Dynamic Contract Trading and Portfolio Optimization in Spectrum Markets

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***Abstract*—We address the question of optimal trading of bandwidth (service) contracts in wireless spectrum markets, for the primary as well as the secondary spectrum providers in this context. We propose a structured spectrum market and consider two basic types of spectrum contracts that can help attain desired flexibilities and trade-offs in terms of service quality, spectrum usage efficiency and pricing – long-term guaranteed-bandwidth contracts, and short-term opportunistic-access contracts. A primary provider (seller) and a secondary provider (buyer) creates and maintains a portfolio composed of an appropriate mix of these two types of contracts. The optimal contract trading question in this context amounts to how the spectrum contract portfolio of a seller (buyer) in the spectrum market should be dynamically adjusted so as to maximize return (minimize cost) subject to meeting the bandwidth demands of its own subscribers. In this paper, we formulate the optimal contract trading question as a stochastic optimization problem, and obtain structural properties of the optimal dynamic trading strategy that takes into account the current market prices of the contracts and the provider usage (subscriber demand) process in the decision-making. We evaluate and study the optimal dynamic trading strategy numerically, and compare it with a static portfolio optimization strategy where the key trading decision is made in advance, based on the steady-state statistics of the price and usage processes.**

I. Introduction

The number of users of the wireless spectrum, as well as the demand for bandwidth per user, has been growing at an enormous pace in recent years. Since spectrum is limited, its effective management is vitally important to meet this growing demand. The spectrum available for public use can be broadly categorized into the *unlicensed* and *licensed* zones. In the unlicensed part of the spectrum, any wireless device is allowed to transmit. To use the licensed part, however, license must be obtained from appropriate government authority – the Federal Communications Commission (FCC) in the United States, for example – for the exclusive right to transmit in a certain block of the spectrum over the license time period, typically for a fee. The need for bringing market-based reform in spectrum trading, with the goal of ensuring efficient use of spectrum and fairness in allocation and pricing of bandwidth, is being increasingly recognized by both economists and engineers [4], [7], [13], [15], [14], [22]. The literature on the economics of spectrum allocation has so far mostly focused on the debate of spectrum commons [11], [13], [15] and spectrum auction mechanism design [9], [16], [21], [20]. Spectrum sharing games and/or pricing issues have been considered in [5], [6], [12], [8], [19]. A clear design of the spectrum market structure, precise definition of spectrum contracts, or how the different contracts can be optimally traded in a dynamic market environment is yet to emerge. This is the space in which we contribute in this paper.

We consider a spectrum market where the license holders (referred to as *primary providers* henceforth) can potentially sell to the *secondary providers*[1] the spectrum they have licensed from the FCC but do not envision using in near future. Primary providers may either be providers of TV broadcasts, or large providers of wireless service who operate nationwide. Secondary providers are relatively smaller, but larger in number, and can be geographically limited providers, whose access to spectrum occurs through the bandwidth (service) contracts that they buy from primary providers. Providers in both categories have their subscriber (TV or mobile communication subscriber) bases whom they need to serve using the spectrum they respectively license from the FCC or buy in the spectrum market. This spectrum market structure is motivated by, and closely resembles, secondary financial markets used for trading of financial instruments (like stocks, bonds) among investment banks, hedge-funds etc. Like in secondary financial markets, we allow trading in spectrum markets, not only of the raw spectrum (bandwidth), but also of the different kinds of service *contracts* derived from the use of spectrum. A question that is key to the efficient operation of the spectrum market is how the players in the market – the *primary* and the *secondary* providers – should trade spectrum (bandwidth/service) contracts dynamically, based on time-varying demand patterns arising from their subscribers, to maximize their returns while satisfying their subscriber base. This question forms the central focus of this paper.

We formulate and evaluate the solutions for the *spectrum contract trading* problem for the primary and the secondary providers. We consider two basic forms of contracts that are used for selling/buying spectral resources: i) *Guaranteed-bandwidth (Type-G)* contracts, and (ii) *Opportunistic-access (Type-O)* contracts. Under the Type-G contracts, a secondary provider purchases a guaranteed amount of bandwidth (in units of frequency bands or sub-bands) for a specified duration of time (typically a "long term") from a primary provider, and pays a fixed fee (either as a lump-sum or as a periodic payment through the duration of the contract) irrespective of how much it uses this bandwidth. If after buying the contract the secondary can not avail of the promised bandwidth (this may for example happen when the primary is forced to use a band it has sold due to an unexpected rise in its subscriber

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[1]Note that our notion of "primary" and "secondary" spectrum providers must be distinguished from similar terms often associated with users (subscribers in our case) in the spectrum allocation literature.

demand), the primary financially compensates the secondary for contractual violation. Under Type-O contracts, a secondary provider purchases bandwidth from a primary as and when additional bandwidth is needed. Thus Type-O contracts are made "on the spot", are short-term (one time unit, in our model), and the buyer only pays for the bandwidth it uses. Note that the primary may use any part of the spectrum it has sold as a Type-O contract, without incurring any penalty (since the secondary pays only for the bandwidth it uses, it does not pay the primary for the amount the primary retracts though). Thus, this contract essentially provides the secondary the right to use the spectrum if the primary is not using it.

Under time-varying user demand, spectrum is more efficiently utilized through opportunistic contracts; on the other hand, there is no guarantee that the bandwidth promised in an opportunistic contract will be available, or that an opportunistic contract will be available at a "low" price at the time when the buyer requires and seeks additional bandwidth. So these two types of contracts can be viewed as being at the two ends of the guarantee-efficiency curve. Also, whereas a Type-G contract allows both the buyer and the seller to lock in a price for spectrum usage for the entire duration of the contract, Type-O contracts are priced based on the market price at the time the contract is made. Thus, the price locked in through sale (purchase) of Type-G contracts may turn out to be higher or lower than what the seller (buyer) could have received by deferring the sale (purchase) of Type-G contracts (and selling (buying) Type-O contracts meanwhile), depending on the evolution of the prices of the contracts. Drawing upon the financial markets analogy, for a buyer, Type-G contracts resemble low-risk instruments like bonds, while Type-O contracts resemble high-risk instruments like stocks – the two basic financial instruments required to constitute an efficient portfolio. The two types of contracts also provide a clear separation in terms of contracting timescale, and a well-balanced portfolio of these contract types can allow the trading parties to trade-off as desired between maximizing return and minimizing the risk of not meeting their subscribers' demand.

The *spectrum contract trading* problem that we formulate and solve allows the primary (secondary) provider to dynamically adjust its *spectrum contract portfolio*, i.e, choose how much of each type of contract to sell (buy) at any time, so as to maximize (minimize) its profit (cost) subject to satisfying its own subscriber demand that varies with time, and given the current market prices of Type-G and Type-O contracts which also vary with time. The exact nature of the spectrum contract trading (selling/buying) question will depend on whether it is considered from the perspective of the primary provider (seller) or the secondary provider (buyer). We therefore separately address the *Primary's Spectrum Contract Trading (Primary-SCT)* problem (Section II) and the *Secondary's Spectrum Contract Trading (Secondary-SCT)* problem (Section III). We formulate each problem as a finite horizon dynamic program whose computation time is polynomial in the input size. We *prove* several structural properties of the optimum solutions. For example, we show that the optimal number of Type-G contracts, for both primary and secondary providers, are threshold-type functions of the subscribers' demands and the contract prices. These structural results provide more insight in the problems, and also allow us to develop faster algorithms for solving the dynamic programs.

