

where $1$ is a vector of ones.

We know from Section A.4 that the variance in the rate of return on our portfolio, $\sigma^2_P$, is given by

$$\sigma^2_P = w'\Sigma w, \text{ where } \Sigma_{ij} = \text{Cov}(\tilde{r}_i, \tilde{r}_j).$$

5.2 Finding the optimal portfolio

The optimal portfolio for any given rate of return $\bar{r}_P$ is the one that minimizes the associated risk, that is, the associated variance. Thus we have the following constrained optimization problem:

$$\min_w \sigma^2_P = w'\Sigma w, \text{ such that } (5.1) \quad w'r = \bar{r}_P$$

$$w'1 = 1. \quad (5.2)$$

Equivalently we can minimize $\sigma^2_P/2$; this makes the algebra slightly cleaner. Thus, we set up the Lagrangian for the problem:

$$\mathcal{L} = \frac{1}{2} w'\Sigma w - \lambda_1 (w'r - \bar{r}_P) - \lambda_2 (w'1 - 1). \quad (6)$$

Since $\Sigma$ is positive definite, concavity is assured and we minimize by taking first derivatives and setting equal to zero:

$$\frac{\partial \mathcal{L}}{\partial w} : \Sigma w - \lambda_1 r - \lambda_2 1 = 0 \quad (6.1)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} : w'r - \bar{r}_P = 0 \quad (6.2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} : w'1 - 1 = 0 \quad (6.3)$$

Solving (6.1) for $w$ gives
\[ w = \lambda_1 \Sigma^{-1} r + \lambda_2 \Sigma^{-1} 1. \]  
(7)

Now we need to find \( \lambda_1, \lambda_2 \):

Multiply (7) on the left by \( r \) and use (6.2) to get:

\[ \bar{r} P = r' w = \lambda_1 r' \Sigma^{-1} r + \lambda_2 \Sigma^{-1} 1. \]
(8)

Now multiply (7) on the left by \( 1' \) and use (6.3) to get:

\[ 1 = 1' w = \lambda_1 1' \Sigma^{-1} r + \lambda_2 1' \Sigma^{-1} 1. \]
(9)

Now, we can solve for \( \lambda_1 \) and \( \lambda_2 \):

\[ \lambda_1 = \frac{C \bar{r} P - A}{D} \]
\[ \lambda_2 = \frac{B - A \bar{r} P}{D}, \]

where

\[ A = 1' \Sigma^{-1} r = r' \Sigma^{-1} 1 \]
\[ B = r' \Sigma^{-1} r \]
\[ C = 1' \Sigma^{-1} 1 \]
\[ D = BC - A^2. \]

Now plug these values into (7) to get the unique set of optimal portfolio weights having portfolio expected return \( \bar{r} P \):

\[ w_P = g + h \bar{r} P, \]
(10)

where

\[ g = \frac{1}{D} [B(\Sigma^{-1} 1) - A(\Sigma^{-1} r)], \]

and

\[ h = \frac{1}{D} [C(\Sigma^{-1} r) - A(\Sigma^{-1} 1)]. \]

Plugging these back into the original minimization function (5), we get

\[ \frac{\sigma_P^2}{1/C} - \frac{(\bar{r} P - A/C)^2}{D/C^2} = 1, \]
(11)

which a hyperbola in \((\sigma, \bar{r} P)\) space, known as the portfolio frontier, which we plot in Figure 6.

Notice several things about this solution. First, there exists an absolute minimum \( \sigma \) on the frontier, called the minimum variance portfolio. Clearly, it makes no sense to invest in portfolios offering \( \bar{r} \) less than \( \bar{r}_{MV P} \), since there is a corresponding portfolio at the same level of variance (risk) offering a higher level of expected return. Furthermore, \( \sigma_{MV P} > 0 \) in general. This is because of the limits to diversification: you can get rid of most of the individual risks in your portfolio, but what happens if the economy tanks? Clearly, there is some systematic risk that can be minimized but not eliminated.

### 5.3 Adding risk-free borrowing and lending

In the previous section we considered the optimal portfolio problem when we could only invest in risky assets. But this is clearly unrealistic: anyone can put money in a CD or Treasury notes, and in theory most people can get some margin credit from their broker, both of which are basically risk-free.

Assuming there is only one risk free interest rate \( r_f \), the constraint (5.1) becomes

\[ w'r + (1 - w'1)r_f = \bar{r} P, \]
(5.1')

but otherwise the problem remains the same. I won’t go through all the details here; check out Huang & Litzenberger for the rest if you’re interested.
Figure 6: The optimal portfolio frontier for a toy example solved via the process above.
The punch line is that you take the portfolio frontier you calculated in Section 6.2 and mark the risk-free asset at \((\sigma = 0, \bar{r}_P = r_f)\) (remember, it’s called risk-free because it has no variance!), and draw the line through that point that’s tangent to the portfolio frontier, as you can see in Figure 7.

