Optimal Quarantining of Wireless Malware Through Power Control

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***Abstract*—The topic of malware propagation in mobile wireless networks is gaining momentum among the research community, as actual vulnerabilities are revealed through recent outbreaks of worms. We introduce a defense strategy that quarantines the malware by reducing the communication range. This counter-measure faces us to a trade-off: reducing the communication range suppresses the spread of the malware, however, it also negatively affects the performance of the network as the end-to-end communication delay increases. We model the propagation of the malware as a deterministic epidemic. Using an optimal control framework, we select the optimal communication range that captures the above trade-off by minimizing a global cost function. Using Pontryagin's Maximum Principle, we derive structural characteristics of the optimal communication range as a function of time for two different cost functions.**

I. Introduction

Malicious computer softwares, in the forms of viruses and worms have proved to be able to inflict enormous damages on the computer networks in internet. For instance, during an outbreak of Code Red on July 19, 2001, hundreds of thousands of computers were infected in a blazing speed, forcing billions of dollars for repair [1]. Worms, as self-replicating codes, have the potential of exploiting their infected hosts to infect other nodes and exponentially multiply the number of their victims: a phenomenon that we call an epidemic. Thus detection and containment of malware in internet has drawn substantial attention among the internet research community ([1], [2], [3] etc).

However, a new battle-field has emerged: personal mobile devices such as cellphones, smartphones and pocket-PCs are acquiring more computation and communication capabilities, and thence, new vulnerabilities are introduced. The sprouting popularity of these mobile devices combined with their new capabilities has created an ideal prey-ground for future malware [2], [4]. In wireless networks, since resources are scarce, worms can cause new forms of havoc which was not of concern in the wired networks. For instance, the functionality of nodes is limited by the lifetime of their batteries, which can be rapidly depleted by the elevated activity of an infected node which tries to find new hosts. For example, Cabir worm, which hit the mobile phones in June 2004, drains the battery due to constant bluetooth scanning [5]. Also, as the media in wireless networks is common, bandwidth is constrained. The increased rate of attempts to access the media by infected nodes can jam the media and can further disturb the functionality of the network [6]. The dimensions of the threat become more alarming when we consider the huge investments that are directed towards wireless communication infrastructure and the economical liability that is built upon it. The viability of these investments is contingent upon designing effective detection and containment strategies.

In this paper, we focus on the containment of infection in a mobile wireless network. As we pointed out, several wireless properties enhance the severity of the infection. However, these unique features can also be utilized to contrive new countermeasures against the spread of the infection. An infected node can transmit its infection to another node only if they are in communication range of each other. We propose to quarantine an infection by regulating the communication range of the nodes. Specifically, the reception gain of the healthy nodes can be reduced to abate the frequency of contacts between the mobile nodes and thus suppress the spread of the infection. In fact, there is an interesting analogy between the spread of a worm in mobile wireless networks and a biological epidemic in a human community. During a biological virus outbreak, individuals might choose to restrain their contacts with the rest of the society. This abstinence decreases the chance of getting infected at the expense of deterioration in the quality of life: a decrease in the rate of communication between the members of the society hampers their ability to fully perform their daily tasks [7]. Such a trade-off also exists in the case of a mobile wireless network: reducing the communication range of nodes can deteriorate the QoS offered by the network, as the end-to-end communication delay increases.

We present a containment strategy based on power control. We propose an optimal control framework to characterize the trade-off between the containment efficacy and communication capabilities of the nodes (section III). Using Pontryagin's Maximum Principle, we devise a framework for computing the optimal communication range as a function of infection level in the network. We identify several structural characteristics of the optimal solution by examining the analytical properties of the solution (section IV). Specifically, for the important

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case of a linear cost function (subsection IV-A), we show that the optimal solution has the classical bang-bang structure, i.e., it is only at its minimum or maximum values. We prove that the optimal solution in this case has at most two (abrupt) transitions between these extreme values and always returns to its maximum value towards the end of the operation time. Subsequently, we consider a nonlinear cost function (subsection IV-B) and we establish that the optimal solution follows a similar structure, with the exception that transitions are smooth instead of being abrupt.

II. Literature Review

First, we present a concise literature review on the topic of modeling of worm propagation. We focus on the mathematical modeling and analysis of worms and viral epidemics. For an up-to-date discussion of propagation, detection and containment of malware in mobile networks from a practical viewpoint, one can consult with [8]. An engaging historical review of major recent malware outbreaks in networks with a discussion of their trends is provided in [9].

Most of the literature on worm propagation traditionally assume a wired network framework and also chiefly, the underlying network is the internet. Deterministic epidemiological models are used to investigate the propagation of malware in computer networks [1], [2], [10], [11], [12], [13], [14], [15]. [16] combined a deterministic worm propagation model with a game theoretic process that involves learning, in order to incorporate decisions of users about whether to install or uninstall a security patch in a wired network. Controlling the spread of the worm by reducing the rate of communication of nodes (i.e., rate-control-based measures) [17], [18], or the number of communications [3] is the closest analog in the wired networks to reducing the communication range of the nodes in the wireless networks. The work in [17] is based on heuristics and simulations. Also, unlike our work, [18] does not propose a formal framework for attaining desired trade-offs. Moreover, [18] only considers a static choice of the reduced communication rate, whereas we allow the communication range of the nodes to be dynamically modified over time as the level of infection evolves. Recently, [3] has proposed a stochastic branching process to model the early phase of worm propagation. [3] develops a worm containment strategy which limits the total number of distinct contacts per node over the containment cycle. However, this work only applies to the initial phase of infection and their countermeasure is ineffective once the epidemic starts.

A malware propagation model based on queueing theory was proposed in [19] to investigate the problem of malware attack from the viewpoint of the attackers which have energy constraints. The authors in [19] introduced an attack strategy where the attackers dynamically adjust their transmission power and they investigated the trade-offs between the instantaneous attack efficacy and the lifetime of the attackers and proposed heuristic power control strategies for the attackers. Note that in contrast, our communication range control policy is developed as a defense strategy, and it provably minimizes the overall cost of the system.

Interestingly, only a few papers (e.g. [20]) consider the control of worm propagation in communication networks (wired as well as wireless networks) as an optimal control problem. Optimal control has however been used as an effective tool to develop strategies to counter the spread of a biological or social epidemics. Nonetheless, most of these works mainly focus on immunization and/or screening policies and the monetary costs which are inflicted on the system. Introduction of our new countermeasure policy in the framework of mobile wireless network results in a new optimal control problem that requires an original analysis and previous results in [20], [21], [22], [23], [24] do not apply here.