Although the spectrum contract trading problem has been motivated by analogues in financial markets, the actual questions posed and the techniques used to answer them turn out to be quite different owing to the nature of the specific commodity, that is RF spectrum, under consideration. First, both the primary and the secondary must decide their trading strategies considering their subscriber demand which changes with time. For example, a primary (or secondary) can not simply decide to sell (buy) a large number of Type-G contracts at any given time at which their market prices are low. This is because a primary will need to pay a hefty penalty if it can not deliver the promised bandwidth owing to an increase in its subscriber demand, and the secondary will need to pay for the contract even if it does not use the corresponding bands owing to a decrease in its subscriber demand. The portfolio optimization literature in finance does not usually address the demand satisfaction constraint. Next, spectrum usage must satisfy certain temporal and spatial constraints that are perhaps unique. Specifically, a frequency band can not be simultaneously successfully used at neighboring locations (without causing significant interference), but can be simultaneously successfully used at geographically disparate locations. Thus, the spectrum trading solution for the primary provider must also take into account spatial constraints for spectrum reuse, and therefore the computation of the optimal trading strategy requires a joint optimization across all locations. We prove a surprising *separation theorem* in this context: when the subscriber demand is the same across all locations, the Primary-SCT problem can be solved separately for each location and the individual optimal solutions can subsequently be combined so as to optimally satisfy the global reuse constraints, and obtain the same revenue as the solution of a computationally prohibitive joint optimization across locations (Section II).

Finally, using numerical evaluations, we investigate properties of the optimal solutions and demonstrate that the revenues they earn substantially outperform static spectrum portfolio optimization strategies that determine the portfolio based on the steady-state statistics of the contract price and subscriber demand processes (Section IV).

II. Primary's Spectrum Contract Trading Problem

In this section we pose and address *Primary-SCT*, the spectrum contract trading question from a primary provider's perspective. We first formulate the problem when a primary provider owns channels in a single region (Section II-A), solve it using a stochastic dynamic program (Section II-B), and identify the structural properties of the optimal solution (Section II-C). Later we formulate and solve the trading problem when the primary owns channels in multiple locations, considering the spatial reuse of channels across different locations (Section II-D).

A. Spectrum contract trading in a single region

We now define the Primary-SCT problem for a primary provider that owns M frequency bands (channels) in a single region, which it sells as Type-G or Type-O contracts to secondary providers. We assume that each channel corresponds to one unit of bandwidth and each contract is for one unit of bandwidth. Therefore, at most one contract – either a Type-G or a Type-O – can be sold on a channel at any time. We also assume that the market has *infinite liquidity*: there is a large number of buyers, and hence the primary provider can sell any or all of the channels it owns anytime and in any combination of Type-G and Type-O contracts.

We assume that time is slotted; the spectrum portfolio optimization question for the primary then involves determining at each slot t, the number of channels that would be packaged as Type-G contracts ($x_G(t)$) and the number that would be packaged as Type-O contracts ($x_O(t)$). We consider optimization over a finite time horizon of T time slots. A Type-G ("long term") contract can be negotiated (start) anywhere during our optimization time horizon and lasts till the end of the horizon. T therefore represents the maximum duration of a Type-G contract. Type-O contracts are made on the spot, and last for a single slot from the time they are negotiated.

The prices of either type of contracts (i.e, prices at which the Type-G or Type-O contracts can be bought/sold in the spectrum market) vary randomly with time and are determined "by the market", possibly depending on the current supply-demand balance in the market and other factors. The "per-slot" market prices for the two types of contracts at time t are denoted by $c_G(t)$ and $c_O(t)$, respectively. We assume that the process $\{c_G(t)\}$ (respectively, $\{c_O(t)\}$) constitutes a Discrete time Markov chain (DTMC) with a finite number of states and transition probability $H^G_{c,d}$ (respectively, $H^O_{c,d}$) from state c to d. For simplicity, we assume that the DTMCs $\{c_G(t)\}$ and $\{c_O(t)\}$ are independent of each other, although our results readily extend to the case when the joint process $\{c_G(t), c_O(t)\}$ is a DTMC. When a Type-G contract is sold at slot t, it remains active for $T-t+1$ slots (that is until the end of the optimization horizon), and therefore fetches a revenue of $\alpha(T-t+1)c_G(t)$, where $\alpha(n)$ is an increasing function of n. We assume that the spectrum trading decisions (i.e., the choice of spectrum contract portfolios) of individual providers have no effect on the contract price variations in the market or the provider usage processes. This is indeed the case when the market has a large number of providers of both types.

Each primary provider is associated with a randomly time-varying *usage* process, $\{u(t)\}$ which corresponds to its subscriber demand (TV channel subscribers or wireless service subscribers) that it must satisfy. We assume that the process $\{u(t)\}$ constitutes a DTMC with a finite number of states and transition probability Q_{ij} from state i to j, that is independent of the price process; each usage state corresponds to an integral amount of bandwidth consumption in subscriber demand.

The contract trading is done at the beginning of time slot t, and $(x_G(t), x_O(t))$ are determined after the market prices $c_G(t)$, $c_O(t)$ and usage levels $u(t)$ are known. Let $(z_G(t), x_O(t))$ denote the spectrum contract portfolio held by the primary during time slot t, *i.e.* the number of Type-G and Type-O contracts that stand leased. Since Type-G contracts last till the end of the time horizon, we have:

$$z_G(t) \;=\; \sum_{t' \le t} x_G(t) \tag{1}$$

The primary provider attempts to satisfy the usage at time t, $u(t)$, and sells out the rest of the spectrum as either Type-G or Type-O contracts. Thus, at any time,

$$x_O(t) \;=\; K(z_G(t), u(t)) \;=\; \max\{0, M - z_G(t) - u(t)\}. \tag{2}$$

However, for all slots, t, for which $zG(t) + u(t) > M$, the provider will have to use channels already sold under Type-G contracts to satisfy its subscriber demand, due to unavailability of additional bandwidth, at the cost of incurring penalty, $Y(z_G(t), u(t))$, for breaching Type-G contracts. The penalty is proportional to the number of such channels the provider uses for satisfying its subscriber demand. Thus,

$$Y(z_G(t), u(t)) \;=\; \beta \max\{0, z_G(t) + u(t) - M\}, \tag{3}$$

where β is the proportionality constant. We make the natural assumption that the penalty is hefty, that is, β is greater than or equal to the maximum possible price of a Type-O contract.

The Primary-SCT problem then is to choose the primary's trading strategy $((x_G(t), x_O(t)), t = 1, \ldots T)$, so as to maximize its expected revenue, which is expressed as

$$\boldsymbol{E}\left(\sum_{t=1}^{T} (\alpha(T-t+1)c_G(t)x_G(t) + c_O(t)x_O(t) - Y(z_G(t), u(t)))\right), \tag{4}$$

subject to relations (1)-(3). The optimum strategy must be causal in that for each $t \in \{1, \ldots T\}$, $((x_G(t), x_O(t))$ must be chosen by time t, that is before $\{u(t'), c_G(t'), c_O(t') : t' = t+1, \ldots, T\}$ *are known to the primary. From (1)-(2), $x_O(t)$ is a function $K(.)$ of $z_G(t)$, and the current usage $u(t)$. Therefore, the Primary-SCT question as defined above reduces to finding the optimal $(z_G(t), t = 1, \ldots, T)$ and hence equivalently $(x_G(t), t = 1, \ldots, T)$.*

Note that our revenue function described above ignores any revenue earned from the subscribers. Since the subscriber demand process $u(t)$ is unaffected by the trading decisions, such revenue adds a constant offset to the revenue function described in (4), and therefore does not influence the optimal spectrum trading decisions.

