YALE UNIVERSITY
Faculty of Engineering

Communication Complexity of Cooperative Control

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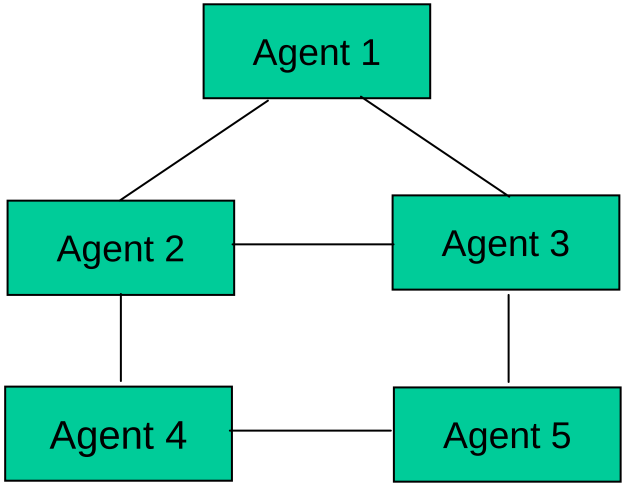
Understanding what is Useful Information



Goal: $x_i(t) \rightarrow$ average for each agent i

Consider consensus problem:

- Traditionally assume perfect observation of neighbor's state (either through sensing or explicit communication)

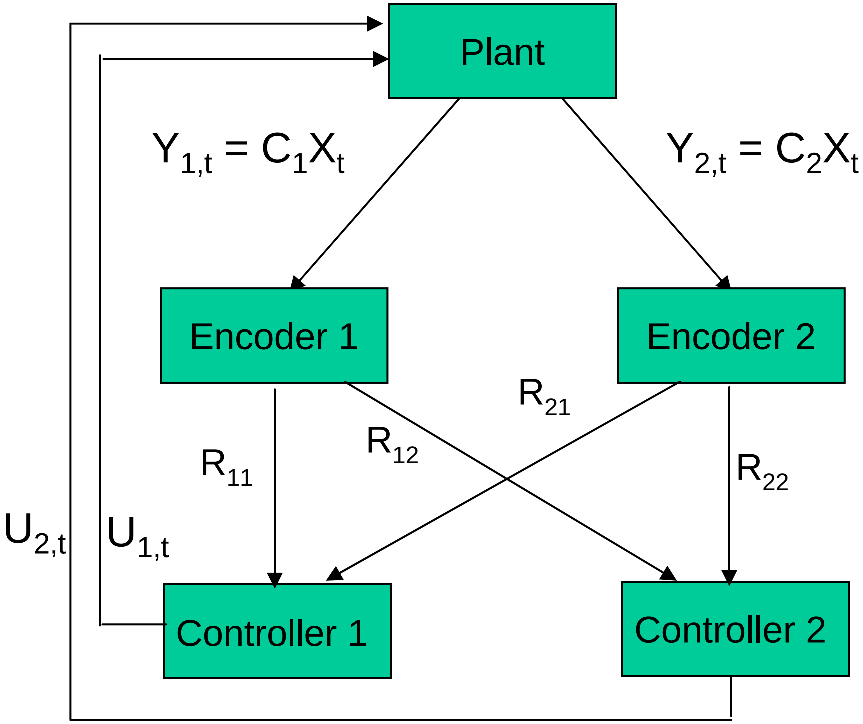
- What information is needed to insure coordination (e.g. achieving consensus)?

- What is information? What is the minimum amount of information needed to achieve goal?

- Information theory: we use the rate distortion methodology

$\rightarrow$ Identifying minimum communication requirements leads to better understanding of coordination

Main Problem



- Plant dynamics:

  $X_{t+1} = AX_t + \sum_k B_k U_{k,t}$

  with S encoders (sensors) and K controllers

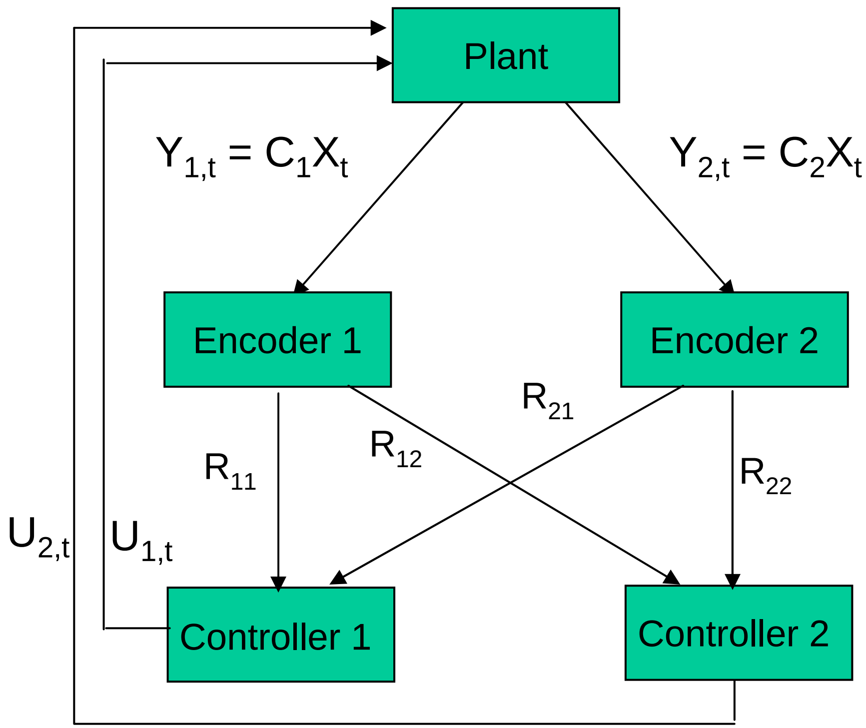
- Sensor: $Y_{s,t} = C_s X_t$.