III. System Model

To begin, let us introduce some terminologies. A node is called **susceptible** if it is not contaminated by the worm, but is prone to infection. A node is **infective** if it has the worm. Infective nodes can propagate the worm through communication with susceptible nodes. Any two nodes can communicate if they are within certain distance of each other which we refer to as their communication range. When two nodes are in communication range of each other, we say they are in *contact.* We assume that the worm cannot change MAC parameters such as transmission or reception gains.

Either the user of an infected device or the network operator removes the infection of the node by installing a security patch, which also grants the node permanent immunity against that threat. However, this does not take place immediately upon infection, but rather after an exponentially distributed random delay with mean $1/\gamma$. We use the term **recovered** for the infective nodes which receive the security patch.

We propose a system level defense policy that is specific to wireless networks. Assume that the reception gain of the susceptible nodes is a variable controlled by the system. Upon detection of malicious behavior, the reception gain of the susceptible nodes can be reduced. This effectively reduces the communication range of the nodes to lessen the frequency of contacts between the infective and susceptible nodes. This reduces the rate of propagation of the infection, thus extending the available time for recovering the infective nodes. Note that the communication range depends on both the transmission and the reception gains of two communicating nodes and reduction of any of the these gains reduces the the communication range.

Reducing the communication range, however, can adversely affect the performance of the network, as it undermines the ability of the nodes to deliver their own traffic and the end-to-end delay increases. This trade-off can be captured through a cost function, as we explain later in this section. After (1) mathematically characterizing the effect of changing the communication range of the nodes on the dynamics of the system and (2) constructing a meaningful cost function which captures the advantages and disadvantages of changing the communication range, the problem will be well-defined. The objective then will be to find the optimum transmission range as a function of the infection level in the system which evolves with time.

Here we address the first task: we analytically investigate the effect of changing the communication range on the propagation dynamics. Nodes are assumed to move in a limited region (of area A) and according to mobility models such as random waypoint or random direction model [25]. Also the communication range (u) is small compared to A, and speed of the movement is sufficiently high. It is shown ([26]) that under such circumstances, the pairwise meeting time is nearly exponentially distributed and the rate of pairwise meeting, which is the rate at which a pair of a susceptible and an infective node contact, is estimated by the following:

$$\text{rate of pairwise meeting} \approx \frac{2wuE[V^*]}{A}$$

where w is a constant factor pertaining to the specific mobility model, and $E[V^*]$ is the average relative speed between two nodes. We assume that when a susceptible and an infective node are in contact, the infection is transmitted to the susceptible node with a fixed probability. We assume that all of the parameters of the system other than the communication range is fixed. Thus, we can represent the rate of transmission of the infection from an infective to a susceptible node as $\hat{\beta}u$.

Let N be the total number of nodes, $n_S(t)$ the total number of susceptible nodes and $n_I(t)$ be the total number of infected nodes at time t. Following the conditions we assumed for the model, the numbers of the infective and recovered nodes evolve according to a Pure Jump Markov Chain. Let the rate between state $\sigma_1(t)$ and $\sigma_2(t)$ in that Markov Chain be denoted by $\rho(\sigma_1(t), \sigma_2(t))$, where the state of the Markov Chain is the triplet $(n_S(t), n_I(t), n_R(t))$. Thus we have:

$$\begin{aligned}\rho((n_S(t), n_I(t), n_R(t)), (n_S(t) - 1, n_I(t) + 1, n_R(t))\\ = \hat{\beta}un_S(t)n_I(t)\end{aligned}$$

and

$$\begin{aligned}\rho((n_S(t), n_I(t), n_R(t)), (n_S(t), n_I(t) - 1), n_R(t) + 1)\\ = \gamma n_I(t)\end{aligned}$$

Let the fraction of the infective nodes at time t be denoted by $I(t)$, i.e., $I(t) = n_I(t)/N$. Likewise, let $S(t) = n_S(t)/N$ and $R(t) = n_R(t)/N$ respectively represent the fraction of susceptible and recovered nodes at time t. Now according to results of [27], if N is large, then $S(t)$ and $I(t)$ converge asymptotically to the solution of the following differential equations:

$$\begin{aligned}\dot{S} &= -N\hat{\beta}uIS\\ \dot{I} &= N\hat{\beta}uIS - \gamma I\\ \dot{R} &= \gamma I\end{aligned}$$

Here as well as in the rest of the paper, whenever not ambiguous, the dependency on t is made implicit for brevity. To make equations more legible, we replace $N\hat{\beta}$ with β. We assume that at time zero, a nonzero portion (I_0) of the nodes, but not all of them, are infective. That is we assume that $0 < I(0) = I_0 < 1$. Adding the initial conditions to the differential equations, the dynamics of the system can be represented as follows:

$$\begin{aligned}\dot{S} &= -\beta uIS & S(0) &= 1 - I_0\\ \dot{I} &= \beta uIS - \gamma I & I(0) &= I_0\\ \dot{R} &= \gamma I & R(0) &= 0.\end{aligned}$$

Here, the control variable is the communication range of the nodes, which is bounded between a maximum and minimum value:

$$u_{min} \leq u \leq 1 \tag{1}$$

These bounds are imposed by the physical constraints of the device as well as the MAC protocol and the minimum acceptable QoS. We make the technical assumption that

$$0 < u_{min}.$$

Note that the actual bounds of the communication range can always be re-scaled and normalized and their impact can be captured by an appropriate β, so that $u_{max} = 1$. Any $u(t)$ that satisfies the above constraint is called *admissible* and the range $[u_{min} \ldots 1]$ is referred to as the *admissible range.*

We also have the following state constraints

$$\begin{aligned}0 \leq S, I, R\\ S + I + R = 1.\end{aligned} \tag{2}$$

The latter follows because $S + I + R$ is equal to $N/N = 1$. Because of constraint (2), the differential system is in fact 2-dimensional. Thus we can reduce the system to the following

$$\begin{aligned}\dot{S} &= -\beta uIS & S(0) &= 1 - I_0\end{aligned} \tag{3a}$$

$$\begin{aligned}\dot{I} &= \beta uIS - \gamma I & I(0) &= I_0\end{aligned} \tag{3b}$$

with the state constraint

$$\begin{aligned}0 \leq S, I\\ S + I \leq 1\end{aligned} \tag{4}$$

where the constraint on the control variable is the same as in (1).

We now get to the second task: we construct a cost function which suitably captures the trade-off which is induced by decreasing the communication range of the nodes. Our cost functions are naturally integration of an instantaneous cost over an operation period. Clearly, for the instantaneous cost to be meaningful, it should grow larger with an increase in the fraction of the infective nodes. We assume a linear dependence on $I(t)$.