The intuition is that, as before, you want to combine the two types of assets to get the lowest variance for any given rate of return, but neatly the optimum is to just linearly combine the two assets to get the line in Figure 7. This result is known as the **two fund separation theorem**, and means that, ideally, everyone everyone just chooses their optimum rate of return or portfolio variance, looks up the corresponding point on the tangent line, and invests accordingly. Note that the point at \(r_f\) means you put all your money in a CD, the point where the line touches the parabolic frontier means you put all your money in the risky asset portfolio, and every point in between means you mix the two, keeping some of your money in the bank and the rest in the risky portfolio.

5.4 Caveats

Again, none of this is perfect (sense a theme here?): in particular, it’s much harder to estimate \(\bar{r}_i\) than \(\Sigma_{ij}\), so it turns out that in general studies trying this technique on historical data have done best by investing in the minimum variance portfolio rather than any other optimal portfolio. Hopefully this makes some sense as the MVP depends only on the covariances \(\Sigma_{ij}\), while any other portfolio will be dependent on your estimates for \(\bar{r}\).

Furthermore, realize that all of this analysis is based on estimates for \(\bar{r}\) and \(\Sigma\) in the future. In all likelihood, the way you’ll go about making these estimates is to look at historical averages, which is relatively easy, and in many cases, the only reasonably feasible thing to do. However, there is no guarantee that the future looks anything like the past! This is particularly problematic for individual securities, etc., which is why people use this model with broad asset classes, where the estimation is better, rather than with individual assets. On top of this, notice that for \(N\) assets, \(\Sigma\) has \(N(N + 1)/2\) elements and \(\bar{r}\) has \(N\) elements, for a total of \((N^2 + 3N)/2\) numbers. Since this total scales as \(N^2\), optimizing a reasonably well diversified portfolio with hundreds or thousands of stocks becomes a nightmare.

As another important note, if you carry out the optimization exactly as detailed above you will generally get some negative portfolio weights. What do these signify? They signify short selling the corresponding assets: that is, borrowing them from someone (usually your broker) and selling them, only to have to buy them back later to pay back your loan. Why would you do this? To profit from an expected decline in prices by selling high and buying low instead of the normal buy low and sell high. However, short selling is a somewhat complicated and risky endeavor, and really not suitable for the individual amateur investor. Thus, you’re likely to want to modify the above optimization problem to constrain the weights to be non-negative by adding

\[ w_i \geq 0 \forall i \quad (6.3) \]

to the original problem, which is easiest done numerically in Excel (frustrating but actually pretty useful for small problems)
Figure 7: Now by adding the risk-free asset we produce an ideal market portfolio. Under this model, it suffices to choose either your desired level of risk ($\sigma$) or average return ($\bar{r}_P$), and you simply look up the corresponding point on the dashed line to find your optimal portfolio. Portfolios between $r_f$ and $r_M$ involve saving some money in a risk-free CD, and portfolios above $r_M$ involve buying the market portfolio on margin by borrowing money at the risk-free rate.
or Matlab (much better for big or repeated problems - try the `fmincon` function).

6 Practical considerations for portfolio construction

Okay. We’ve figured out that we want to diversify our portfolio, and we’ve even come up with an algorithm for identifying the best composition of that portfolio for our needs. However, there remains a significant final step: actually buying that portfolio.

Actual portfolio construction is difficult for an individual investor not only because there are so many stocks and bonds to keep track of (easily 6000 stocks alone that one might consider), and because stocks are generally sold in blocks of 100, which means that any individual investor would run out of money way before they got to buy anywhere near a number of stocks that would be representative of the market as a whole. How can we avoid this problem? Enter the mutual fund and its cousin, the exchange traded fund (ETF).

Mutual funds and ETFs work by pooling lots of peoples’ assets and constructing a single big, diversified portfolio. They can do this more efficiently than individual investors because of their larger size, and so are a valuable tool for us little guys. Of course, they charge some fees for this service, but it’s still worth what you pay for it, particularly if you’re choosy about what you buy. Understanding their lingo and the types of products you can buy are essential to investing wisely and can save you lots of money in the long run.

This is possibly the most important section in the article, as investors have lost literally hundreds of millions of dollars to investment companies in the form of excessive fees. However, these fees are easy to avoid if you know what you’re looking at. So pay attention, otherwise it will cost you.

6.1 Active vs. Passive strategies

One important distinction among funds is between active and passive investment strategies. This concerns how the portfolio is maintained, rather than its composition, and is a fundamental difference in the way a fund is run.