Generalizations:

1) For a Type-O contract, the secondary provider pays the primary only for the amount of bandwidth it uses. Thus, the expected revenue earned by a primary on selling such a contract equals the secondary's expected usage of such a channel times the market price of such a contract. We can incorporate this by considering the revenue from a Type-O contract in slot t as $\kappa c_O(t)$, where κ is the secondary's expected usage of such a channel. The formulation and the results extend to this case.

2) Our formulation and results can be extended to consider the case that $u(t)$ is only an estimate of the demand in slot t, and the estimation error in each slot is an independent, identically distributed random variable whose distribution is known to the primary. Then, $x_O(t)$ must be selected so that $M - x_O(t) - z_G(t)$ is greater than or equal to the actual demand with a desired probability. Thus, $x_O(t)$ will be a function, $K(z_G(t), u(t))$, of $(z_G(t), u(t))$, which may be different from that in (2), but can nevertheless be determined from the knowledge of the distribution of the estimation error. Also, in this case, the lack of exact knowledge of the demand, will force the primary to use part or whole of the bandwidth it has sold as Type-O contracts in satisfying its demand. This will not incur any penalty for the primary owing to the nature of

the contract, but will reduce the secondary's expected usage κ of each channel sold as a Type-O contract, and thereby reduce the expected amount $\kappa c_O(t)$ the secondary pays the primary for each such channel.
3) For clarity of exposition, we assumed integral demands $u(t)$. However, in practice, the demands may be fractional. For example, when a set of subscribers intermittently access the Internet on a channel, a fraction of the bandwidth on a channel is used every slot. In this case, a Type-G or Type-O contract may be sold on the channel (while incurring a penalty proportional to the fraction used on the channel for the former). All our results apply without change in this case.
4) Our formulation can be extended to consider the case that the maximum duration of a Type-G contract T is strictly less than the optimization horizon. But, then the portfolio at any slot t will comprise of $(z_1^G(t), \ldots, z_T^G(t), z^O(t))$, where $z_i^G(t)$ is the number of Type-G contracts that would last for i slots more. Thus, the portfolio has M^{T+1} possible values, which is exponential in T.

B. Polynomial time computation of optimal trading strategy

We show that the Primary-SCT problem defined in Section II-A can be solved as a stochastic dynamic program (SDP). A *policy* [17] is a rule, which specifies the $x_G(t)$ at each slot t. Now, since the demand and prices are Markovian, the statistics of the future evolution of the system from slot t onwards is completely determined by the vector $(z_G(t-1), u(t), c_G(t), c_O(t))$, which we call the *state* at slot t, and the primary's decisions $\{x_G(t') : t' = t, \ldots, T\}$ under the policy being used. Now, in general, a policy may determine $(x_G(t), x_O(t))$ based on all past states and actions at slot t, *i.e.* based on

$$\{u(t'), c_G(t'), c_O(t') : t' = 1, \ldots, t\}$$
$$\cup\{x_G(t'), x_O(t') : t' = 1, \ldots, t-1\}$$

However, a well-known result (Theorem 4.4.2 in [17]) shows that there exists an optimal policy which specifies the optimal $x_G(t)$ at any slot t only as a (deterministic) function of the current state and n [2]. We next compute such an optimal policy by solving a SDP.

For a given t, let $n = T - t + 1$ be the number of slots remaining until the end of the horizon, and $V_n(a, i, c_G, c_O)$ denote the maximum possible revenue from the remaining n slots, under any policy, when the current state is (a, i, c_G, c_O). In particular, note that $V_T(0, i, c_G, c_O)$ is the maximum possible value of the expected revenue in (4) under any policy when $u(1) = i$, $c_G(1) = c_G$ and $c_O(1) = c_O$. The function $V_n(.)$ is called the *value function* [17].

We have:

$$V_n(a, i, c_G, c_O) = \max_{0 \le x \le M-a} W_n(a, i, c_G, c_O, x) \qquad (5)$$

where

$$W_n(a, i, c_G, c_O, x) = \alpha(n) c_G x + J(x + a, i, c_O) + \sum_{d_G} \sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij} V_{n-1}(a + x, j, d_G, d_O) \qquad (6)$$

and

$$J(z_G(t), u(t), c_O(t)) = c_O(t) K(z_G(t), u(t)) - Y(z_G(t), u(t)), \qquad (7)$$

and the maximum in (5) is over integer values of x in $[0, M - a]$. Equation (5) is called *Bellman's optimality equation* [17] and holds because, by definition of $V_{n-1}(.)$, $W_n(a, i, c_G, c_O, x)$ defined by (6) is the maximum possible expected revenue when n slots remain until the end of the horizon and $x_G(t) = x$ is chosen. We get (5) by taking the maximum over all permissible values of x. Denote the (largest) x that maximizes $W_n(a, i, c_G, c_O, x)$ by $x_n^*(a, i, c_G, c_O)$. The function $x_n^*(.)$ provides the optimal solution to the Primary-SCT problem.

Now, the value function and optimal policy can be found from (5) using *backward induction* [17]. Note that $V_0(.) = 0$. Thus, $W_1(.)$ can be computed using (6), and $V_1(.)$ using (5), and $W_2(.), V_2(.) \ldots W_n(.), V_n(.)$ can be successively computed. This backward induction consumes $O((N_G N_O M^2)^2 T)$ time, where N_G (N_O) is the number of states in the Markov Chain $\{c_G(t)\}$ ($\{c_O(t)\}$) - the computation time is therefore polynomial in the input size.

C. Properties of the optimal solution

We analytically prove a number of structural properties of the optimal policy which provide insight into the nature of the optimal solution. Our results are quite general in that they hold not only for the $K(.), Y(.)$ functions defined in (2), (3), but also for any functions that satisfy the following properties (which are of course satisfied by those in (2), (3)). This loose requirement allows our results to extend to the first three generalizations described at the end of Section II-A.

Property 1: $K(a, i)$ decreases in a and $Y(a, i)$ increases in a for each i. Hence, by (7), for each i and c_O, $J(a, i, c_O)$ decreases in a.

Property 2: The $K(.), Y(.)$ functions should be such that $J(a, i, c_O)$ is concave [3] in a for fixed i, c_O.

Property 3: The $K(.), Y(.)$ functions should be such that, for each a, $J(a, i, c_O) - J(a+1, i, c_O)$ is an increasing function of i.

We next state a technical assumption on the statistics of the demand and price processes that we need for our proofs.

Assumption 1: If X_i is the demand in the next slot given that the present demand is i, or, if X_i is the price of a Type-G (respectively, Type-O) contract in the next slot given that the present price is i, then for $i \le i'$, $X_i \le_{st} X_{i'}$ (X_i is *stochastically smaller* [18] than $X_{i'}$), *i.e.*, for each $b \in R$, $Pr(X_i > b) \le Pr(X_{i'} > b)$.

Intuitively, this assumption says that the primary's demand and the prices do not fluctuate very rapidly, and the future demand (or price) is more likely to be high when the initial demand (or price) is high as opposed to when the initial demand (or price) is low.

We are now ready to state the structural properties of the optimum trading policy. We defer the proofs of these properties to Appendix A.

[2] Such a policy is called a *deterministic Markov* policy.