- Action: $U_{k,t}$ depends on information sent from different encoders.

- Find rate region and encoder and controller policies that insure, for example, stability.

- Consensus: 2E + V encoders

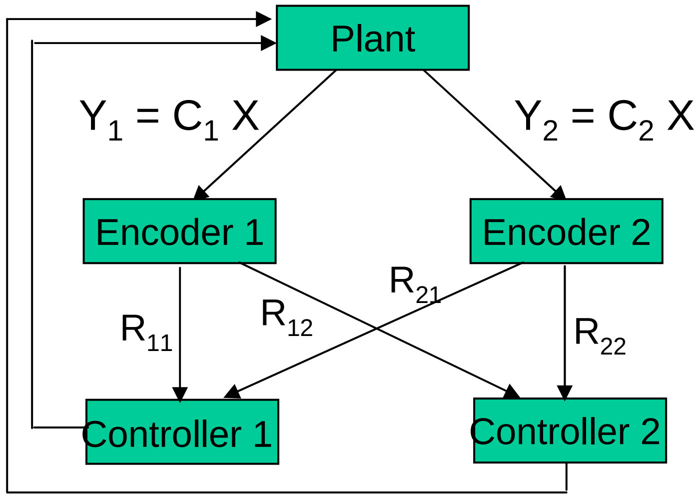
Main Problem Refined



- Let S = the number of sensors and K = the number of controllers.
- $X_{t+1} = (A + \sum_k B_k (\sum_s I_{s,k} K_{s,k} C_s)) X_t$
- Assume $\exists$ stabilizing controllers $\{K_{s,k}\}$ (under $R_{s,k}=\infty$)

- Find rate region $\mathcal{R}$ such that system is asymptotically stable under "certainty equivalent" controllers. (Separation between estimation and control. Not source and channel.)

Information Theoretic Techniques



- Converse:
  - Directed data processing inequality
  - In interest of time we will only talk about achievable schemes

- Direct:
  - High rate quantization and successive refinement
  - Lossy source coding with side-information at the receiver
  - Slepian-Wolf coding

Outline

- S=1, K=1
- S is general, K=1
- S=1, K is general
- S and K are general
- S=K is general and nested connectivity

Achievability: No Control

$X_{t+1} = A X_t$ with full observation: $Y_t = X_t$ Goal: state estimation

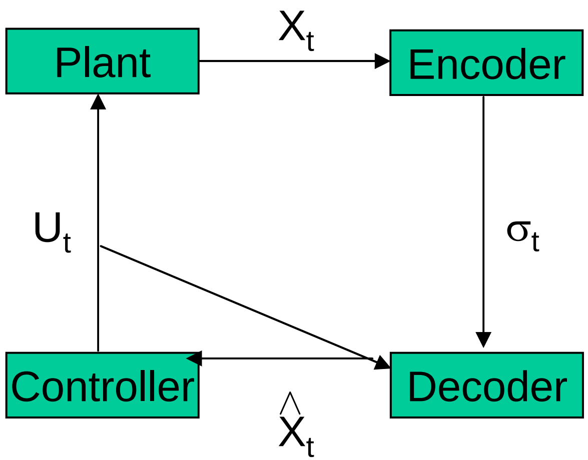
Main idea: compute innovation at encoder. Encoder knows decoder's state estimate:

$$X_t - \hat{X}_{t|t-1} = A X_{t-1} - A \hat{X}_{t-1} = A e_{t-1}$$

Proposition: For bounded initial set Λ_0 a sufficient condition for asymptotic observability is $R > \sum_{\lambda(A)} \max \{0, \log |\lambda(A)| \}$.

Rate of convergence: $|| e_t ||_2 \leq \kappa\, 2^{-\alpha t}$
where $\alpha = \min_i (R_i - \log |\lambda_i(A)|)$

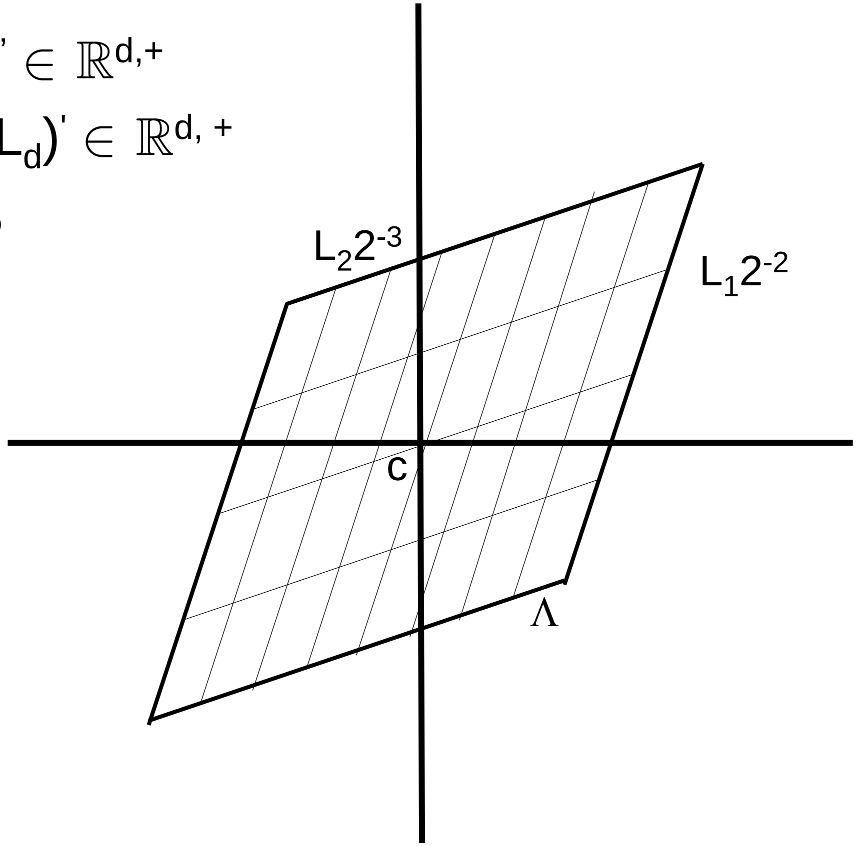
WLOG use uniform quantizer



Quantizer

A *quantizer* is a four-tuple $(c, \underline{R}, \underline{L}, \Phi)$:

- Centroid: $c \in \mathbb{R}^d$
- Rate vector: $R = (R_1, \ldots, R_d)' \in \mathbb{R}^{d,+}$
- Dynamic range: $L = (L_1, \ldots, L_d)' \in \mathbb{R}^{d,+}$
- Coordinate transformation: Φ



$R = \sum_i R_i$

Boxes? View as high-rate (low distortion) lossy source coding.