We now explore the relation between the instantaneous cost and the communication range. As mentioned before, as the communication range of the nodes decreases, the ability of nodes to which in turn deteriorates the QoS.[1] Note that u_{max} is the normal communication range of the nodes which has been calculated as an optimum operating work in natural circumstances. Thus a decrease in the communication range decreases the QoS, implying that the instantaneous cost must be a decreasing function of $u(t)$. The exact relation of the communication range of the nodes, u, to the cost function depends on the implemented MAC and routing policies. In this paper, we consider two different instantaneous cost functions, both of which are decreasing in u and are linearly increasing in I.

In subsection IV-A the cost function is as follows:

$$J = \int_{t=0}^{T} (CI - u)\, dt \tag{5}$$

Coefficient C determines the relative importance (hazard) of the infection. The linearity in u is justified by the fact that any (undesirable) effect of reduction of the communication range on the QoS of the network can be approximated by its linearization around an operation point $u \in [u_{min} \dots u_{max}]$. The approximation is good whenever the range of changing of the communication range is small compared to its value at the operating point.

The second cost function in this paper, which we consider in subsection IV-B is as follows:

$$J = \int_{0}^{T} [CI + u^{-1}]\, dt \tag{6}$$

Relation between J and u in this case is motivated by the work of [26] and [28] in which they show that the expected delivery delay of the network is proportional to the inverse of the communication range of nodes for a number of different routing protocols.

The problem is now well-defined. The task at hand is to find the optimum $u(t)$ that minimizes the cost function over all admissible $u(t)$s.

S	fraction of susceptible nodes
I	Fraction of infective nodes
R	Fraction of recovered nodes
u	Communication Range
γ	Recovery rate of Infective nodes

TABLE I
TABLE OF IMPORTANT NOTATIONS.

IV. STRUCTURAL RESULTS

In this section, we obtain structural results for the optimal communication range, u as a function of time, that minimizes the overall system cost which captures desired trade-offs between communication efficacy (and hence QoS) and containment of the worm.

[1] Note that assuming a bi-directional communication, u is in fact the communication range between a susceptible and an infective node or between two susceptible nodes. Specifically, the communication range between the infective nodes is unaltered. However, as far as QoS is concerned, we only care about the communication range between the susceptible nodes, which is indeed u.

In subsection IV-A we analyze the optimal control problem that seeks to minimize the cost function (5), which is linear both in u and I. We present the structural characteristics of the optimal solution for this case in Theorem 1. We show that the optimal communication range is of *bang-bang* form, that is, it possesses only two possible values u_{max} and u_{min} and switches abruptly between them. It has at most two such jumps and the final jump necessarily culminates at u_{max}.

Next, in subsection IV-B, we assume the cost function (6) which is linear in I but nonlinear in u. We establish the structural properties of the optimal communication range as a function of time for this case in Theorem 2. The optimal solution in this case is not bang-bang anymore, but follows a similar pattern. Specifically, it has at most two switches between u_{max} and u_{min}, with the difference that here the transitions are smooth.

We now state Lemmas 1 and 2 and Corollary 1 which will be used throughout the proofs.

Lemma 1. *I and S are continuous functions of time.*

Proof: According to (3), both S and I are integrals of bounded functions and thus are continuous functions of time. ■

Note that as a consequence, any continuous function of I and S is also a continuous function.

Lemma 2. *(S, I) strictly satisfy the constraints of (4) for the entire interval of $(0 \dots T)$, for any $u(t)$ that satisfies (1).*

This lemma allows us to deal with an optimal control problem with no state constraints, since the state constraints are never active. In other words, even though (4) might imply that we have an optimal control with state constraints, we can ignore the state constraints since they are automatically satisfied provided that the initial infection is neither zero nor is equal to the entire population.

Proof: Note that at $t = 0$, by assumption we have $0 < I = I_0 < 1$ and also $0 < S = S_0 = 1 - I_0 < 1$. Hence the first two constraints in (4), i.e., $0 \le S, I$ are strictly met. The last constraint, i.e., $S + I \le 1$ is active at $t = 0$, however, by summing equations (3a) and (3b) we have $\frac{d}{dt}(S + I)$ at time zero is equal to $-\gamma I_0$, which by assumption is negative. Therefore, there exists an interval after time zero on which the constraint $S + I \le 1$ is strictly met. Now suppose that the statement of the Lemma is not true. Then let $0 < t_0 \le T$ be the first time that (at least) one of the three state constraints of (4) becomes active. Thus for the interval $(0 \dots t_0)$ the constraints are strictly met. For $0 < t < t_0$, from (3a) we have $\dot{S} \ge -\beta S$, thus $\frac{d}{dt}(Se^{\beta t}) \ge 0$. Hence $(Se^{\beta t}) \ge S_0$, and thus, $S(t) > 0$ if $S_0 > 0$. Thus the condition $S \ge 0$ could not have become active at t_0. Similarly, for $0 < t < t_0$ from (3b) we have $\dot{I} \ge -\gamma I$, thus $\frac{d}{dt}(Ie^{\gamma t}) \ge 0$. Hence $(Ie^{\gamma t}) \ge I_0$, and thus, $I(t) > 0$ if $I_0 > 0$. This means that the constraint $I \ge 0$ could not have become active at t_0 either. Now by summing the equations (3a) and (3b), we obtain $\frac{d}{dt}(S+I) = -\gamma I$ which due to what was just shown is strictly negative for $0 < t < t_0$. Thus at t_0, $S + I < S_0 + I_0$ and hence, $S + I < 1$. Thus none of the constraints could have become active, a contradiction. ■

Corollary 1. *S is a strictly decreasing continuous function of time, i.e., $S \searrow S(T)$*

Proof: Form equation (3a) and Lemma 2 and since by assumption $0 < u_{min} \leq u$ we observe that $\dot{S}$ is strictly negative . The result follows. ■

A. A linear cost function

In this subsection, we assume the cost function (5) as the cost of the system and derive the optimal controller accordingly. The structural properties of the optimal controller are expressed in Theorem 1.

Theorem 1. *The optimal $u(t)$ has one of the three following structures: (Fig.1)*

1) $u(t) = u_{max}$ *for* $0 < t < T$;
2) $\exists t_1$ *such that* $u(t) = u_{min}$ *for* $0 < t < t_1$ *and* $u(t) = u_{max}$ *for* $t_1 < t < T$;
3) $\exists t_1, t_2$ *such that* $u(t) = u_{max}$ *for* $0 < t < t_1$ *and* $u(t) = u_{min}$ *for* $t_1 < t < t_2$ *and* $u(t) = u_{max}$ *for* $t_2 < t < T$.

In words, the optimal $u(t)$ is bang-bang and it either has no jump and is fixed at u_{max}; or has only one jump of the form $u = u_{min} \uparrow u_{max}$; or has only two jumps of the form $u = u_{max} \downarrow u_{min} \uparrow u_{max}$.