A passive investment strategy can be summed up as buy and hold. The idea is that you put together a portfolio at the beginning of your investment period, then basically just let it grow over time with occasional updates. A very common passive investment strategy is the use of index funds, which use some predetermined index like the Dow Jones Industrial Average as the portfolio. For example, a Dow index fund would create a portfolio exactly the same composition as the index by buying one share of each company in the Dow for each share of the overall fund. The advantage of this strategy is that it is cheap (you don’t have to pay anyone to put together the portfolio), and you’ll get the same return as the index itself (minus some minor trading costs).

An active investment strategy involves continuously updating your portfolio to try to take advantages of mispricings in the market. For example, you might think that shares in Boeing are currently undervalued, so you buy more of them in the hopes that they’ll go up more than the market as a whole. The difficulty with this type of strategy is that it is relatively difficult to find and
take advantage of such mispricings (if it were easy, everyone would do it, thereby putting the prices back to where they should be). Since it’s difficult, you have to spend a lot of effort to do it, which means paying lots of people and racking up often considerable expenses.

Since active strategies are expensive to implement, they’re only profitable if the portfolio managers have some considerable skill that can deliver much higher returns to cover their expenses. So the relevant question is, does the active portfolio manager have skill? The answer, on average, is a resounding no: French (2008) carries out an analysis that shows that, after taking into account expenses, the average active portfolio returned 0.67% less than a similar passive portfolio [4]. This may not seem like that much, but over time it’s huge: consider again the 44 year retirement savings example from Section 3.2. With a 6% annual return, you were left with a final value of $199,758.03, but if you were paying those 0.67% for active management, you’d end up with an annual return of 5.33% and a final value of $165,554.89, which is a huge 17% difference. So, when you invest,

Beware the effect of a fund’s expense ratio!

Now, let’s learn how to find a fund’s expense ratio and how they actually work.

6.2 Lingo: practical details about mutual funds and ETFs

We know that mutual funds and ETFs both work by putting together assets from lots of smaller investors into one big portfolio and that this results in economies of scale by allowing for better diversification. However, there are important differences between the two. Let’s compare them and in so doing learn about the way they actually function and the relevant numbers for an investor who is looking to buy shares of one of these funds.

Probably the most important number for either type of fund is the Net Asset Value, or NAV. NAV is pretty much exactly what it sounds like, the value of the fund’s portfolio minus its liabilities (for example, management fees), usually expressed on a per-share basis.

The company running a fund calculates its NAV at the end of each business day; in the case of mutual funds you buy shares directly from the management company at the NAV calculated at the end of the first business day after you made the investment (so if you buy at 6 pm on a Tuesday, you buy at Wednesday’s closing NAV). ETFs, on the other hand, are actively traded on exchanges (hence the name) just like stocks. Thus you can buy them at any time during the day and their current market price can deviate (usually only slightly) from current NAV.

Both types of fund can invest in a variety of assets, but ETFs are almost always passively managed and track some sort of index, while mutual funds can be either passively or actively managed. In either case you’ll have to pay some sort of expenses summarized in the expense ratio, and actively managed funds of course generally have higher expense ratios.

There are additional fees, however: since ETFs trade like stocks, you generally have to pay some sort of brokerage fee to buy or sell them. It’s pretty much unavoidable, but not a big deal if you don’t trade very often and
trade in sufficiently large volume to make the fees small relative to the amount of money you’re moving around. Some mutual funds have similar costs, called loads or back end charges. Here, the advice is simple:

**Beware the effect of loads!**

A load is not a performance-based fee, it is a preset cost. If you pay a load, it works like this: suppose you invest $10,000 in a mutual fund with a 5% load (this is a pretty typical number). Then the mutual fund company charges you 5% and invests the rest, $9,500 in the mutual fund. This is basically a guaranteed -5% return in your first year, which is terrible. You’ll have to earn slightly more than a 5% return on the remaining money just to break even. If you’re planning on keeping the money in the fund for a long time, this might not matter much, but if you’re only going to keep it there for a year or two, you’re probably better off with a simple savings account.

Back end charges are similar to loads, but occur when you want to take your money out of the fund. Most funds to charge these, but they generally decrease to zero as long as you keep the money in the fund for a couple of years, so don’t matter if you plan to invest the money for the medium or long term.

The fee structures on mutual funds can be somewhat confusing, but generally if you avoid loads you’re doing all right. Often a given fund will have several different “classes” of funds, each with a slightly different fee structure: for example, one might have heavier upfront loads but lower back end charges, or vice versa. Take a look at the different classes before making a final choice.