Achievability: With Control

$X_{t+1} = AX_t + BU_t$ with full observation $Y_t = X_t$

Idea: source-coding with side-information at the decoder.

Before we quantized the innovation. Now we should *bin* X_t:

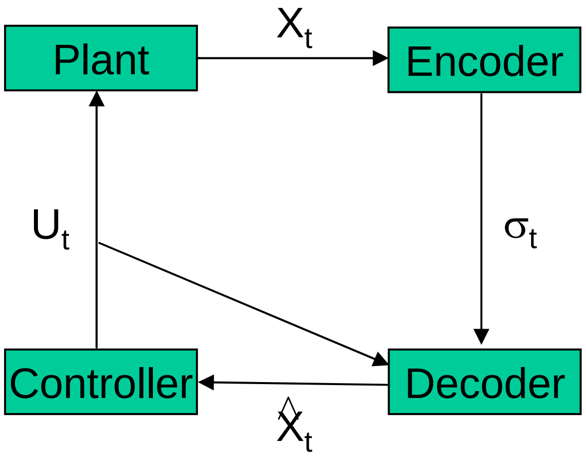
$$X_t = (A\,\hat{X}_{t-1} + B\,U_{t-1}) + A\,e_{t-1}$$

Term in parenthesis known to Rx.

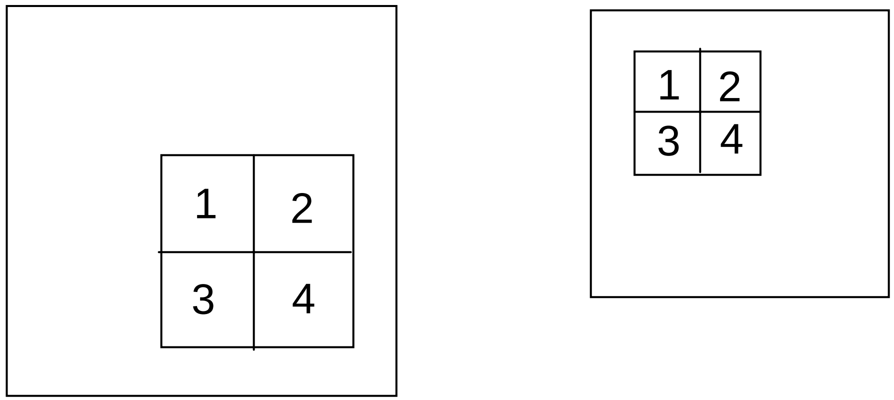
Rate condition sufficient for asymptotic observability and stabilizability:

$R > \sum \max\{0, \log|\lambda(A)|\}$.

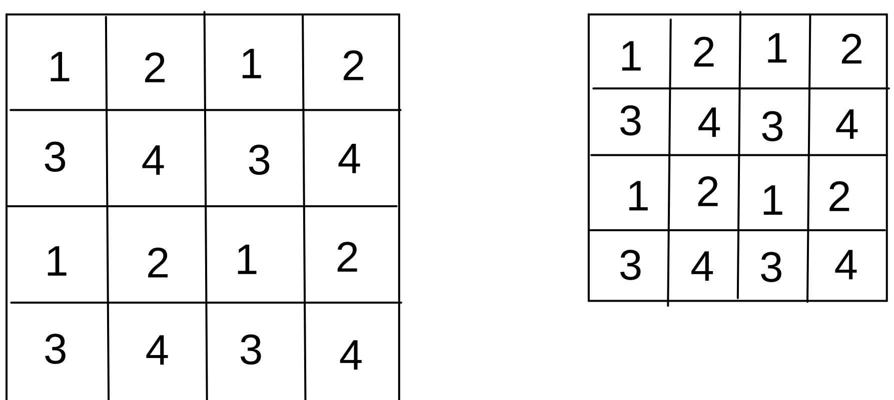
Non-nested information patterns!



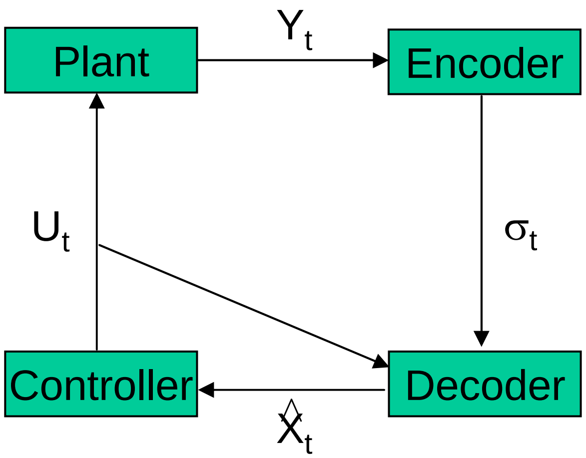
Quantizer Example – Binning



Encoder: dynamic range, primitive quantizer tiling, resolution



Output



$Y_t = CX_t$

(A, C) observable

$\exists$ matrices β, γ:

$\gamma\, [Y_{t-d+1}, \ldots, Y_t] = (A\, \hat{X}_{t-1} + \beta\, [U_{t-d+1}, \ldots, U_{t-1}]) + Ae_{t-1}$

Term in parenthesis known to Rx. Hence bin:

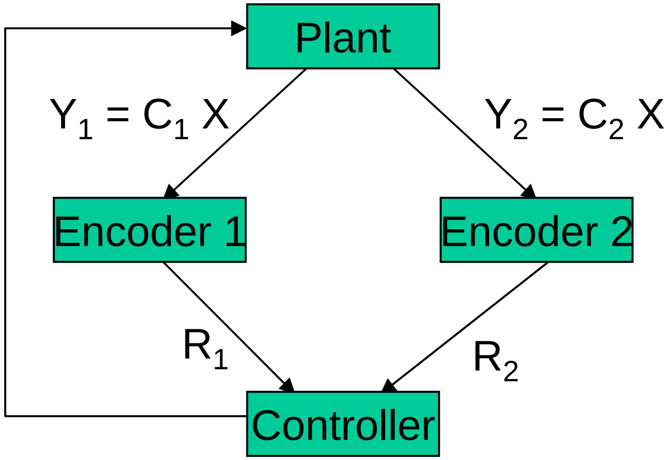
$\gamma\, [Y_{t-d+1}, \ldots, Y_t]$

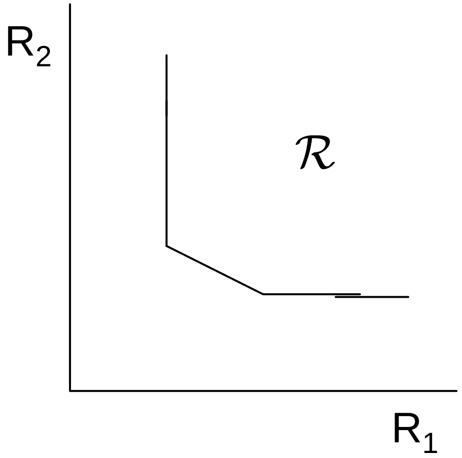
One can also treat process disturbances

Outline

- S=1, K=1
- **S is general, K=1**
- S=1, K is general
- S and K are general
- S=K is general and nested connectivity

Multiple Sensors





S sensors and one controller:

$X_{t+1} = A X_t + BU_t$, $Y_{s,t} = C_s X_t$, $s=1,\ldots,S$

The system is jointly observable but each individual (A,C_s) may not be observable.

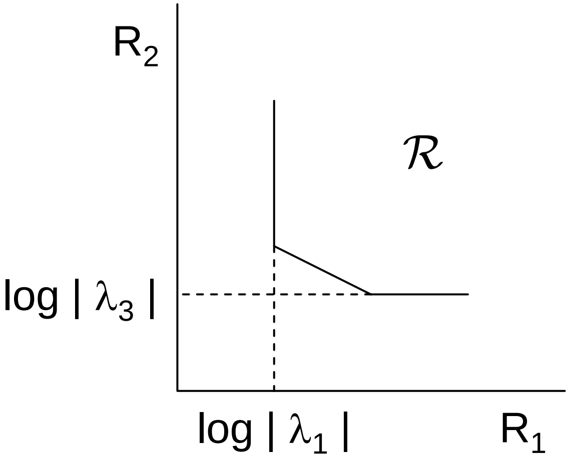
What rates are needed?

Rate Region: Example

- Example:

  $A = \text{diag}[\lambda_1, \lambda_2, \lambda_3]$.

- Let $C_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$ $C_2 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$



- Encoder 1 sees modes: λ_1, λ_2 and encoder 2 sees modes: λ_2, λ_3

- Then: $R_1 + R_2 > \log |\lambda_1| + \log |\lambda_2| + \log |\lambda_3|$

  $R_1 > \log |\lambda_1|$

  and $R_2 > \log |\lambda_3|$

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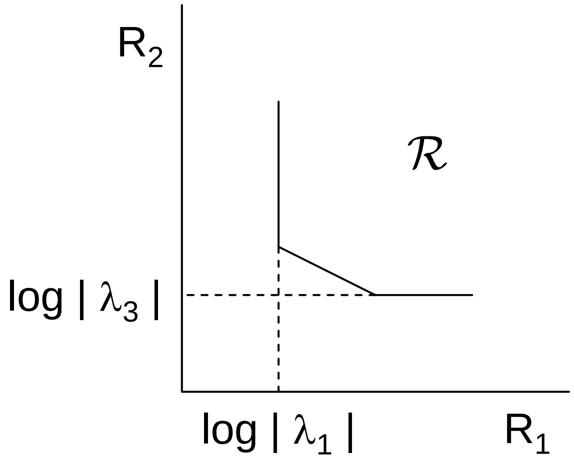
Rate Region: General Case

For each m let $\mathcal{O}_s$ be the observable subspace (quotient space corresponding to A-invariant unobservable subspace)