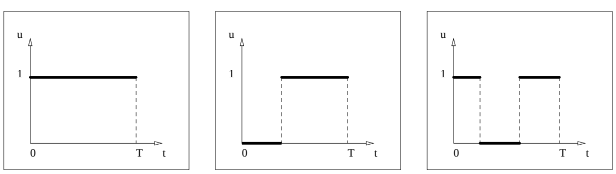

Fig. 1. All three bang-bang controllers stipulated in Theorem 1.

Proof: Throughout the proof, the variables pertain to the optimum solution. This optimal control is a constrained input problem, i.e., the optimal controller is bounded (1). We need to apply *Pontryagin's Maximum Principle* [29, P.232].

We construct H as a scaler function of variables λ_1 and λ_2, that satisfy the following:

$$H = CI - u - \lambda_1 \beta u I S + \lambda_2 \beta u I S - \lambda_2 \gamma I. \tag{7}$$

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial S} = \lambda_1 \beta u I - \lambda_2 \beta u I \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial I} = -C + \lambda_1 \beta u S - \lambda_2 \beta u S + \lambda_2 \gamma \end{aligned} \tag{8}$$

with the final value constraints

$$\begin{aligned} \lambda_1(T) &= 0 \\ \lambda_2(T) &= 0. \end{aligned} \tag{9}$$

In the terminology of Pontryagin's Maximum Principle, H is referred to as the *Hamiltonian* of the system and λ_1 and λ_2 are called the *co-state* variables. Then, according to Pontryagin's Maximum Principle, absolutely continuous functions λ_1 and λ_2 exist that satisfy (8) and (9) and the optimal control u is the minimization of the Hamiltonian (7) over all admissible controls, assuming that all of the state and co-state variables are according to their value for the optimum u [29, P.232].

Let φ, called the *switching function*, be defined as the following:

$$\varphi \triangleq \frac{\partial H}{\partial u} = -1 + \beta I S(\lambda_2 - \lambda_1). \tag{10}$$

This allows us to rewrite the Hamiltonian in (7) as follows:

$$H = CI + \varphi u - \lambda_2 \gamma I. \tag{11}$$

Therefore, we get the following property for the optimal control:

$$u^*(t) = \begin{cases} 1, & \varphi(t) < 0 \\ u_{min}, & \varphi(t) > 0 \end{cases} \tag{12}$$

To affirm that the optimal u takes only its maximum and minimum values (i.e., is bang-bang), we need to establish that the set of times on which $\varphi(t) = 0$ has a zero Lebesgue measure. If the latter was not true, then we had to deal with a *singular optimal controller*. The organization of the proof is as follows:

Step 1 First we prove that the optimal controller is indeed bang-bang by negating the possibility of a singular optimal controller. Specifically, we argue that the switching function φ can be zero at at most three distinct time epochs.

Step 2 Next we show that φ can have at most two zero-crossing points (which are the time epochs at which φ crosses zero and changes its sign). Note that from (12), these are the time epochs at which u switches between its extreme values, and therefore, the optimal controller has at most two jumps.

Step 3 Finally, we use a terminal value condition of φ to evince the nature of the jumps of the bang-bang optimal controller.

Proof of Step 1. From (10), (3) and (8), we obtain

$$\begin{aligned} \frac{\dot{\varphi}}{\beta} &= \dot{I}S(\lambda_2 - \lambda_1) + I\dot{S}(\lambda_2 - \lambda_1) + IS(\dot{\lambda_2 - \lambda_1}) \\ &= (\beta u I S - \gamma I)S(\lambda_2 - \lambda_1) + I(-\beta u I S)(\lambda_2 - \lambda_1) \\ &\quad + IS(-C + \lambda_1 \beta u S - \lambda_2 \beta u S + \lambda_2 \gamma - \lambda_1 \beta u I + \lambda_2 \beta u I) \\ &= -IS(C - \lambda_1 \gamma). \end{aligned}$$

Thus,

$$\dot{\varphi} = -\beta I S(C - \lambda_1 \gamma). \tag{13}$$

The Hamiltonian is autonomous, i.e., does not have an explicit dependency on the independent variable t. When the final time T is fixed and the Hamiltonian is autonomous (i.e., $\frac{\partial H}{\partial t} \equiv 0$), then ([29, P.236]):

$$H(S(t), I(t), u(t), \lambda_1(t), \lambda_2(t)) \equiv \text{constant}. \tag{14}$$

Let a *null point* t_s be a time when $\varphi = 0$. Also let the variables with tilde denote their values at t_s. From (10) by equating $\varphi(t_s) = 0$ we obtain

$$\beta \tilde{I} \tilde{S}(\tilde{\lambda}_2 - \tilde{\lambda}_1) = 1. \tag{15}$$

Using $\tilde{\varphi} = 0$ in (11) yields

$$H(t_s) = \tilde{H} = \tilde{I}(C - \gamma\tilde{\lambda}_2). \tag{16}$$

Now we can obtain

$$\begin{aligned}\dot{\tilde{\varphi}} &= -\beta\tilde{I}\tilde{S}(C - \tilde{\lambda}_1\gamma) && \text{[from (13)]}\\ &= -\beta\tilde{I}\tilde{S}(C + \gamma(\frac{1}{\beta\tilde{I}\tilde{S}} - \tilde{\lambda}_2)) && \text{[from (15)]}\\ &= -\beta\tilde{I}\tilde{S}(C - \gamma\tilde{\lambda}_2) - \gamma\\ &= -\beta\tilde{S}\tilde{H} - \gamma. && \text{[from (16)]}\end{aligned} \tag{17}$$

Here, we state a general property of differentiable functions which we prove in the appendix.

Property 1. *Assume $f(t)$ is a differentiable function of t. Assume t_1 and t_2 to be its two consecutive L-Level points, that is, $f(t_1) = f(t_2) = L$ and $f(t) \neq L$ for all $t_1 < t < t_2$. Now if $\dot{f}(t_1) \neq 0$ and $\dot{f}(t_2) \neq 0$, then $\dot{f}(t_1)$ and $\dot{f}(t_2)$ must have different signs.*

We investigate the case of $H = 0$ first. According to (17), $\dot{\tilde{\varphi}} = -\gamma < 0$. Thus, first of all, $\tilde{\varphi}$, and thus φ, cannot be zero over an interval of nonzero length. Now suppose that there were more than one null point and call the first two consecutive ones t_{s1} and t_{s2}. We have $\dot{\tilde{\varphi}}(t_{s1}) = \dot{\tilde{\varphi}}(t_{s2}) = -\gamma \neq 0$. However, according to Property 1, $\dot{\tilde{\varphi}}(t_{s1})$ and $\dot{\tilde{\varphi}}(t_{s2})$ must have had different signs, which is a contradiction. Thus there is at most one null point in this case.