Finally, the funds will have different tax characteristics. Since mutual funds must continually buy and sell shares to cover investors’ buying and selling of shares of the fund, they generate considerable realized capital gains that are considered taxable, even if the individual investor in the fund hasn’t touched his shares. Thus, the individual investor will owe taxes at the end of the year, even though he hasn’t gotten any cash. Shares of passive ETFs, on the other hand, merely trade on their relevant exchange and so individual investors do not realize taxable capital gains until they sell their shares. It should be noted that actively managed ETFs sacrifice most of this tax advantage through their active trading.

I have summarized the differences between the two types of funds in Table 1.

### 6.3 “Target Date” funds and why you might want them for retirement

As a small investor, it’s often difficult to produce a well-diversified portfolio even with individual mutual funds. That’s because most mutual funds are focussed on one thing: bonds, large-cap stocks, or perhaps an index portfolio of the stock market as a whole.

You could create a fully diversified portfolio by buying shares in a fund, a stock market index fund, an emerging markets fund, etc... However, mutual funds generally have a minimum investment of several thousand dollars, so if you have a relatively small amount of money invested (where small means less than $30-40,000), it’s essentially impossible for you to create a well diversified portfolio on your own.

Fortunately, most large brokerage firms provide a class of products to fill this hole in the form of target date funds, which are
mutual funds of mutual funds. Basically, they take the date you hope to retire and carry out the sort of optimization I cover in Section 6 to get approximate portfolio weights for various broad asset classes. Then they construct this portfolio by investing in the firm’s corresponding asset class mutual funds. They generally use a passive management strategy and update the portfolio weights roughly once a year, with the idea that you should take on less risk as you get closer to retirement.

These are great because there’s only the one minimum investment amount and you purchase a reasonably well diversified portfolio. Once you’ve accumulated a larger pile of money in your retirement account you can split the money up into your own preferred portfolio, but until then you have a simple and cheap investment scheme.

Personally, I have found that the Vanguard Target Date funds are very good, offering no loads and very reasonable expense ratios, but more and more companies are coming out with similar products, so check around.

### Table 1: Comparison of mutual funds and ETFs.

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Mutual Funds</th>
<th>ETFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>End-of-day price</td>
<td>NAV</td>
<td>NAV</td>
</tr>
<tr>
<td>Administrative costs</td>
<td>Expense ratio</td>
<td>Expense ratio</td>
</tr>
<tr>
<td>Intra-day trading</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Transaction costs</td>
<td>Loads or back end charges</td>
<td>Brokerage fees</td>
</tr>
<tr>
<td>Where traded</td>
<td>With management co.</td>
<td>As stocks in exchange</td>
</tr>
<tr>
<td>Taxes</td>
<td>During holding period</td>
<td>Only at final sale</td>
</tr>
</tbody>
</table>

#### 6.4 What’s a bad fund to choose?

In this section I’ll just compare three hypothetical but realistic funds based on historical returns, fee structure (loads), and expense ratios to give you an idea of what’s out there and what you should pay attention to.

For simplicity I first compare two funds that attempt to track a large blend index with a passive management style. Then we compare these two passive funds against an actively managed fund that explicitly attempts to beat the index. See Table 2 for the details of the various funds.

First, start with Fund A. It is an index fund based on the base index, a broad market index that focusses on large US stocks. As an index fund, Fund A simply tries to replicate the composition of the index, a nearly totally passive strategy. Because of this passive strategy the fund has low expenses: 0.15%. It’s difficult to do much better than this, and by comparing the returns on Fund A to those on the index, you see that the returns are quite close to those of the index minus the expense ratio.

Now, examine Fund B. It also invests in assets similar to those of the index, but it employs a semi-active investment strategy
that means it tries to beat the index. This strategy is costly, as reflected in the 1.20% expense ratio; does this cost pay off? Looking at the returns, the answer is a resounding no, since it failed to beat Fund A over any of the time periods shown and failed to beat the index over any but the 10 year period.

Finally, look at Fund C, which uses an active strategy to attempt to beat the index. Its expense ratio is lower than that of Fund B but still nearly six times that of Fund A. Plus it charges a 5.75% upfront load compared to the other funds’ 0, which means that it would take you about two years of the average 10-year return just to break even. Even without the loads it barely beats Fund B and Fund A beats it handily. Which one would you choose for your retirement income?

This is a simple example: all three funds invest in the same types of assets and differ only through fees and investment strategies. As we’ve seen, different types of assets have different risk and return characteristics, which makes it difficult to compare apples to apples. For example, a fund could generally beat the index just by investing in small rather than large stocks, but this doesn’t mean the fund manager is a genius, it just means he chose a different type of asset. Most critically, he also took on more risk. Thus, to see if a particular fund is successful, we have to consider risk-adjusted performance. We’ll consider this in more detail in the next section.