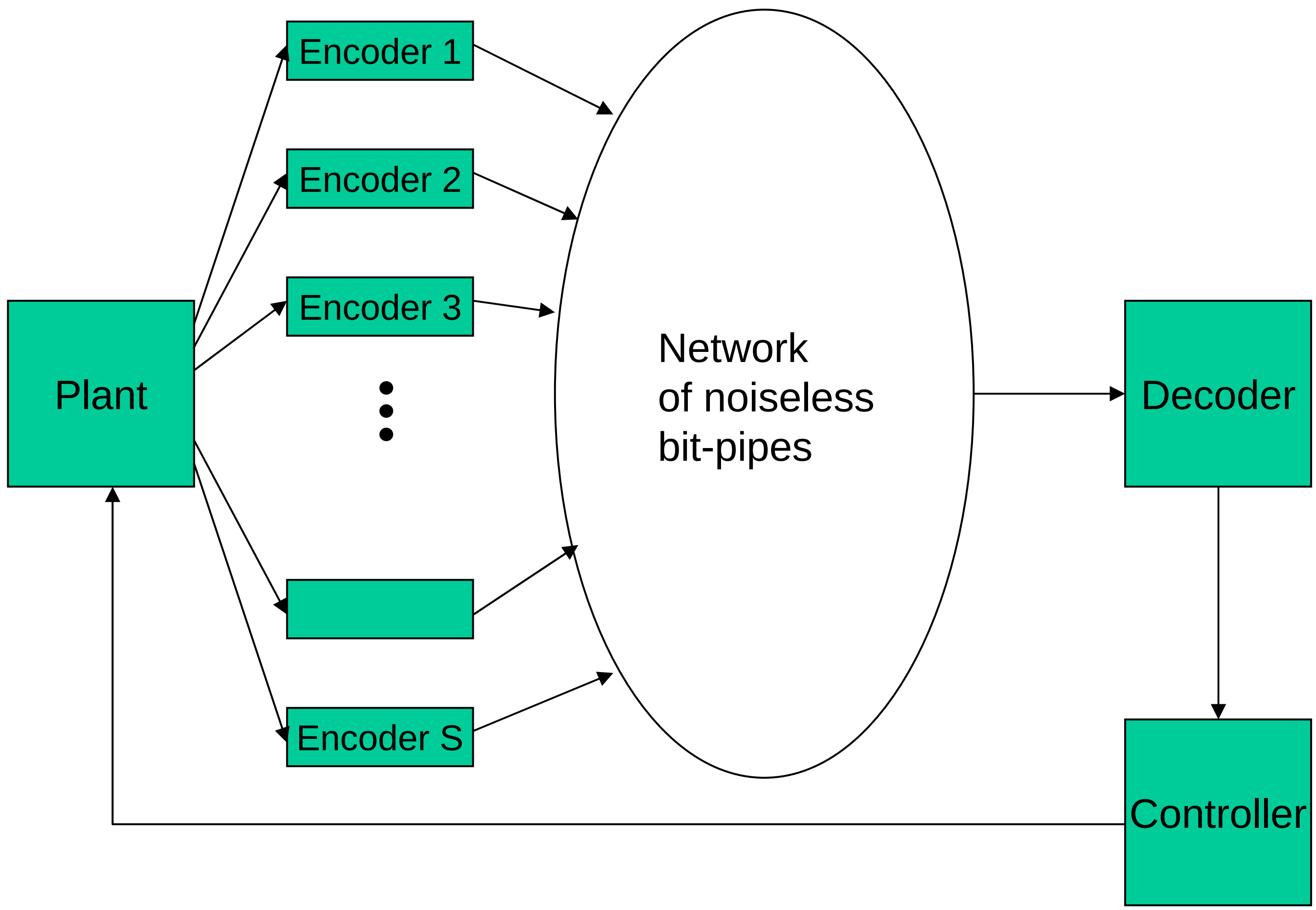
Λ_s = { $\lambda(A)$: those eigenvalues of A corresponding to the subspace $\mathcal{O}_s$ }.



Let $\mathcal{R} = \{(\underline{R}_1,\ldots,\underline{R}_S) : \sum_{s:\ \lambda\in \Lambda_s} R_{s,\lambda} > \max\{0, \log|\lambda|\}, \forall\ \lambda(A)\}$ where $R_{s,\lambda}$ is the rate assigned by sensor s to mode λ. (Slepian-Wolf conditions.)

Proposition: A necessary and sufficient condition on the rate vector for asymptotic observability and stabilizability is $(\underline{R}_1,\ldots,\underline{R}_S) \in \mathcal{R}$

General Multiple Sensor Set-up



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- Review S=1, K=1
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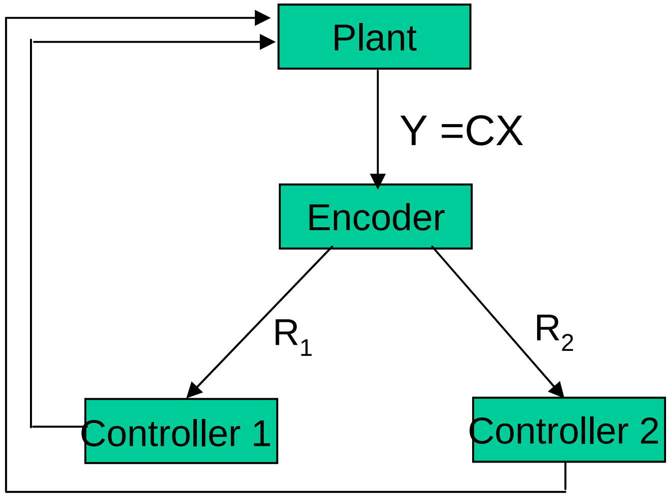
S=1, K is General



$X_{t+1} = (A + \sum_k B_k K_k C) X_t$

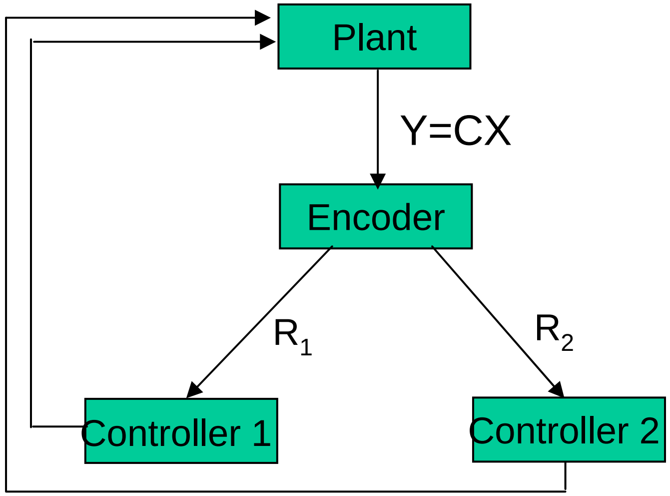
Assume stablizing controllers $\{K_k\}$ (under $R=\infty$)

- Lower bound on rate is $\sum_k R_k > \sum \max\{0, \log|\lambda|\}$. Is this achievable? What about each $R_k > \sum \max\{0, \log|\lambda|\}$?