[3] A function $f(k)$ with domain being a subset of the integers is concave [3] if $f(k+2) - f(k+1) \le f(k+1) - f(k)$ for all k [18]. If the inequality is reversed, $f(.)$ is convex.

The first property identifies the relation between $x_n^*(a, i, c_G, c_O)$ and a:

Theorem 1: For each n, i, c_G, c_O,

$$x_n^*(a+1, i, c_G, c_O) = \max(x_n^*(a, i, c_G, c_O) - 1, 0). \quad (8)$$

Intuitively, this theorem suggests that for each n, i, c_G, c_O, there exists an optimal portfolio level of Type-G contracts, $z_G^*(t)$, such that for a given $z_G(t-1) = a$, $x_G(t)$ should be chosen to be equal to $z_G^*(t) - a$ (if the latter is non-negative). Also, due to Theorem 1, for each n, i, c_G and c_O, it is sufficient to find $x_n^*(a, i, c_G, c_O)$ only for $a = 0$ while performing backward induction, and $x_n^*(a, i, c_G, c_O)$ for other a can be deduced from (8). This reduces the overall computation time by a factor of M: the optimal policy can now be computed in $O((N_G N_O)^2 M^3 T)$ time.

The next two results identify the nature of the dependence between $x_n^*(i, a, c_G, c_O)$ and the demand i and prices c_G, c_O.

Theorem 2: For each n, a, c_G and c_O, $x_n^*(i, a, c_G, c_O)$ is monotone decreasing in i.

Theorem 2 confirms the intuition that when the primary's usage is high, it should sell fewer Type-G contracts so as to reserve bandwidth to meet its demand. At the same time, note that this result is not obvious– when the demand is lower, more bandwidth is freed up, which can be sold as Type-G *or as Type-O* contracts. Theorem 2 asserts that the primary should sell at least as many Type-G contracts as before (that is, as for the high demand state), while possibly also increasing the number of Type-O contracts to sell.

Theorem 3: $x_n^*(a, i, c_G, c_O)$ is monotone increasing in c_G for fixed a, c_O and monotone decreasing in c_O for fixed a, c_G.

Theorem 3 confirms the intuition that the primary should preferentially sell the type of contract (G or O) with a "high" price.

D. Spectrum contract trading across multiple locations

We now consider spectrum contract trading across multiple locations from a primary provider's point of view. Wireless transmissions suffer from the fundamental limitation that the same channel can not be successfully used for simultaneous transmissions at neighboring locations, but can support simultaneous transmissions at geographically disparate locations. Thus, a primary provider can not trade contracts in the same channel at neighboring locations, but can do so at far off locations. Thus, the spectrum contract trading problem at different locations is inherently coupled, and must be optimized jointly. We now extend the problem formulation to consider the case of multiple locations, taking into account possible interference relationships between adjacent regions.

We model spatial reuse of the channels across multiple locations by considering an undirected graph $\mathcal{G}$ with the set of nodes S, with each node representing a location (each location represents a certain area in the overall region under consideration). There is an edge between two nodes if and only if transmissions at the corresponding locations on the same channel interfere with each other. A primary provider can sell contracts in M channels at each location. At any given time and at a given node, on each channel (a) either a Type-G contract can be sold, (b) a Type-O contract can be sold or (c) no contract can be sold, subject to the constraint that at no point in time, a contract can stand leased at neighbors on the channel, that is, the set of nodes at which a contract stands leased constitutes an independent set [23].

A primary provider needs to satisfy its subscriber demand which is also subject to certain reuse constraints. We consider the case where the subscribers of a primary provider require broadcast transmissions. This, for example, happens when the primary is a TV transmitter which broadcasts signals over different channels across all regions. At any given slot t, the primary needs to broadcast over a certain number, say $u(t)$, channels which randomly varies with time depending on subscriber demands. Whenever the primary broadcasts on a channel, the broadcast reaches all nodes, and thus the channel can not be used by the secondaries at any node, and hence if the primary has sold a Type-G contract in the channel at any node it incurs a penalty of β at the node. Thus, at slot t, $u(t)$ represents the primary's demand at all nodes. Note that the set of nodes at which the primary uses a given channel for demand satisfaction does not constitute an independent set (as opposed to the set of nodes at which contracts stand leased). Also, the primary's usage status (regarding subscriber demand satisfaction) in any given channel at any given time is the same across all nodes.

The durations of Type-G and Type-O contracts are as described in Section II-A. We assume that at any slot t Type-G (Type-O, respectively) contracts have equal prices $c_G(t)$ ($c_O(t)$, respectively) at all nodes. The processes $(u(t), c_G(t), c_O(t))$ evolve as per independent DTMCs as stated in Section II-A.

The spectrum contract trading problem across multiple locations for a primary (Primary-SCTM) is to optimally choose at each slot t, the type of contract to sell (if any) at each location on each channel so as to maximize the total expected revenue from all nodes over a finite horizon of T slots.

Theorem 4: Primary-SCTM is NP-Hard.

We next present a separation result that shows that an optimal solution for the Primary-SCTM is to (i) compute its optimum contract trading policy for $[1, \ldots, T]$ at each node, and (ii) in each slot $t \in [1, T]$ sell contracts as per this solution, at each node of a maximum size independent set in $\mathcal{G}$. We refer to this policy as the *Separation policy*. Note that this policy clearly ensures that the contracts in each channel are sold only at nodes that constitute an independent set. It is however surprising that this solution optimally solves the Primary-SCTM problem as well:

Theorem 5 (Separation Theorem): The Separation policy optimally solves the Primary-SCTM problem.

We now prove the above Theorem using the following result which we prove in Appendix B.

Lemma 1: Consider the class of policies $\mathcal{F}$, such that a policy $f \in \mathcal{F}$ operates as follows. At the beginning of the horizon, it finds a maximum independent set, $I(S)$, in $\mathcal{G}$. Then, in each slot, it sells contracts only at nodes in $I(S)$. There exists a policy in $\mathcal{F}$ that optimally solves the Primary-SCTM problem.

Proof of Theorem 5: By Lemma 1, we can restrict our search for an optimal policy to the policies in $\mathcal{F}$. Now, the total revenue of a policy in $\mathcal{F}$ is the sum of the revenues at the nodes in $I(S)$. Clearly, the total revenue is maximized if the stochastic dynamic program for the single node case is executed at each node. Note that this solution satisfies the spatial reuse constraints since $I(S)$ is an independent set. ■

Note that the optimum solution at any node can be computed in polynomial time using the SDP presented in Section II-A. However, computation of a maximum size independent set is an NP-hard problem [10]. This computation therefore seems to be the basis of the NP-hardness of Primary-SCTM. Also, the following theorem, which is a direct consequence of Theorem 5, shows that Primary-SCTM can be approximated in polynomial time within a factor of μ if the maximum independent set problem can be approximated in polynomial time within a factor of μ.

Theorem 6 (Approximate Separation Theorem): Consider a μ-separation policy that differs from a separation policy in that it sells contracts as per the individual node optimum solution, at each node of an independent set whose size is at least $\frac{1}{\mu}$ times that of a maximum independent set. This policy's expected revenue is at least $\frac{1}{\mu}$ times the optimal expected revenue.