7 Measuring risk-adjusted performance

We can apply some of the ideas explored in Section 6 to develop some basic theory that predicts the market price of risk, thus allowing us to adjust portfolio performance for the varying levels of risk that a given portfolio can incur. This allows us to see if a portfolio is a good buy and summarize the information with a handful of statistics.

7.1 CAPM

If we assume that all investors employ the Markowitz portfolio optimization scheme of Section 6 and make several other assumptions, we can come up with a set of results known as the Capital Asset Pricing Model, or CAPM. The CAPM follows fairly simply from the Markowitz model once we assume that all investors use the Markowitz model and that they all have the same estimates for the covariances and expected returns of stocks. These assumptions (particularly the latter one) may seem excessive, but the model works well enough to give useful predictions, so people do use it.

Recall the fundamental result of the Markowitz model with risk-free lending: everyone holds a combination of the same market portfolio and the risk-free asset. This holds for every individual investor under the CAPM, but when we aggregate over all investors we see that the overall portfolio is the market portfolio itself. This is because for every dollar lent out at the risk free rate there has to be a dollar saved, so the individual weights on the risk-free asset cancel out and we are left with the market portfolio.

Recall further our definition of risk premium: the expected returns in excess of the risk-free rate required to encourage investors to hold risky assets. With the CAPM we can find the risk premium on the market by noting that the market standard deviation
Table 2: Comparison of three “large blend” funds: a large blend index itself and two hypothetical large blend funds employing a variety of strategies. The returns are annualized average returns calculated by dividing the 3 year return by 3, etc.

<table>
<thead>
<tr>
<th>Fund</th>
<th>Index</th>
<th>Fund A</th>
<th>Fund B</th>
<th>Fund C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expense ratio</td>
<td>N/A</td>
<td>0.15%</td>
<td>1.20%</td>
<td>0.90%</td>
</tr>
<tr>
<td>Upfront load</td>
<td>N/A</td>
<td>None</td>
<td>None</td>
<td>5.75%</td>
</tr>
<tr>
<td>Return 1 year</td>
<td>-7%</td>
<td>-5.00%</td>
<td>-6.00%</td>
<td>-13.50%</td>
</tr>
<tr>
<td></td>
<td>3 year</td>
<td>4.00</td>
<td>6.72</td>
<td>3.90</td>
</tr>
<tr>
<td></td>
<td>5 year</td>
<td>11.20</td>
<td>11.18</td>
<td>8.10</td>
</tr>
<tr>
<td></td>
<td>10 year</td>
<td>2.00</td>
<td>3.50</td>
<td>2.50</td>
</tr>
</tbody>
</table>

is $\sigma_M$ and its expected return $\bar{r}_M$. Thus the risk premium is $r_M - r_f$, and the premium per unit of standard deviation is

$$\text{Risk premium} = \frac{r_M - r_f}{\sigma_M},$$

often called the market price of risk.

Thus under the CAPM the riskiness of the stock is the only determinant of its expected returns, entering through the market price of risk, and for security $i$ we have

$$E[r_i] - r_f = \beta_i(E[r_i] - r_f),$$

where $\beta_i = \frac{\text{Cov}(r_i, r_M)}{\sigma_M^2}$ is the measurement of security $i$’s riskiness relative to the market portfolio. Of course it is impossible to directly measure the returns and standard deviation on the actual market portfolio, as it would include returns on all risky investments: stocks, privately held companies, real estate, etc. To facilitate the application of the CAPM we generally proxy the market portfolio with a broad index of stocks, such as the S&P 500 or the Wilshire 5000. Often a portfolio trying to beat some benchmark index, such as the Oppenheimer MSIGX fund above that tries to beat the S&P 500 will use the benchmark as its market portfolio.

Once we’ve decided on a market portfolio, we proceed by fitting the CAPM equation to the portfolio returns:

$$E[r_P] - r_f = \alpha_P + \beta_P(r_M - r_f),$$

which is just a line relating the excess returns on the portfolio to the excess returns on the market. Now we’re interested in the parameters from the equation, $\alpha_P$ and $\beta_P$, plus some statistics from when we fit the data to the model, in particular the standard errors on $\alpha$ and $\beta$ and the $R^2$ value.

The $R^2$ value is related to the magnitude of the residuals in the model, that is, the distance between the line and the data points we’ve used to fit it. Generally speaking, the value of $R^2$ tells you how much of the variation in the data is captured in the fitted model. That is, an $R^2$ of 0.88 or 88% means that 88% of the variation in the data is captured in the model. Higher $R^2$ values of course means the model more accurately captures its risk.