- Find rate region $\mathcal{R}$ such that system is asymptotically stable.

- For convenience assume A is diagonal and the $\{C_k\}$ matrices project onto standard coordinates.

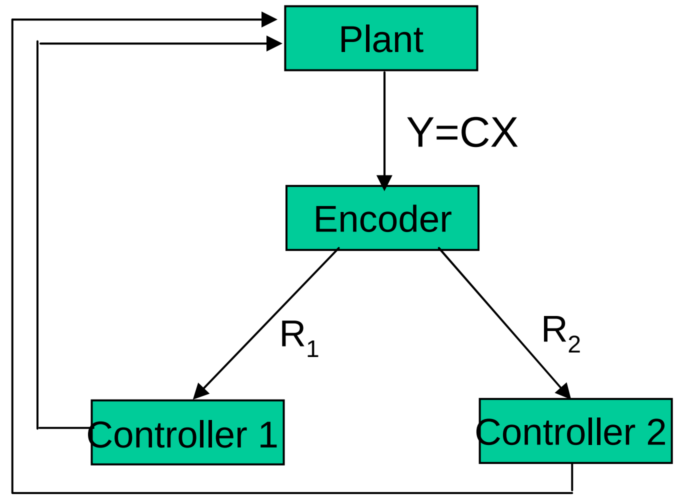
S=1, K is General -- part 2



Potential problem: controller 1 does not know action 2. Will binning story work?

- Idea let controller i: $\hat{x}(i)_{t+1|t} = A\hat{x}(i)_t + \sum_k B_k K_k C\ \hat{x}(i)_t$
- Hence $e(i)_{t+1} = F\,[\,Ae(i)_t + \sum_k B_k K_k C\ (\hat{x}(k)_t - \hat{x}(i)_t)\,]$
- Where $F = \mathrm{diag}(\{\,2^{-R\lambda}\,\})$
- FA stable. What about the other term?

S=1, K is General -- part 3



- $e(i)_{t+1} = F\,[\,Ae(i)_t + \sum_k B_k K_k C\,(e(i)_t - e(k_t))\,]$
- Can show relative error: $\hat{x}(k)_t - \hat{x}(i)_i = e(i)_t - e(k)_t$
- If FA stable then absolute error $e(i)_t \to 0$
- **Proposition:** $R_k > \sum_\lambda \max\{0, \log|\lambda|\}\ \forall k$ is sufficient for stabilizability under the controllers $\{K_k\}$.

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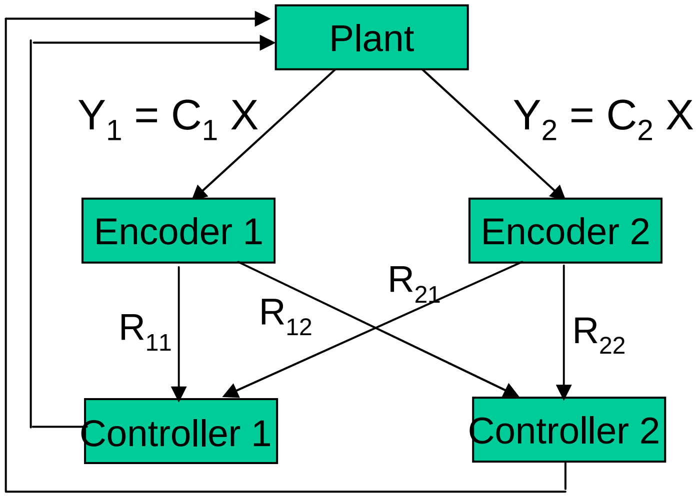
S and K are General



- Combine Slepian-Wolf coding with binning (source-coding with side-information) technique. Now many controllers. Each controller should update its estimate as before.

- As before each encoder needs to send information about its observable modes.

Proposition: If $\{R_{s,k}\}$ satisfy the Slepian-Wolf conditions for each k then the system is stabilizable under the controllers $\{K_{s,k}\}$.

Outline

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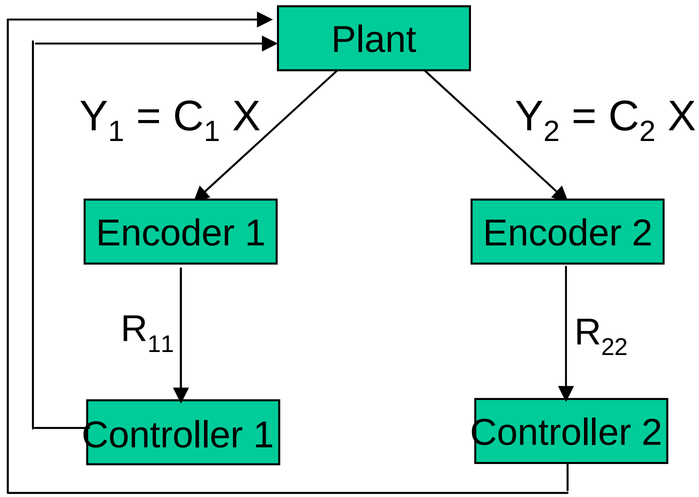
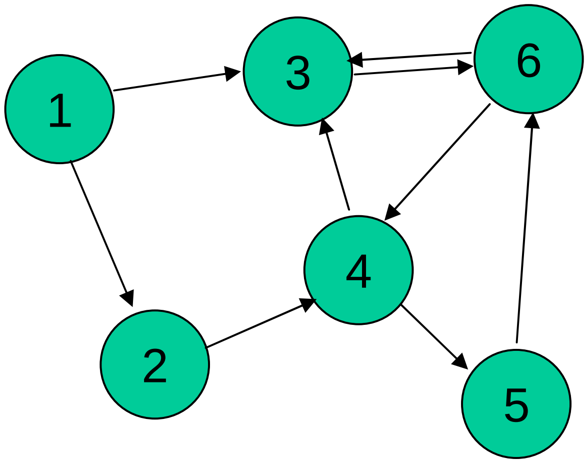
Who sees what?