However, in a graph with N nodes, the maximum size independent set problem can not in general be approximated to within a factor of $O(N^{\epsilon})$ for some $\epsilon > 0$ in polynomial time unless $P = NP$ [1]. Nevertheless, *polynomial time approximation algorithms* (PTAS) i.e., algorithms that compute an independent set whose size is within $(1-\epsilon)$ of the maximum size independent set, for any given $\epsilon > 0$, using a computation time of $O(N^{1/\epsilon})$ are known in important special cases, e.g., when the degree of each node is upper-bounded [2] (this happens in our case when the number of locations each location interferes with is upper-bounded). Thus, in view of Theorem 6, for any given $\epsilon > 0$, the Primary-SCTM problem can be approximated within a factor of $1-\epsilon$ using a computation time of $O(N^{1/\epsilon})$ in such graphs.

III. Secondary's Spectrum Contract Trading Problem

In this section we pose and address *Secondary-SCT*, the spectrum contract trading question from a secondary provider's (buyer's) perspective. First note that the Secondary-SCT problem need not consider the reuse constraints for channels since it is buying the spectrum bands that are offered in the market (presumably in a manner that satisfies the reuse constraints), and also because the secondary providers are usually localized (i.e., operate in small regions). Thus, the secondary's spectrum trading decisions in different regions can be separately optimized. So henceforth in this section, we restrict ourselves to the case of a single region.

A. Formulation

We consider an arbitrary secondary provider that is interested in buying contracts in the secondary spectrum market. Our assumptions regarding the optimization horizon T, the durations of Type-G and Type-O contracts and their price processes $(c_G(t), c_O(t))$ remain the same as in Section II-A. Let $\tilde{u}(t)$ denote the usage level (subscriber demand) of the provider at time t – our assumptions on the statistics of $\{\tilde{u}(t)\}$ are again the same as those for $\{u(t)\}$ presented in Section II-A.

The secondary decides the number of Type-G and Type-O contracts it will buy (from primary providers) at slot t, $(\tilde{x}_G(t), \tilde{x}_O(t))$, after it learns the market prices and usage levels at t. We continue to assume that the market has infinite liquidity, which now implies that the market has a lot of sellers (i.e., primary providers), and hence the secondary can buy as many contracts of any type by paying their market price. Let $(\tilde{z}_G(t), \tilde{x}_O(t))$ denote the spectrum contract portfolio held by the secondary during slot t. Then we again have

$$\tilde{z}_G(t) \;=\; \sum_{t' \le t} \tilde{x}_G(t). \tag{9}$$

The secondary provider's spectrum trading goal is to meet its time-varying usage (subscriber demand) in every time slot at the minimum cost, by choosing an appropriate portfolio of Type-G and Type-O contracts, $\{(\tilde{z}_G(t), \tilde{x}_O(t))\}$, adjusted dynamically.

Note that there are uncertainties on how much bandwidth the secondary actually ends up getting from each contract at a time t during its duration, since a Type-O contract only allows the secondary the right to use the channel when the owner (primary) is not using it, and there maybe a non-zero probability of contract violation for a Type-G by the primary due to its usage level plus the amount of Type-G contracts sold exceeding its total owned spectrum (see the Primary-SCT formulation in Section II-A). Due to this, the subscriber demand $\tilde{u}(t)$ can be met only in statistical terms, e.g., in expectation, or with a certain probability, by any spectrum contract portfolio. (We assume that statistics on such contract violations are available (possibly from historical data) to the buyers, and can be incorporated in the corresponding contract trading decision.) We generalize this notion by associating with each value of usage level (subscriber demand) δ, a *usage satisfaction set* $\mathcal{F}_\delta$ within which a spectrum contract portfolio $(\tilde{z}_G, \tilde{x}_O)$ must lie for meeting the usage level δ satisfactorily. A portfolio $(\tilde{z}_G(t), \tilde{x}_O(t))$ is said to be *usage-satisfactory* at time t if it can meet the usage level at time t satisfactorily, i.e., $(\tilde{z}_G(t), \tilde{x}_O(t)) \in \mathcal{F}_{\tilde{u}(t)}$.

Thus, the Secondary-SCT problem is to minimize the expected contract trading cost subject to the spectrum contract portfolio being usage-satisfactory at all times t. The objective is thus to minimize

$$\boldsymbol{E}\left(\sum_{t=1}^{T} \left(\alpha(T-t+1)c_G(t)\tilde{x}_G(t) + c_O(t)\tilde{x}_O(t)\right)\right), \tag{10}$$

subject to (9) and

$$(\tilde{z}_G(t), \tilde{x}_O(t)) \;\in\; \mathcal{F}_{\tilde{u}(t)},\ \forall t, \tag{11}$$

and such that for each $t \in \{1, \ldots T\}$*,* $(\tilde{x}_G(t), \tilde{x}_O(t))$ *must be chosen by time t, and hence* before $\{\tilde{u}(t'), c_G(t'), c_O(t') : t' = t+1, \ldots, T\}$ *are known, but* after $\{\tilde{u}(t'), c_G(t'), c_O(t') : t' = 1, \ldots, t\}$ *are known, to the secondary provider.*

We assume that the sets $\mathcal{F}_\delta$ for different δ are given. (Typically, we will have $\mathcal{F}_{\delta'} \subseteq \mathcal{F}_\delta$, for $\delta \le \delta'$.) Also, we make the natural assumption that if $(\tilde{z}_G, \tilde{x}_O) \in \mathcal{F}_\delta$ for some δ, then $(\tilde{z}_G, \tilde{x}'_O) \in \mathcal{F}_\delta\ \forall \tilde{x}'_O \ge \tilde{x}_O$. Accordingly, let $L(\tilde{z}_G(t), \tilde{u}(t))$ be the minimum number of Type-O contracts $\tilde{x}_O$ required for a portfolio $(\tilde{z}_G(t), \tilde{x}_O)$ to be in $\mathcal{F}_{\tilde{u}(t)}$, for a given $(\tilde{z}_G(t), \tilde{u}(t))$.

It is easy to see that for a given $(\tilde{z}_G(t), \tilde{u}(t))$, it is optimal to select $\tilde{x}_O = L(\tilde{z}_G(t), \tilde{u}(t))$ (not more).

For example, suppose the secondary seeks to meet the current usage level in expectation. Also, due to the uncertain amount of bandwidth available on Type-G and Type-O contracts, suppose the expected amount of bandwidth obtained from a Type-G contract is γ $(0 < \gamma \leq 1)$. Also, η Type-O contracts are required, on average, to meet one unit of demand, where η is a positive integer. Also, for simplicity, assume that the product $\gamma\eta$ is an integer. Then:

$$L(\tilde{z}_G(t), \tilde{u}(t)) = \max\{\eta(\tilde{u}(t) - \gamma\tilde{z}_G(t)), 0\} \qquad (12)$$

Remarks: 1) Note that in (10), we do not consider the revenue earned from the penalties paid by the primary due to Type-G contract violations. Such penalties lead to a net decrease in the price of a Type-G contract, and their effects can be incorporated by considering the price process of Type-G contracts as $\{\tilde{c}_G(t)\}$, where $\tilde{c}_G(t) = c_G(t) - \kappa(t)$, where $\kappa(t)$ is i.i.d and independent of $\{c_G(t)\}$. Subsequent formulations and analysis do not change owing to the above modification. 2) Like for the Primary-SCT problem, our results can be extended to the case where the secondary knows only an estimate of $\tilde{u}(t)$ at the beginning of time slot t.

B. Analysis

We formulate the secondary's problem as a stochastic dynamic program (SDP) and prove a number of structural properties of the optimal solution and value function. The formulation and analysis are very similar to that for the primary; hence we only provide a brief outline.