Now consider the case of $H \neq 0$. Since β, H, γ are constants, (17) is linear in $\tilde{S}$. Also recall from Corollary (1) that S is a strictly monotonic function of time. Thus $\tilde{S}$, as samples of S, is strictly monotonic in t_s. Therefore, $\dot{\tilde{\varphi}}$ is strictly monotonic in t_s. First of all, this implies that $\tilde{\varphi}$, and thus φ, cannot be zero over an interval of nonzero length.

Strict monotonicity of $\dot{\tilde{\varphi}}$ in t_s implies that there is at most one t_s at which $\dot{\varphi} = 0$ (Fact-I).

Now suppose that there were more than three null points. Let the first four consecutive null points be t_{s1} to t_{s4}. Based on Fact-I, at most one of them can have $\dot{\varphi} = 0$. Suppose for example that $\dot{\varphi}(t_{s2}) = 0$. Now either $\dot{\varphi}(t_{s1}) > 0$ or $\dot{\varphi}(t_{s1}) < 0$. If $\dot{\varphi}(t_{s1}) > 0$, then according to the strict monotonicity of $\dot{\tilde{\varphi}}$ in t_s and since $\dot{\varphi}(t_{s2}) = 0$, we must have $\dot{\varphi}(t_{s4}) < \dot{\varphi}(t_{s3}) < 0$. However, strict negativity of $\dot{\varphi}$ on consecutive null points t_{s3} and t_{s4} is in violation of Property 1. Now on the other hand, if $\dot{\varphi}(t_{s1}) < 0$, then similarly according to strict monotonicity of $\dot{\tilde{\varphi}}$ in t_s and since $\dot{\varphi}(t_{s2}) = 0$, we conclude $\dot{\varphi}(t_{s4}) > \dot{\varphi}(t_{s3}) > 0$. The latter violates Property 1.

Similar arguments apply if $\dot{\varphi} = 0$ for any of the other three null points instead of t_{s2}, or if $\dot{\varphi}$ is nonzero at every four of them. Hence, we conclude that there are at most three distinct time epochs at which $\varphi = 0$ for the case $H \neq 0$.

Hence, the set of times at which $\varphi = 0$ is finite (it has at most three members) and thus, the optimal controller is not singular. This concludes Step 1.

Proof of Step 2. In this step, we show that u has at most two jumps. According to (12), the points at which there is a jump in u are the time epochs at which φ crosses 0 and changes its sign. We refer to such time epochs as *zero-crossing points* or *switching times.* In what follows, we suppose that there are more than two switching times and we arrive at a contradiction.

First, we present another property of differentiable functions whose proof can be found in the appendix.

Property 2. *Let $f(t)$ be a differentiable function of time. Let t_1, t_2, t_3 be three consecutive L-level points that are also L-crossing points, that is, $f(t_1) = f(t_2) = f(t_3) = L$, $f(t) \neq L$ for all $t_1 < t < t_2$ and $t_2 < t < t_3$, and $(f(t) - L)$ changes its sign at these point. Now if we have $\dot{f}(t_1) \neq 0$ and $\dot{f}(t_2) = 0$ and $\dot{f}(t_3) \neq 0$, then $\dot{f}(t_1)$ and $\dot{f}(t_3)$ must be of the same sign.*

In step 1, we showed that there are at most three distinct null points, say t_{s1} to t_{s3}. Note that, a zero-crossing point must also be a null point. Thus, if there are more than two zero-crossing points, then they have to be t_{s1} to t_{s3}. Following a similar argument that we used in step 1, we conclude that the only arrangement that is feasible by Fact-I and Property 1 and the strict monotonicity of $\dot{\tilde{\varphi}}$ in t_s, is when $\dot{\varphi}(t_{s2}) = 0$ and furthermore, $\dot{\varphi}(t_{s1})$ and $\dot{\varphi}(t_{s3})$ have opposite signs. But then, t_{s2} cannot be a zero-crossing point because according to Property 2 that requires $\dot{\varphi}(t_{s1})$ and $\dot{\varphi}(t_{s1})$ to have identical signs. Therefore, there cannot be more than two distinct switching times, and thus, u has at most two jumps.

Proof of Step 3. Note that $\varphi(t)$ is a continuous function that following (9), ends at

$$\varphi(T) = -1. \tag{18}$$

Hence from (12), the optimal controller is at its maximum for a subinterval towards the end of $[0 \dots T]$. Now if φ has no zero-crossing point then the optimal controller is always at its maximum for the entire interval (case 1). If φ has one zero-crossing point, then it starts from its minimum value and jumps to its maximum at some $t_1 \in (0 \dots T)$ and stays at its maximum for the remainder of the interval (case 2). Finally, if φ has two zero-crossing points, since it has to finally be equal to -1, then it would have necessarily changed its sign from negative to positive at some time $0 < t_1 < T$ and then back to negative at some later time $0 < t_1 < t_2 < T$. Referring to (12) we conclude that the optimal controller is at its maximum until t_1, then it is at its minimum until t_2 when it jumps up to its maximum at t_2 and stays there (case 3). ■

Remark 1. *We can differentiate the following two cases in the light of equation* (17).

(I): $H \geq -\dfrac{\gamma}{\beta(1 - I_0)}$. *Then $\dot{\tilde{\varphi}} < 0$. To see this note that from Corollary 1, $0 < S < S_0 = 1 - I_0$ and S is a monotone continuous function. Now by referring to* (17)*, it is easy to check that the stated condition on H guarantees $\dot{\tilde{\varphi}} < 0$ for both $\tilde{S} = S_0$ and $\tilde{S} = 0$, and hence for any value of $\tilde{S}$ in between. The negativity of $\dot{\tilde{\varphi}}$ along with the fact that φ is a continuous function of time shows that we can have at most one switch in the sign of φ.*

In order to qualitatively distinguish these two sub-cases where we have no jump in the optimal controller and where we have one jump from u_{min} to u_{max}, recall from (18) that $\varphi(T) = -1 < 0$. Thus, if $\varphi(0) = -1 + \beta I_0 S_0(\lambda_2(0) - \lambda_1(0)) > 0$, then by the Intermediate Value Theorem (IVT), we have such a jump. Also if $\varphi(0) < 0$ then there is no jumps in the optimal controller and $u = u_{max}$ for the entire interval.

Therefore, referring to the shadow price interpretation of the co-state variables, we have the following intuitive interpretation.