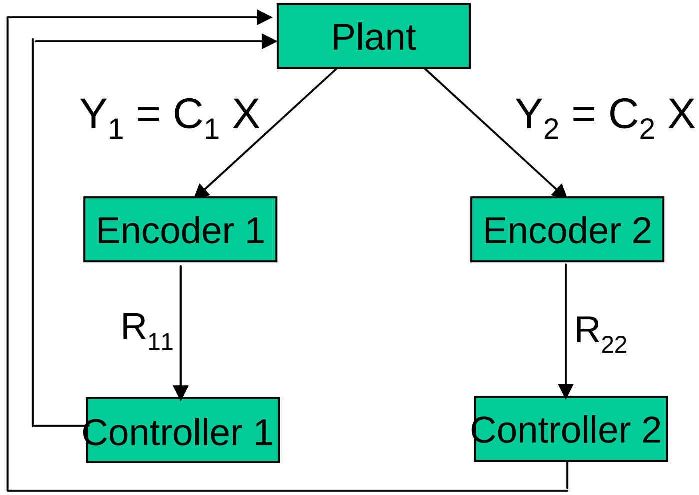


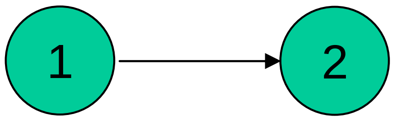


- Can we do better than full rate on each link? Hard question in general. Don't need full state estimate at each controller.
- Here $x_{t+1} = (A + \sum_k B_k K_k C_k)\, x_t$ under perfect channels.
- Examine connectivity graph of controllers: specifically $i \rightarrow j$ if

  $C_j (zI - A)^{-1} B_i \neq 0$. The action of i is observed by j.

Nested Connectivity – Example



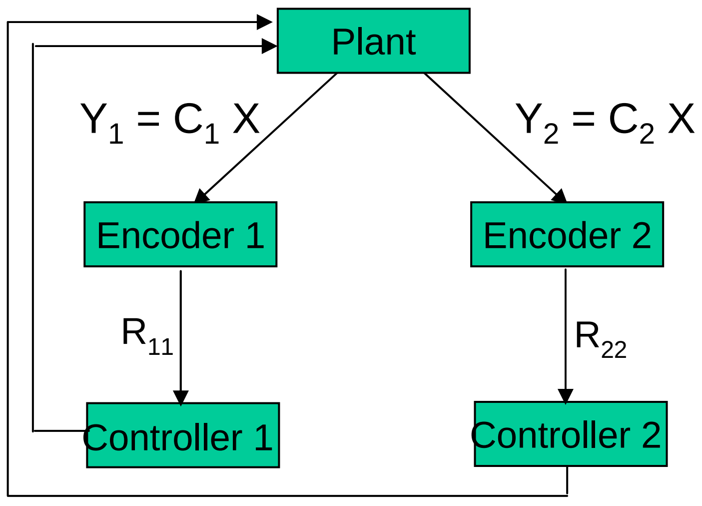


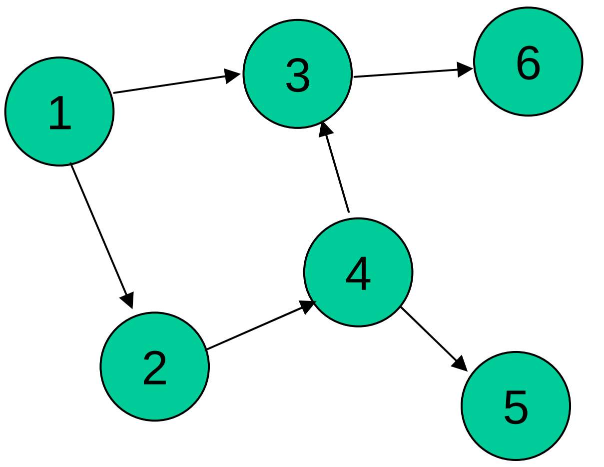
- Assume: $C_1 (zI - A)^{-1} B_2 = 0$. The actions of controller 2 are not observed by controller 1. $A + B_1 K_1 C_1 + B_2 K_2 C_2$ is stable.

- Controller 1: $e(1)_{t+1} = F_1 [\, A\, e(1)_t + B_2 K_2 C_2 \, (e(1)_t - e(2)_t) \,]$
  Hence $C_1 e(1)_{t+1} = C_1 F_1 A\, e(1)_t \rightarrow 0$

- Controller 2: $e(2)_{t+1} = F_2 [\, A\, e(2)_t + B_1 K_1 C_1 (e(2)_t - e(1)_t) \,]$
  Hence $e(2)_{t+1} = F_2 [\, A + B_1 K_1 C_1]\, e(2)_t \rightarrow 0$