Let $(\tilde{z}_G(t-1), \tilde{u}(t), c_G(t), c_O(t))$ be the state at the beginning of slot t, $n = T - t + 1$ and $V_n(a, i, c_G, c_O)$ denote the value function, *i.e.*, the minimum possible cost over the remaining slots, starting from slot t. In particular, note that $V_T(0, i, c_G, c_O)$ is the minimum possible value of the expected cost in (10) under any policy when $\tilde{u}(1) = i$, $c_G(1) = c_G$ and $c_O(1) = c_O$. Then the optimality equation is given by:

$$V_n(a, i, c_G, c_O) = \min_x W_n(a, i, c_G, c_O, x) \qquad (13)$$

where

$$W_n(a, i, c_G, c_O, x) = \alpha(n) c_G x + c_O L(x + a, i) + \sum_{d_G} \sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j P_{ij} V_{n-1}(a + x, j, d_G, d_O) \qquad (14)$$

and the minimum in (13) is over nonnegative integer values of x. Denote the (smallest) x that minimizes $W_n(a, i, c_G, c_O, x)$ by $\tilde{x}^*_n(a, i, c_G, c_O)$. The value function and optimal policy can be found from (13) using *backward induction* [17] in $O((N_G N_O D^2)^2 T)$ time, where D is the number of states in the Markov Chain $\{\tilde{u}(t)\}$.

We now identify the structure of the optimal trading strategy: $\{\tilde{x}^*_n(a, i, c_G, c_O), n = 0, \ldots, T\}$ for the following properties of the $L(.)$ function, which are analogous to Properties 1, 2 and 3 of the $J(.)$ function for the Primary-SCT problem. (i) For each i, $L(a, i)$ decreases in a, (ii) $L(a, i)$ is convex in a for fixed i, (iii) For each a, $L(a, i) - L(a + 1, i)$ is an increasing function of i. It can be checked that these properties are true for the function $L(.)$ in (12). We also assume that the price and demand processes satisfy Assumption 1.

We have the following structural results, which closely parallel Theorems 1 to 3. The proofs are similar to those of Theorems 1 to 3, and hence omitted.

Theorem 7: For each n, i, c_G, c_O, $\tilde{x}^*_n(a + 1, i, c_G, c_O) = \max(\tilde{x}^*_n(a, i, c_G, c_O) - 1, 0)$.

Theorem 8: For each n, a, c_G and c_O, $\tilde{x}^*_n(i, a, c_G, c_O)$ is monotone increasing in i.

Theorem 9: $\tilde{x}^*_n(a, i, c_G, c_O)$ is monotone decreasing in c_G for fixed a, c_O and monotone increasing in c_O for fixed a, c_G.

IV. Numerical Studies

Now, we study the properties of the optimal trading strategy using numerical investigations, and explore how the expected revenue varies as a function of key system parameters. Due to space limitations, and the similarity in the results for Primary-SCT and Secondary-SCT, we only present our results for the former. We consider 20 channels, penalty parameter $\beta = 3.0$ and a birth-death demand process with 20 states and integral state values. The price process c_G (c_O) is again a birth-death process that varies between 1.0 and 4.0 (1.0 and 2.0, respectively) with a total of 10 uniformly-spaced states. For both the demand and price processes, we assume that the forward and backward transition probabilities equal p (a parameter).

In Theorems 2 and 3, we have established the monotonicity properties of the optimal solution $x^*_n(a, i, c_G, c_O)$ with respect to the demand level i and prices c_G, c_O. Recall that $n = T - t + 1$ at slot t, and represents the duration of a Type-G contract made at slot t. Now, our numerical evaluations suggest that the optimal solution $x^*_n(.)$ is decreasing in n, and when n is close to T, $x^*_n(.)$ is zero (Figure 1). Thus, the primary prefers Type-G contracts towards the end of the optimization horizon, and Type-O towards the beginning. This is because when n is close to T, Type-G contracts are very long-term, and hence likely to incur hefty penalties since demand and prices may be difficult to predict long-term.

The two plots in Figure 2 show the variation in the primary's average (expected) revenue per slot with respect to p and T. For these results, the initial state for the demand and price processes are chosen according to the steady state distributions of these processes. The average revenue obtained from the optimal dynamic trading strategy is compared with that of an optimal *static* strategy. In the latter strategy, the number of Type-G contracts is chosen only once (optimally, based on the steady state distribution of the demand and price processes), at the very beginning of the time horizon; the number of Type-O contracts made is adjusted dynamically to the amount of "free bandwidth" available at any slot (i.e., the number of channels minus the sum of the demand and Type-O contracts made). We observe that the average revenue for the optimal static strategy is invariant to changes in p or T – this happens because the initial states for the demand and price processes follow their steady state distributions, which in our case is uniform and does not depend on p or T. We observe that the optimal dynamic contract trading strategy significantly outperforms the optimal static strategy, demonstrating the benefits of dynamic choice of the number of Type-G contracts. Note that if the static strategy buys a Type-G contract, it must buy one that is really long-term (i.e., one that lasts for the entire T slots),

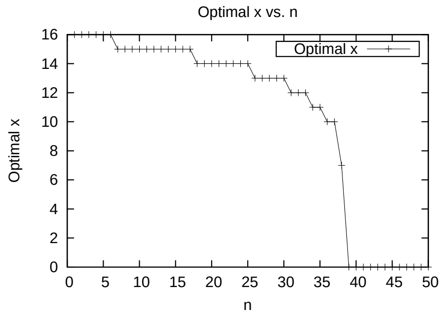

Fig. 1. $x_n^*(a, i, c_G, c_O)$ versus n for $a = 0$, $i = 4$, $c_G = 2.0, c_O = 1.0$ and $T = 50$.

whereas the dynamic can choose the duration of Type-G contracts it buys by deciding when they are purchased, based on its demand and prices of the contracts that evolve dynamically. The figures also show that the primary's average revenue per slot under dynamic choice increases with an increase in p and T (for the same value of the other parameter). Note that a larger p (T respectively) implies larger temporal variation in the prices, making it easier for the primary to "lock in" a good price for a contract. From the bottom plot in Figure 2, we also observe that the average per-slot revenue shows diminishing returns as T increases, and appears to stabilize eventually (at a faster rate for a larger p). This is intuitive since the revenue earned per unit time is upper bounded, and also because very long-term Type-G contracts offer small returns.

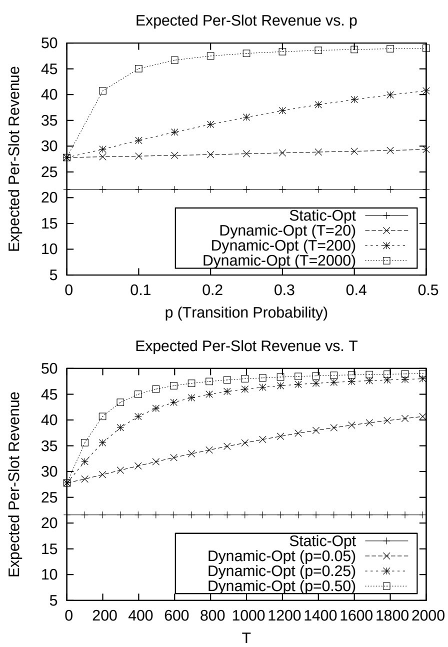

Fig. 2. The top plot shows the average per-slot revenue vs transition probability p. The bottom plot shows the average per-slot revenue vs time horizon T.