> *If the virus spreads slowly (small β) and/or its relative cost is low (i.e., not a serious threat), then we might never reduce the communication ranges of the nodes as a counter-measure.*

(II): $H < -\dfrac{\gamma}{\beta(1-I_0)}$. *Then depending on $\tilde{S}$, the sign of $\tilde{\dot{\varphi}} = -\beta\tilde{S}\tilde{H} - \gamma$ can be either positive or negative, and based on what we showed the optimal bang-bang controller can display up to two jumps.*

According to (9) and (14) $H = H(T) = CI(T) - 1$. Also from (3b) we have $I(T) \geq I_0 e^{-\gamma T}$. Thus a necessary condition for having two jumps in the optimal controller is as follows

$$CI_0 e^{-\gamma T} - 1 < -\frac{\gamma}{\beta(1-I_0)}.$$

Also, another (weaker) necessary condition, derived from the above inequality, can be the following:

$$\frac{\gamma}{\beta(1-I_0)} < 1.$$

B. A non-linear cost function

In this subsection, we consider the cost function in equation (6). We provide the structural properties of the optimal controller for this case in Theorem 2, which relies on Lemmas 3 and 4. Lemma 3 discusses the number of intervals over which the optimal controller is at its maximum value. Lemma 4 sheds light on the behavior of the optimal controller during its transitions between its extrema.

In order to state Theorem 2, we need to define *phases* as follows (Fig.2):

Phase 1:

- $u(t) = k$, on $0 \leq t \leq t_1 < T$ for some $t_1 \geq 0$; $k = u_{max}$ if $t_1 > 0$;
- $u(t)$ strictly and continually decreases on $t_1 < t \leq t_2 < T$ for some $t_2 \geq t_1$;
- $u(t) = u_{min}$ on $t_2 < t \leq t_3 < T$ for some $t_3 \geq t_2$.

Phase 2:

- $u(t)$ strictly and continually increases on $\tau_1 < t \leq \tau_2 < T$ for some $0 \leq \tau_1 \leq \tau_2$;
- $u(t) = u_{max}$ on the interval $\tau_2 < t \leq T$.

Theorem 2. *The optimal $u(t)$ that minimizes the cost function in (6) is a continuous function which is necessarily composed of one the following cases:*

- *Only Phase 2.*
- *Phase 1 followed by Phase 2.*

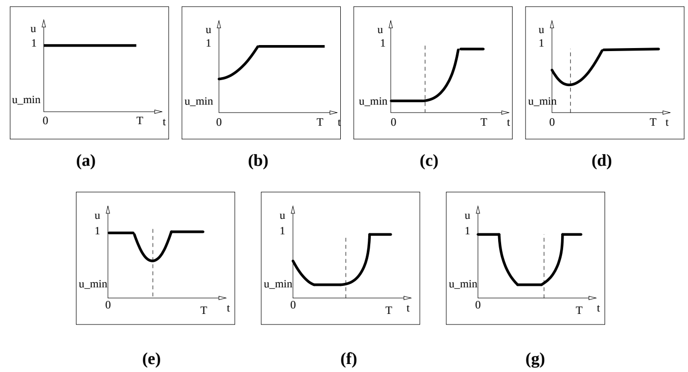

Fig. 2. All possible structures for the optimal controller considered in Theorem 2. The dotted lines designate the beginning of Phase 2 in each case. (a) u is entirely Phase 2 with $\tau_1 = \tau_2 = 0$; (b) u is entirely Phase 2 with $0 = \tau_1 < \tau_2$; (c) u is composed of Phase 1 with $0 = t_1 = t_2 < t_3$, followed by Phase 2; (d) u is composed of Phase 1 with $0 = t_1 < t_2 = t_3$, followed by Phase 2; (e) u is composed of Phase 1 with $0 < t_1 < t_2 = t_3$, followed by Phase 2; (f) u is composed of Phase 1 with $0 = t_1 < t_2 < t_2$, followed by Phase 2; (g) u is composed of Phase 1 with $0 < t_1 < t_2 < t_3$, followed by Phase 2.

Proof: Throughout the proof, the variables with no underline denote their values according to the optimal solution. As we did in subsection IV-A, we will apply the Pontryagin's Maximum Principle. Following the Pontryagin's Maximum Principle, we construct the Hamiltonian and the co-state variables for this problem as follows:

$$H = CI + u^{-1} - \lambda_1\beta uIS + \lambda_2\beta uIS - \lambda_2\gamma I, \tag{19}$$

and

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial S}, \quad \dot{\lambda}_2 = -\frac{\partial H}{\partial I}$$

with the final value conditions:

$$\lambda_1(T) = \lambda_2(T) = 0 \tag{20}$$

The resulting differential equations for the co-state variables λ_1 and λ_2 turn out to be the same as in (8). According to Pontryagin's Maximum Principle, we must have [29, P.232]:

$$H(S, I, u, \lambda_1, \lambda_2) \leq H(S, I, \underline{u}, \lambda_1, \lambda_2) \quad \text{over all admissible } \underline{u}.$$

This leads to the following condition for the optimal controller

$$u^{-1} + [\lambda_2 - \lambda_1]\beta uIS \leq \underline{u}^{-1} + [\lambda_2 - \lambda_1]\beta\underline{u}IS. \tag{21}$$

Let ψ be defined as follows:

$$\psi \triangleq [(\lambda_2 - \lambda_1)\beta IS], \tag{22}$$

which is a differentiable function of time. The Hamiltonian in (19) can be rewritten as follows

$$H = CI + u^{-1} + \psi u - \lambda_2\gamma I. \tag{23}$$

Also, the condition for the optimal u in (21) can be restated as follows

$$u^{-1} + \psi u \le \underline{u}^{-1} + \psi \underline{u}.$$

Note that the function $f(\underline{u}) = \dfrac{1}{\underline{u}} + \psi \underline{u}$ with $\underline{u} \in [u_{min} \dots 1]$ for an arbitrary ψ has the following minimization:

for $\psi > 0$

$$u = \begin{cases} u_{min}, & \psi^{-1/2} < u_{min} \\ \psi^{-1/2}, & u_{min} \le \psi^{-1/2} < 1 \\ 1, & 1 \le \psi^{-1/2}, \end{cases}$$

and for $\psi \le 0$

$$u = 1.$$

Therefore, the optimal controller is obtained as

$$u = \begin{cases} u_{min}, & u_{min}^{-2} < \psi \\ \psi^{-1/2}, & 1 < \psi \le u_{min}^{-2} \\ 1, & \psi \le 1. \end{cases} \tag{24}$$

Note that the optimal u is a continuous function of ψ, which itself is a continuous function of time. Hence, the optimal u is a continuous function of time.

Lemma 3. *(A) There exists a time $0 \le t_1 < T$ such that $u = u_{max}$ for $t_1 < t < T$.*
(B) Also if there exists times $t_3 < t_2 < t_1$ such that $u = u_{max}$ on $t_3 < t < t_2$ and $u \ne u_{max}$ on $t_2 < t < t$, then $u = u_{max}$ for all $0 \le t \le t_2$.