First, notice that $\beta_P$ tells us how risky the portfolio is compared to the benchmark: a
\( \beta \) of 1 means the portfolio is just as risky, less than one less risky and greater than one more risky. \( \alpha_P \) is a measurement of the risk-adjusted performance of the portfolio in that \( \alpha \) tells us how much better or worse than the market we did: \( \alpha > 0 \) means that the portfolio delivered excess returns that were greater than those on a market portfolio with the same level of risk. Thus, \( \alpha \) is a measurement of the portfolio manager’s skill. If you look at these statistics, you’ll generally find that \( \alpha \), whether positive or negative, is generally small, on the order of 0.5%. However, as we have seen, small changes in the rate of return can drastically affect the growth of your investments over time.

There is a caveat to these numbers in the form of their standard errors, which give an idea of how uncertain we are about \( \alpha \) and \( \beta \). Remember that the excess returns won’t fit the model of Equation 12 perfectly, so if the data were to move around a bit our estimates of \( \alpha \) and \( \beta \) could change. The standard error gives a simple idea of the magnitude of this uncertainty. \( \beta \) is generally relatively well estimated, but \( \alpha \) can be quite variable. Couple this with the small values of \( \alpha \) generally seen in practice and it can be difficult to say that \( \alpha \) is above zero with much confidence. Unfortunately, no one ever reports the standard errors in a prospectus, so it is generally impossible to make this correction. The best advice I can give is to look at \( \alpha \) over a long period of time.

There are several other risk-to-reward measures that exist, which I explain here because they are often included in portfolio risk profiles. First is the Sharpe ratio, given by

\[
S = \frac{\bar{r}_P - \bar{r}_f}{\sigma_P},
\]

which looks at the excess returns compared to the total risk of the portfolio. The other common ratio is the Treynor ratio, given by

\[
T = \frac{\bar{r}_P - \bar{r}_f}{\beta_P}.
\]

Since \( \beta_P \) captures the relative volatility of the portfolio compared to the market \( T \) is a measure of reward to relative risk. In either case, higher is better, since that implies greater rewards for a given level of risk.

### 7.2 Fama-French three factor model

The CAPM is a good model, but earlier we did notice some patterns in the returns on stocks, in particular big versus small and growth versus value securities. This could make the above results problematic if, say, a portfolio invested in value stocks but used a growth stock index as its benchmark. Then they’d come up with a positive alpha for no better reason than that they can look at simple historical data and take on different types of risk. Can we incorporate these two trends into our equation and do better than the CAPM?

The answer is yes, and professors Fama and French have added these factors, meaning that they use the model

\[
r_P - r_f = \alpha_P + \beta_3(r_M - r_f) + \beta_{SMB}SMB + \beta_{HML}HML,
\]

where \( SMB \) is the returns on small firms minus those on large firms and \( HML \) is the returns on high (book-to-market value, or value) minus low (book-to-market value, or growth) stocks. The \( \beta_3 \) of this model is similar to \( \beta_P \) from the CAPM Equation 12, but
is different since we have added the other two elements to the right hand side of the equation.

It is somewhat unusual to find the statistics from the Fama-French model in a portfolio prospectus, but again the portfolio performance is measured by $\alpha_P$, with values greater than zero meaning good performance. Most importantly, more and more sources of investment information are using these factors to classify mutual funds, index funds and ETFs, and many portfolios are designed to be “large value” or “small growth”, so the Fama-French model is important to understanding these products.

8 Making sense of the retirement plan soup

I’ve mostly been focussing on investing for retirement, since it’s the one thing that we all know we’ll want to be able to fund and have some idea when we’ll need the money. Of course all the information so far has been applicable to investing in general, but here I’ll cover retirement plans in detail. I know when I first waded in, the whole thing was pretty confusing.

These details vary widely from country to country, but I’ll focus exclusively on plans available for US residents.

8.1 What’s available

The available plans come in two types, each of which has two flavors: 401(k) plans and Individual Retirement Accounts (IRAs), each of which can be of the Traditional or so-called Roth variety.

The first difference is that 401(k)s are set up by employers for their employees, while IRAs are set up by individuals for themselves. Table 3 shows this distinction visually. For either type of plan, you invest in the Traditional flavor with pre-tax money, which reduces your taxable income for the year you make the investment, but you pay income taxes on the money when you withdraw it in retirement. With the Roth flavor of either type of plan, you pay taxes on the income now, before you invest it, and then it grows and is withdrawn tax-free.

Both plans have limits on how much money you can invest in any given year; for IRAs it’s the minimum of $5,000 or your total income for that year (at least for those under 50 for 2008), and for 401(k)s it’s the minimum of $15,500 or your total income for that year (again, for those under 50; you can contribute more if you’re over 50 as what’s termed a ‘catch up’ contribution).