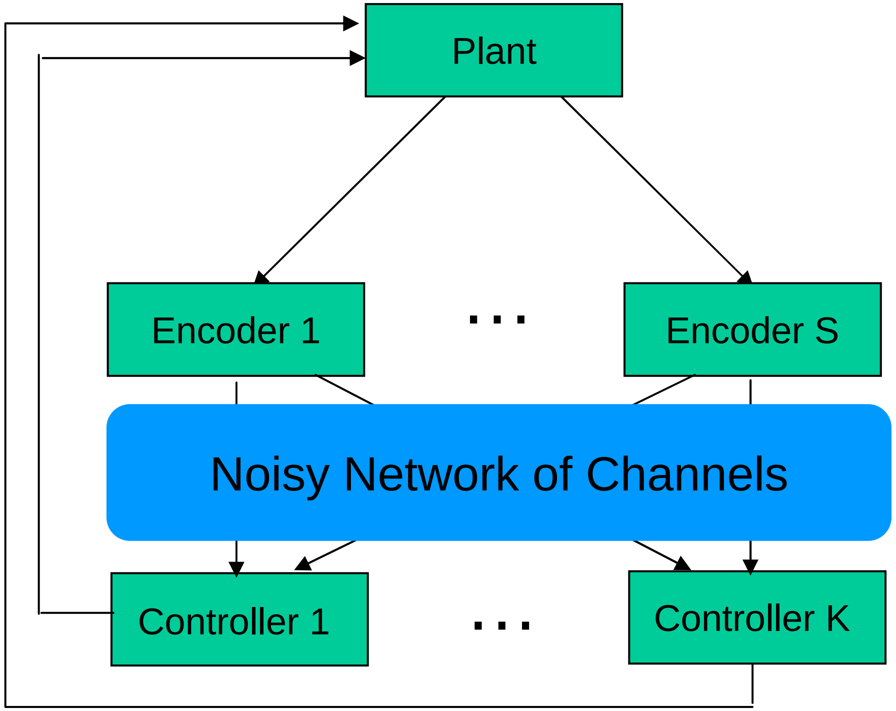
Nested Connectivity – General Case





- Assume that connectivity graph is a DAG (no directed cycles.)
- $e(i)_{t+1} = F_i [A\, e(i)_t + \sum_k B_k K_k C_k (e(i)_t – e(k)_t)]$
- For $C_i\, e(i)_t \rightarrow 0$ we will need:
  $F_i(A + \sum_{k \text{ in } pa(i)} B_k K_k C_k)$ to be stable over the modes of C_i.
- Could be that some of the rates are zero.

Open Challenge



Thank You

http://www.pantheon.yale.edu/~sct29

For More Information:

“Markov Control Problems Under Communication Constraints," Borkar, Mitter, and Tatikonda. CIS, January 2001.

“Optimal Sequential Vector Quantization of Markov Sources," Borkar, Mitter, and Tatikonda. SICON, January 2001.

“Control Under Communication Constraints," Tatikonda and Mitter. IEEE-TAC, July 2004.

“Control Over Noisy Channels," Tatikonda and Mitter IEEE-TAC, July 2004.

“Stochastic Linear Control Over a Communication Channel," Tatikonda, Sahai, and Mitter. IEEE-TAC, September 2004.

“Control Over Networks," Tatikonda. 2002 CDC

“The Sequential Rate Distortion Function and Joint Source-Channel Coding with Feedback," Tatikonda. 2003 Allerton Conference

“Some Scaling Properties of Large Distributed Control Systems," Tatikonda. 2003 CDC

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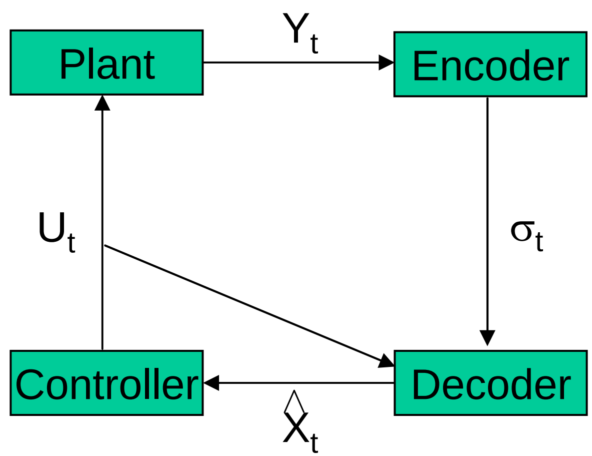
S=K=1, Lower Bounds on the Rate

Let the *state estimation error* be $e_t = X_t - \hat{X}_t$

At time t we can only distinguish between 2^{tR} initial positions hence the need for these definitions:

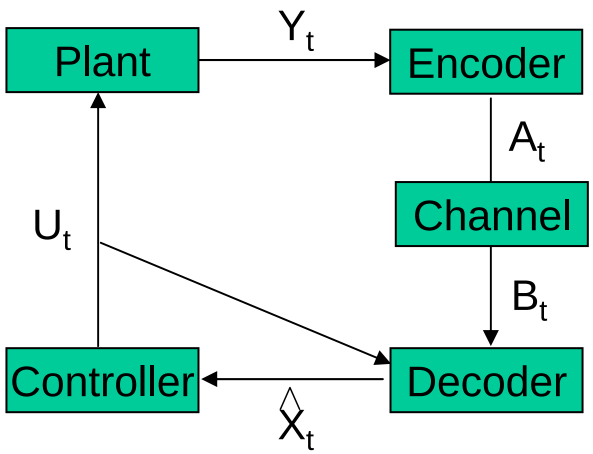


Asymptotic observablility: if there exists an encoder and decoder such that the following holds for any control sequence: $\| e_t \|_2 \to 0$.

Asymptotic stabilizability: if there exists an encoder, decoder, and controller such that: $\| X_t \|_2 \to 0$.

Proposition: A necessary condition for asymptotic observability and asymptotic stabilizability is: $R \geq \sum_{\lambda(A)} \max \{0, \log | \lambda(A) | \}$.

Lower Bounds on the Rate – Part 2



Asymptotic observability can be viewed as a high rate rate-distortion problem with $R_t(D) \approx \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\} - 1/t \log \mathrm{vol} S(D)$.

Standard DPI: $C \geq R(D)$. Here we have feedback via control (and potential explicit feedback).
The directed DPI: $I(X^T \to \hat{X}^T) \leq I(A^T \to B^T)$.

Proposition: A necessary condition for asymptotic observability and asymptotic stabilizability is $C \geq \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$