This lemma states that there is at least one interval of nonzero length on which $u = u_{max}$ and that extends until the end of the operation period. And there are at most two distinct intervals of nonzero length on which $u = u_{max}$, which must occur at the beginning and at the end of the optimization period.

Proof: We prove this lemma by first verifying a claim.

Claim 1. *ψ crosses 1 at at most two distinct time epochs in $(0 \dots T)$. Also ψ cannot be equal to 1 over an interval of nonzero length.*

Proof: From (3), (8) and identical to the derivation in (13) we obtain:

$$\dot{\psi} = -\beta I S (C - \lambda_1 \gamma). \tag{25}$$

Let t_s be a time when $\psi = 1$. Indicating the values at t_s with a tilde, we have:

$$\begin{aligned} \tilde{\psi} = 1 &\Leftrightarrow (\tilde{\lambda}_2 - \tilde{\lambda}_1)\beta \tilde{I}\tilde{S} = 1 \\ \Rightarrow \tilde{\dot{\psi}} &= -\beta \tilde{I}\tilde{S}(C - \gamma(\tilde{\lambda}_2 - \frac{1}{\beta \tilde{I}\tilde{S}})) \\ &= -\beta \tilde{S}(C\tilde{I} - \gamma \tilde{\lambda}_2 \tilde{I}) - \gamma \end{aligned}$$

The system is again autonomous and thus the Hamiltonian is a constant [29, P.236], and is therefore equal to its value at t_s. From (24) $\tilde{u} = 1$. Replacing for $\tilde{u}$ and $\tilde{\psi}$ in (23) yields:

$$H = C\tilde{I} + 2 - \gamma \tilde{\lambda}_2 \tilde{I}.$$

Hence,

$$\tilde{\dot{\psi}} = -\beta \tilde{S}(H - 2) - \gamma. \tag{26}$$

This shows that $\tilde{\dot{\psi}}$ is linear in $\tilde{S}$. Since ψ is a differentiable function, similar arguments used after equation (17) ins step 2 of proof of Theorem 1 applies here. Thus the claim follows. ■

Now, we can prove the first part of the lemma. According to (20) and the definition of ψ in (22) we have

$$\psi(T) = 0. \tag{27}$$

Thus, as ψ is a continuous function and referring to (24) the optimal controller must be at u_{max} for a sub-interval that expands until the end of the optimization period.

For proving the second part of the lemma, suppose that there were more than two distinct intervals of nonzero length on which $u = u_{max}$, or the first interval on which $u = u_{max}$ occurred somewhere in the middle of $(0 \dots T)$. Then by referring to (24) and as ψ is a continuous function of time, ψ must have had more than two 1-crossing points or must have been equal to 1 over an interval, both of which are impossible by Claim 1. ■

The next lemma describes the behavior of the optimal u during its transition intervals.

Lemma 4.
(A) *$\dot{u} = 0$ over an interval of nonzero length only if $u = u_{min}$ or $u = u_{max}$ over that interval.*
(B) *If u is strictly increasing over $(t_1 \dots t_2)$ and over $(t_3 \dots t_4)$ for some $0 \le t_1 < t_2 < t_3 < t_4 \le T$, then u is stricly increasing over $(t_1 \dots t_4)$.*

In words, part (A) of the lemma states that flat sections of u can only occur for $u = u_{min}$ or $u = u_{max}$, and part (B) says that u is a strictly increasing function of time on at most one interval of nonzero length during $[0 \dots T]$.

Proof: Let's calculate the derivative of u with respect to time:

$$\frac{du}{dt} = \begin{cases} -\frac{1}{2}\psi^{-3/2}\dot{\psi}, & 1 < \psi < u_{min}^{-2} \\ 0, & otherwise \end{cases} \tag{28}$$

where $\dot{\psi}$ is given by (25).

Note that there exists at most one interval of nonzero length on which $\psi > 1$. This is because otherwise ψ, as a continuous function of time that has to satisfy $\psi(T) = 0$, had to cross 1 more than twice or had to be at 1 for an interval of positive length. However, that would be in contradiction with Claim 1.

In order to prove the lemma, we need to establish the following claim:

Claim 2. *ψ has zero time-derivative for at most one time epoch during the (only possible) interval on which $\psi > 1$.*

Proof: Suppose that $\dot{\psi}$ is zero at more than one time epoch during the interval on which $\psi > 1$. Let those time epochs be t_1 and t_2. Since from Lemma 2, IS is never zero, from (25) we have:

$$C - \lambda_1(t_1)\gamma = 0 = C - \lambda_1(t_2)\gamma$$

and hence

$$\lambda_1(t_1) = \lambda_1(t_2). \tag{29}$$

The relation for $\dot{\lambda_1}$ in (8) can be rewritten as follows:

$$\dot{\lambda}_1 = -\frac{\psi u}{S}.$$

Note that by assumption, $\psi > 1$ during the interval $[t_1 \dots t_2]$. Therefore, since by assumption $u \geq u_{min>} > 0$, λ_1 is strictly decreasing during the interval $[t_1 \dots t_2]$. This contradicts (29). ■

According to (28) and (24), $\dot{u} = 0$ and $u \notin \{u_{min}, u_{max}\}$ only when $\dot{\psi} = 0$ and $1 < \psi < u_{min}^{-2}$. Now according to Claim 2, this cannot take place over an interval of nonzero length. Hence, Part (A) of the lemma follows.

Now suppose Part (B) was not true, and there were more than one distinct intervals on which u is strictly increasing. Let these intervals be $(t_1 \dots t_2)$ and $(t_3 \dots t_4)$ where $t_3 > t_2$. Note that according to (28) u can be a strictly increasing function of time only if $\psi > 1$. Since we argued that there is only one interval of nonzero length on which $\psi > 1$, we must have $\psi > 1$ for the entire $t_1 < t < t_4$. We now argue that u must be a strictly decreasing function of time over at least one sub-interval of the $(t_2 \dots t_3)$. This is because otherwise, u is flat throughout $(t_2 \dots t_3)$. Thus, according to part (A) of Lemma 4 that we just showed, $u = u_{min}$ or $u = u_{max}$ over $(t_2 \dots t_3)$. Recall that after equation (24), we verified that u is a continuous function of time. Now, $u = u_{min}$ contradicts u being a strictly increasing function of time over $(t_1 \dots t_2)$, and $u = u_{max}$ contradicts u being a strictly increasing function of time over $(t_3 \dots t_4)$.