The employer matching contributions I talked about in Section 3.1 generally apply to 401(k)s, not IRAs. Remember to contribute, if at all possible, enough to maximize the employer contribution. Roth 401(k)s are relatively new and so many employers are yet to offer them; if your employer doesn’t yet, bug them to start for the reasons I detail below.
8.2 Why Roth accounts are attractive

Generally Roth plans are better if you’re younger and have a smaller income since you’ll avoid paying higher taxes in the future (see Section 3.4 for a full discussion). Essentially, if you have a relatively small income now, your marginal tax rate is also small, so you’d be taxed higher on income in retirement. Furthermore, (and this seems obvious, but isn’t 100% guaranteed) we’ll have to pay down a seriously huge amount of government debt in the coming decades, so income tax rates are bound to go up, probably considerably. This only increases the attractiveness of the Roth account, particularly if you’re young.

9 Summary of what’s important

- Keep 3 months of salary on hand in case of emergency.
- Consider an overall life plan as part of formulating investment goals.
- If you have a plan with an employer contribution, contribute enough to max out the employer contribution.
- Start saving and investing early! Time is valuable when it comes to investing.
- Each person has a tolerance for risk which drives individual investment preferences. There is no universal right choice for a portfolio.
- Look at both returns and risk; that is, compare risk-adjusted returns.

- Avoid paying loads! In general, don’t pay excessive management fees on funds.
- Consider Roth retirement account options, particularly if you’re young.

10 Conclusion

This concludes our whirlwind tour of the world of personal investing. I’ve done my best to cover the things that are relevant to the majority of people, but there are definitely holes, which reflect the gaps in my personal knowledge. As one example, older readers will be particularly interested in the details of various types of life insurance, which I currently know nothing about.

As I’ve mentioned several times, I’ve focused mostly on investing for retirement because it’s the one expense we all know (hope?) we’ll have, and roughly when we’ll need to have the money ready. However, almost everything covered in the sections up to 7 are relevant to any kind of investing.

As always, one’s personal optimal investment strategy is dependent on personal preference for (or dislike of) risk, so use this article as a set of tools to evaluate your investment options rather than a recommendation for one sure-fire plan over another. I’ve done my best to give you everything you need.

Finally, there’s a great deal of further information out there for those of you who’d like to go farther: use the bibliography as a starting point for the literature. Best of luck to everyone, and I hope this article has been useful.
A Statistics review

A.1 Random variables

We all know what a variable is, you know, just a symbol say, \( x \), that stands for a quantity in an equation. However, generally, a variable stands for some quantity that we know for sure. For example, when we say \( y = x^2 \), we assume that we know what \( x \) is, say, 4.

What if we don’t know exactly what the value of the variable is, but only that it can take on some certain values and have some idea of the probability of each value? As an example, take a coin, and write the number 1 on the heads side and 0 on the other. Then let’s flip the coin and let \( h \) represent the value of the number that comes up. Then we have

\[
    h = \begin{cases} 
        1 & \text{if heads,} \\
        0 & \text{otherwise.} 
    \end{cases} \tag{14}
\]

If it’s a fair coin each of these possibilities has an equal probability \( p = 0.5 \), and then we can forget about heads and tails and rewrite (14) as

\[
    h = \begin{cases} 
        1, & p = 0.5 \\
        0, & p = 0.5. 
    \end{cases} \tag{15}
\]

Thus, a random variable is a variable that can take on various values, each of which has an associated probability of occurring.

We can extend this concept arbitrarily to any number of discrete outcomes, each with an associated probability, or even to a random variable that can take on any real number value. One such continuous random variable might obey the standard normal distribution, the well-known bell curve, which means that the probability that such a variable, call it \( y \), falls between \( y_1 \) and \( y_2 \) is

\[
    P[y_1 \leq y \leq y_2] = \int_{y_1}^{y_2} \phi(x) \, dx, \quad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]

See Figure 8 for a visual representation of this.

In investing we care about random variables because most of what we deal with, such as the rate of return on a particular stock, interest rates, or the inflation rate, is uncertain, but we can assign some probability to each possible outcome. Thus, we think of these rates as random variables. A good way to think of the accompanying probability distribution is how likely you think it is that any given outcome will happen.

A final note on notation: when I want to emphasize that a particular variable is random (another word you’ll see is stochastic) rather than known I use a tilde: \( \tilde{x} \).

A.2 Mean

There are a bunch of ways of describing what statisticians call central tendency, that is, the “average” of some set of numbers, including the mean, median and mode. The one we’re most concerned with is the mean, or what you probably think of “average”: add up all the numbers and divide by the number of numbers in your set. Thus, the mean of the set \( \{1,2,3,4,5\} \) is \( 15/5 = 3 \).