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APPENDIX A
PROOFS OF RESULTS IN SECTION II-C

First, we note that with X_i as in Assumption 1, the assumption $X_i \leq_{st} X_{i'}$ for $i \leq i'$ is equivalent to the following condition [18]:

Condition 1: For every increasing function $f(i)$,

$$E(f(X_i)) \leq E(f(X_{i'})) \;\; \forall i \leq i'$$

i.e., $\sum_j Q_{ij} f(j)$, $\sum_j H^G_{ij} f(j)$ and $\sum_j H^O_{ij} f(j)$ are increasing functions of i.

Note that in the summations in Condition 1, as well as in those in the rest of this section, the summation is over all possible states of the respective Markov Chain.

A. *Proof of Theorem 1*

An outline of the proof is as follows. We first prove a simple lemma (Lemma 2), which allows us to prove that the value function is concave in a (Theorem 10). Then, using Theorem 10, we prove Theorem 1.

Lemma 2: For fixed i, c_G, c_O, $V_n(a, i, c_G, c_O)$ decreases in a.

Proof: We prove the result by induction. Let $V_0(a, i, c_G, c_O) = 0$. Then the claim is true for $n = 0$. Suppose $V_{n-1}(a, i, c_G, c_O)$ decreases in a for each i, c_G, c_O. Now, let $a_1 \geq 1$ and $x_n^*(a_1, i, c_G, c_O) = x_1$ for some x_1. Then, by (5):

$$V_n(a_1, i, c_G, c_O) = W_n(a_1, i, c_G, c_O, x_1) \tag{15}$$

Now,

$$\begin{aligned}
& V_n(a_1 - 1, i, c_G, c_O) \\
\geq \; & W_n(a_1 - 1, i, c_G, c_O, x_1) \\
= \; & \alpha(n) c_G x_1 + J(x_1 + a_1 - 1, i, c_O) \\
+ \; & \sum_{d_G} \sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij} V_{n-1}(a_1 + x_1 - 1, j, d_G, d_O) \\
\geq \; & \alpha(n) c_G x_1 + J(x_1 + a_1, i, c_O) \\
& + \sum_{d_G} \sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij} V_{n-1}(a_1 + x_1, j, d_G, d_O) \\
= \; & W_n(a_1, i, c_G, c_O, x_1) \\
= \; & V_n(a_1, i, c_G, c_O) \qquad \text{(by (15))}
\end{aligned}$$

where the second inequality follows from induction hypothesis and Property 1. The result follows. ■

The following theorem shows that the value function is concave.

Theorem 10: For each n, $V_n(a, i, c_G, c_O)$ is concave in a for fixed i, c_G, c_O.

Proof: We prove the result by induction. $V_0(a, i, c_G, c_O)$ is concave in a since it is equal to 0. Suppose $V_{n-1}(a, i, c_G, c_O)$ is concave in a for fixed i, c_G, c_O. Recall that $V_{n-1}(a, i, c_G, c_O)$ is defined for integer values of a. Now, for fixed i, c_G and c_O, define $\tilde{V}_{n-1}(a, i, c_G, c_O)$ for a real as the function obtained by linearly interpolating $V_{n-1}(a, i, c_G, c_O)$ between each pair of adjacent integers a_0 and $a_0 + 1$. Similarly, define $\tilde{J}(a, i, c_O)$.

Now, $J(x + a, i, c_O)$ (respectively, $V_{n-1}(x + a, i, c_G, c_O)$) is concave decreasing in $x + a$ for fixed i, c_O (respectively, for fixed i, c_G, c_O) by Properties 1 and 2 (respectively, by Lemma 2 and induction hypothesis). Hence, we get:

Property 4: $\tilde{J}(x + a, i, c_O)$ (respectively, $\tilde{V}_{n-1}(x + a, i, c_G, c_O)$) is concave decreasing in $x + a$ for fixed i, c_O (respectively, for fixed i, c_G, c_O).

Now, consider the function

$$\begin{aligned}
\tilde{W}_n(a, i, c_G, c_O, x) &= \alpha(n) c_G x + \tilde{J}(x + a, i, c_O) \\
+ \sum_{d_G} \sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} & \sum_j Q_{ij} \tilde{V}_{n-1}(a + x, j, d_G, d_O)
\end{aligned} \tag{16}$$

as a function of the two real variables a, x, *i.e.* the vector (a, x).

Recall the following property of composition of functions [3]:

Property 5: Let $h : R \to R$, $g : R^k \to R$, where $k \geq 1$ and R^k denotes the k-dimensional Euclidean space. Let $f : R^k \to R$ be defined by $f(\mathbf{v}) = h(g(\mathbf{v}))$. If $h(.)$ is concave and decreasing, and $g(\mathbf{v})$ is convex in $\mathbf{v}$, then $f(\mathbf{v})$ is concave in $\mathbf{v}$.

By the fact that $a + x$ is convex in (a, x), Property 4 and Property 5, we get that $\tilde{J}(x + a, i, c_O)$ (respectively, $\tilde{V}_{n-1}(a + x, j, d_G, d_O)$) is concave in (a, x) for fixed i, c_O (respectively, for fixed j, d_G, d_O). Also, x is clearly concave in (a, x). Hence, $\tilde{W}_n(a, i, c_G, c_O, x)$ being a nonnegative weighted linear combination of these functions, is concave in (a, x) for fixed i, c_G, c_O.

Now, define:

$$\tilde{V}_n(a, i, c_G, c_O) = \sup_{x \in R, 0 \leq x \leq M - a} \tilde{W}_n(a, i, c_G, c_O, x) \tag{17}$$

Note that $\{x : x \in R, 0 \leq x \leq M - a\}$ is a non-empty convex set. Recall the following property [3]:

Property 6: If $f(a, x)$ is concave in (a, x) and C is a convex nonempty set, then the function

$$g(a) = \sup_{x \in C} f(a, x)$$

is concave in a, provided $g(a) < \infty$ for some a.

Now, $\tilde{V}_n(a, i, c_G, c_O) < \infty$ (assuming that c_G and c_O are upper bounded). So by Property 6, $\tilde{V}_n(a, i, c_G, c_O)$ is concave in a for fixed i, c_G, c_O.

Now, we will show that $V_n(a, i, c_G, c_O) = \tilde{V}_n(a, i, c_G, c_O)$ for a integer, which will imply that $V_n(a, i, c_G, c_O)$ is concave.

Fix i, c_G, c_O and an integer a. Note that $V_n(a, i, c_G, c_O)$ is the maximum of $\tilde{W}_n(a, i, c_G, c_O, x)$ over integer x, whereas $\tilde{V}_n(a, i, c_G, c_O)$ is the maximum over real x in the range $[0, M - a]$. Hence, to prove that $V_n(a, i, c_G, c_O) = \tilde{V}_n(a, i, c_G, c_O)$, it will suffice to show that the supremum over real x occurs at integer x.

Now, by the definition of the functions $\tilde{J}(.)$ and $\tilde{V}_{n-1}(.)$, $f(x) = \tilde{W}_n(a, i, c_G, c_O, x)$ is continuous and piecewise linear in x, with breakpoints at the integers. Also, note that the endpoints of the domain of $f(x)$, viz. 0 and $M - a$ are integers that are contained in the domain. As a result, it can be checked that the maximum of $f(x)$ must occur at an integer. This completes the proof. ■

Note that $W_n(a, i, c_G, c_O, x)$ is concave in (a, x) and $V_n(a, i, c_G, c_O)$ is the maximum of $W_n(.)$ over a *non-convex* set, namely the set of integers in $[0, M - a]$. This makes the above proof more involved, since had the maximum been over a convex set, the concavity of $V_n(a, i, c_G, c_O)$ would have simply followed from Property 6.