According to (28), u is a strictly increasing function of time only if (a) $\dot{\psi} \leq 0$ and (b) $\dot{\psi} = 0$ only at singular points. Also similarly, u is a strictly decreasing function of time only if (a) $\dot{\psi} \geq 0$ and (b) $\dot{\psi} = 0$ only at singular points. Therefore, over the interval $(t_1 \dots t_4)$, $\dot{\psi}$ must have changed its sign more than once. Thus, since $\dot{\psi}$ is a continuous function of time, $\dot{\psi} = 0$ for at least two distinct time epochs in $(t_1 \dots t_4)$. This contradicts Claim 2, since we argued that $\psi > 1$ for $(t_1 \dots t_4)$. ■

Now we focus on the proof of Theorem 2. The proof is best perceived if we investigate the optimal u from time T backwards. First recall from the argument that followed (24) that the optimal u is a continuous function of time. Now, from Lemma 3 we conclude that $u = u_{max}$ for the interval $(\nu_1 \dots T)$ for some $0 \leq \nu_1 < T$. If $\nu_1 = 0$ then $u = u_{max}$ for the entire $[0 \dots T]$. Referring to the definition of Phases, this corresponds to the case where u is entirely Phase 2 and $\tau_1 = \tau_2 = 0$.

On the other hand, if $\nu_1 > 0$ then the interval of $u = u_{max}$ is precedented by an interval of $(\nu_1 \dots \nu_2)$ for some $0 \leq \nu_1 < \nu_2$ on which u is an increasing function of time, simply because u cannot exceed u_{max}. Thus, we have established that the optimal controller necessarily ends with Phase 2.

Subsequently, from Lemma 4 part (B), we conclude that the optimal controller on the interval $(0 \dots \nu_1)$ is composed of only flat or decreasing sections. By part (A) of Lemma 4 we know that the flat parts can only occur for $u = c$ where $c \in \{u_{min}, u_{max}\}$. Therefore, there cannot be multiple decreasing sections either. Thus, the only possiblities for the interval $(0 \dots \nu_1)$ are (i) $u = u_{max}$ for some subinterval starting from $t = 0$, then it is strictly decreasing to reach $u = u_{min}$, and then it is flat on $u = u_{min}$ until $t = \nu_1$; or (ii) $u = u_{max}$ for some subinterval starting from $t = 0$, then it is strictly decreasing until $t = \nu_1$; or (iii) u is strictly decreasing over that interval. All of these three possiblities are captured by Phase 1. ■

Remark 2. *In the light of equation* (26)*, we can differentiate the following two cases:*

(I): $H \geq 2 - \frac{\gamma}{\beta(1-I_0)}$. *Then following a similar argument as in Remark 1, we conclude that $\dot{\tilde{\psi}}$ is always negative and therefore ψ can cross 1 at most once. If moreover $\psi(0) < 1$, it never does. In the latter case, the optimal controller is at u_{max} for the entire interval.*

(II): $H < 2 - \frac{\gamma}{\beta(1-I_0)}$. *Then depending on the value of $\tilde{S}$, $\dot{\tilde{\psi}}$ can be positive or negative. From the fact that Hamiltonian is a constant and equations* (19)*,* (20) *and* (24) *we obtain $H = constant = H(T) = CI(T) + 1$. Thus, following a similar argument as in Remark 1, a necessary condition for ψ having two 1-crossing points (and thus, the optimal control having two intervals of $u = u_{max}$) is*

$$CI_0 e^{-\gamma T} + 1 < 2 - \frac{\gamma}{\beta(1-I_0)}$$

which can be expressed by the following (weaker) condition:

$$\frac{\gamma}{\beta(1-I_0)} < 1.$$

V. Conclusion

We proposed reduction of reception gains of susceptible nodes as a containment strategy in case of a malware outbreak in mobile wireless networks. We framed the trade-offs introduced by this countermeasure in a cost function. Using optimal control tools, we identified the optimum policy of controlling the communication ranges as a function of time so as to minimize the above cost functions. We analytically derived the structural properties of the optimal policy for special cases of linear and nonlinear cost functions.

APPENDIX

Proof of Property 1. We prove the property for $\dot{f}(t_1) > 0$. The proof follows similarly if $\dot{f}(t_1) < 0$. We have,

$$f(t_1) = L, \quad \dot{f}(t_1) > 0 \quad \text{and} \quad f(t) \neq L \text{ for } t_1 < t < t_2$$
$$\Rightarrow \exists \delta_1 \in (0 \ldots \frac{1}{2}(t_2 - t_1)) \text{ such that } f(t_1 + \delta_1) > L.$$

Suppose that Property 1 did not hold, and $\dot{f}(t_2) > 0$. Then,

$$f(t_2) = L, \quad \dot{f}(t_2) > 0 \quad \text{and} \quad f(t) \neq L \text{ for } t_1 < t < t_2$$
$$\Rightarrow \exists \delta_2 \in (0 \ldots \frac{1}{2}(t_2 - t_1)) \text{ such that } f(t_2 - \delta_2) < L.$$

But now, by the Intermediate Value Theorem (IVT), there must exist a time $t_1 + \delta_1 < \tau < t_2 - \delta_2$ such that $f(\tau) = L$. This contradicts the assumption that $f(t) \neq L$ for all $t_1 < t < t_2$. □

Proof of Property 2. Assume $\dot{f}(t_1) > 0$. The proof follows similarly if $\dot{f}(t_1) < 0$. We have,

$$f(t_1) = L, \quad \dot{f}(t_1) > 0 \quad \text{and} \quad f(t) \neq L \text{ for } t_1 < t < t_2$$
$$\Rightarrow \exists \delta_1 \in (0 \ldots \frac{1}{2}(t_2 - t_1)) \text{ such that } f(t_1 + \delta_1) > L.$$

Also, $(f(t)-L)$ must change its sign from positive to negative at t_2. This is because otherwise, $\exists \delta_2 \in (0 \ldots \frac{1}{2}(t_2 - t_1))$, such that $f(t_2 - \delta_2) < L$. But then, following IVT, $\exists \tau_1 \in (t_1 + \delta_1 \ldots t_2 - \delta_2)$ such that $f(\tau_1) = L$. This contradicts the assumption that $f(t) \neq L$, for all $t_1 < t < t_2$. Thus,

$$\exists \delta_2 \in (0 \ldots \frac{1}{2}(t_3 - t_2)) \text{ such that } f(t_2 + \delta_2) < L.$$

Now suppose that Property 2 was not true, and $\dot{f}(t_3) < 0$. Then,

$$f(t_3) = L, \quad \dot{f}(t_3) < 0 \quad \text{and} \quad f(t) \neq L \text{ for } t_2 < t < t_3$$
$$\Rightarrow \exists \delta_3 \in (0 \ldots \frac{1}{2}(t_3 - t_2)) \text{ such that } f(t_3 - \delta_3) > L.$$

However, now by the Intermediate Value Theorem (IVT), there must exist a time $t_2 + \delta_2 < \tau < t_3 - \delta_3$, such that $f(\tau) = L$. This contradicts the assumption that $f(t) \neq L$ for all $t_2 < t < t_3$. □

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