For a random variable \( x \) the mean \( \bar{x} \) is

\[
    \bar{x} = E[x] = \sum_{\text{outcomes}} P_{\text{outcome}} x_{\text{outcome}}.
\]
Figure 8: The standard normal curve and the corresponding probability that a standard normal random variable falls between -1.5 and -0.5.
or for the continuous case,
\[ \bar{x} = E[x] = \int_{-\infty}^{\infty} f(x)x \, dx, \]
where \( f(x) \) is the probability density function, for example \( \phi(x) \) for the normal distribution. In both of the previous equations I’ve used the expectations operator, \( E[\cdot] \), whose name gives you a good feeling of what it does: it tells you the value you should expect to get of a variable. Take for example our coin (14) and corresponding random variable \( h \). We have
\[ h = E[h] = \sum_{i=0,1} h_ip(h_i) = 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5. \]

We generally use the mean or expectations operator when answering questions about how we expect one of our random variables to perform; perhaps we have a range of possible values for interest rates and want to know what we should, on average, expect interest rates to be.

Finally, a useful property about the mean and the expectations operator is that it is linear: that is, if you have two random variables \( x \) and \( y \) and two known constants \( a, b \in \mathbb{R} \), then
\[ E[ax + by] = E[ax] + E[by] = aE[x] + bE[y]. \]
This property will come in handy when discussing portfolios of different assets.

A.3 Variance

While the mean was a measure of how the average bit of data behaved, it is also useful to have a measure of how spread out the set of numbers are around the mean. One such measure is the variance, defined as
\[ \sigma^2 = \text{Var}(x) = E[(x-E[x])^2] = E[x^2]-(E[x])^2. \]

Again working with our coin example, we have
\[ \text{Var}(h) = E[h^2] - (E[h])^2 = (0.5(1)^2 + 0.5(0)^2) - (0.5)^2 = 0.25. \]

To understand the practical significance of variance, suppose \( r \) is the rate of return on some investment. Then \( E[r] \) is the expected rate of return, and \( \text{Var}(r) \) is its variance, or spread around the expected rate of return. The higher the variance, the more wildly the rate of return is expected to vary. If you have an investment whose rate of return is varying wildly, you may get a queasy stomach watching the value of your asset fluctuating in time. Thus the variance of the rate of return on an asset is a measure of the risk of that asset. The square root of the variance, designated \( \sigma \), is the perhaps better known standard deviation. Notice that \( \sigma \) has the same units as \( x \), while \( \sigma^2 \) has units of \( x^2 \).

A.4 Variance of multiple variables

In the previous section we assumed we just had one random variable, for example the return on only one stock. Now what if we had several random variables, such as the returns on multiple stocks? We might want to buy some of those different stocks and then try to estimate the amount of risk we’re running with our portfolio of stocks. Thus, we need to calculate the variance of a linear combination of several random variables, for example
\[ y = a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_Nx_N. \] (16)

If the variance operator were linear like the expectations operator this would be easy. However, notice from the previous section that it isn’t: \( \text{Var}(2x) = 2^2\text{Var}(x) = 4\text{Var}(x) \).

If we define \( a \) as the column vector such that \( (a)_i = a_i \) and \( (x)_i = x_i \), then we can show that

\[ \text{Var}(y) = a^\prime \Sigma a, \] where (17)

\[ \Sigma_{ij} = E[(x_i - E[x_i])(x_j - E[x_j])] = \text{Cov}(x_i, x_j) = \sigma_{ij}. \]

We call \( \text{Cov}(x_i, x_j) \) the covariance of variables \( x_i \) and \( x_j \).\(^7\) Note that \( \text{Cov}(x_i, x_i) = \text{Var}(x_i) \). One can easily show (17) by applying the formula for variance to the linear combination of variables (16). Finally, \( \Sigma \) has at least two nice properties: it is 1) symmetric, and 2) positive-definite. The first property follows obviously from the definition (17), and the second comes from the fact that variances must be non-negative.

We can use the covariance to get one final useful measure for how two random variables are related: the correlation coefficient, defined as

\[ \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}. \]

It can be shown that \( \rho \) is bounded by \(-1\) and 1, with 1 (or \(-1\)) only being possible when the two variables move in the exact same (or exactly opposite) directions. Thus, \( \rho_{xx} = 1 \) and \( \rho_{x,-x} = -1 \), but in general \( \rho \) will be in the interval \((-1, 1)\).

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\(^7\)For those of you out there who took general relativity or differential geometry, obviously this is a different concept.

References