We are now ready to prove Theorem 1.

Proof of Theorem 1: From (6), we have:

$$W_n(a, i, c_G, c_O, x) = W_n(a+1, i, c_G, c_O, x-1) + \alpha(n) c_G, \;\; \forall x \geq 1 \tag{18}$$

Now, by optimality of $x_n^*(a, i, c_G, c_O)$:

$$W_n(a, i, c_G, c_O, x_n^*(a, i, c_G, c_O)) \geq W_n(a, i, c_G, c_O, x) \;\; \forall x \geq 1 \tag{19}$$

If $x_n^*(a, i, c_G, c_O) \geq 1$, then putting $x = x_n^*(a, i, c_G, c_O)$ in (18), we get:

$$\begin{aligned}
& W_n(a, i, c_G, c_O, x_n^*(a, i, c_G, c_O)) = \\
& W_n(a + 1, i, c_G, c_O, x_n^*(a, i, c_G, c_O) - 1) + \alpha(n) c_G, \;\; \forall x \geq 1
\end{aligned} \tag{20}$$

Putting (18) and (20) in (19) and cancelling the term $\alpha(n)c_G$, we get:

$$W_n(a+1,i,c_G,c_O,x_n^*(a,i,c_G,c_O)-1) \geq W_n(a+1,i,c_G,c_O,x-1)\ \forall x \geq 1$$

which shows that $x_n^*(a+1,i,c_G,c_O) = x_n^*(a,i,c_G,c_O)-1$ if $x_n^*(a,i,c_G,c_O) \geq 1$.

Now, suppose $x_n^*(a,i,c_G,c_O) = 0$. Recall that from the concavity of $V_n(a,i,c_G,c_O)$ in a (Theorem 10), it follows that $W_n(a,i,c_G,c_O,x)$ is concave in x. For $x \geq 2$, we have:

$$\begin{aligned} & W_n(a+1,i,c_G,c_O,x-1) - W_n(a+1,i,c_G,c_O,0) \\ = \ & W_n(a,i,c_G,c_O,x) - W_n(a,i,c_G,c_O,1) \text{ (by (18))} \\ \leq \ & W_n(a,i,c_G,c_O,x-1) - W_n(a,i,c_G,c_O,0) \\ & \text{(by concavity)} \\ \leq \ & 0 \end{aligned}$$

which shows that $x_n^*(a+1,i,c_G,c_O) = 0$. ■

B. Proofs of Theorems 2 and 3

The proofs of Theorems 2 and 3 are based on the concepts of submodularity and supermodularity, which we briefly review. Let $X \subseteq R$ and $Y \subseteq R$ be two sets. A function $g(x,y) : X \times Y \to R$ is called *supermodular* [17] if for $x^+ \geq x^-$ in X and $y^+ \geq y^-$ in Y,

$$g(x^+,y^+) + g(x^-,y^-) \geq g(x^+,y^-) + g(x^-,y^+)$$

If the inequality is reversed, g is called *submodular* [17].

We will require the following key result [17].

Theorem 11: If $g(x,y)$ is supermodular (submodular) on $X \times Y$, then

$$f(x) = \max\{y' : y' \in \operatorname*{argmax}_Y g(x,y)\}$$

is increasing (decreasing) in x.

To prove Theorem 2, we show that $W_n(a,i,c_G,c_O,x)$ is submodular in (i,x). The monotonicity of $x_n^*(a,i,c_G,c_O)$ in i then follows from Theorem 11. First, we prove some lemmas.

The following lemma provides a necessary and sufficient condition for submodularity.

Lemma 3: Let $g(i,x)$ be a function with domain being integer values of x and real values of i. $g(i,x)$ is submodular in (i,x) if and only if $g(i,x) - g(i,x+1)$ is an increasing function of i for all x.

Proof: The necessity is obvious. We now prove sufficiency. For an integer $z > 0$:

$$g(i,x) - g(i,x+z) = [g(i,x) - g(i,x+1)] + \ldots + [g(i,x+z-1) - g(i,x+z)]$$

So $g(i,x) - g(i,x+z)$, being the sum of increasing functions, is increasing in i.

Hence, for $x^- < x^+$, $g(i,x^-) - g(i,x^+)$ is increasing in i. So for $i^- < i^+$:

$$g(i^-,x^-) - g(i^-,x^+) \leq g(i^+,x^-) - g(i^+,x^+)$$

Hence, $g(i,x)$ is submodular in (i,x) by definition. ■

For $m \geq 1$, define [4]

$$i_n^m(a,c_G,c_O) = \max\{i : x_n^*(a,i,c_G,c_O) \geq m\}. \tag{21}$$

Lemma 4: If $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i, then

$$i_n^1(a,c_G,c_O) \geq i_n^2(a,c_G,c_O) \geq \ldots \geq i_n^{M-a}(a,c_G,c_O)$$

Also, $x_n^*(a,i,c_G,c_O) = m$ if and only if $i_n^m(a,c_G,c_O) \geq i > i_n^{m+1}(a,c_G,c_O)$.

Proof: The result follows by definition of $i_n^m(.)$. ■

The next lemma establishes a sufficient condition for monotonicity of $x_n^*(i,a,c_G,c_O)$.

Lemma 5: Fix n. Suppose $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ is an increasing function of j for each a, d_G, d_O, j. Then $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i for each a, c_G, c_O and i.

It is important to note that the lemma requires $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ to be increasing in j *for a fixed* n, and asserts that $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i *for that* n.

Proof: By (6):

$$\begin{aligned} & W_n(a,i,c_G,c_O,x) - W_n(a,i,c_G,c_O,x+1) \\ & = -\alpha(n)c_G + [J(a+x,i,c_O) - J(a+x+1,i,c_O)] \\ & + \sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij} \left(V_{n-1}(a+x,j,d_G,d_O)\right. \\ & \left. -V_{n-1}(a+x+1,j,d_G,d_O)\right) \end{aligned}$$

The first term on the right hand side is constant, the second term is increasing in i by Property 3 and the third term is increasing in i since $V_{n-1}(a+x,j,d_G,d_O) - V_{n-1}(a+x+1,j,d_G,d_O)$ is increasing in j and by Condition 1.

So $W_n(a,i,c_G,c_O,x) - W_n(a,i,c_G,c_O,x+1)$ is increasing in i. Hence, by Lemma 3, $W_n(a,i,c_G,c_O,x)$ is submodular in (i,x) and so by Theorem 11, $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i. ■

The next lemma is a simple consequence of (8).

Lemma 6: If $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i for each a, c_G, c_O, then $i_n^{m+1}(a,c_G,c_O) = i_n^m(a+1,c_G,c_O)$ for $m = 1, 2, \ldots$.

Proof: Let $m \geq 1$. Separately with a and with $a+1$, start with $i = M$ (the highest demand state) and keep decreasing it to the next lower state, one at a time. By (8), the maximum i at which $x_n^*(a,i,c_G,c_O) \geq m+1$ is precisely the maximum i at which $x_n^*(a+1,i,c_G,c_O) \geq m$. So $i_n^{m+1}(a,c_G,c_O) = i_n^m(a+1,c_G,c_O)$ by definition of $i_n^m(.)$. ■

Lemma 7: For each n, $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$ is an increasing function of i for each a, c_G, c_O.

Proof: We prove the claim by induction. Since $V_0(a,i,c_G,c_O) \equiv 0$, the claim is true for $n = 0$.

Suppose the statement is true for $n-1$, *i.e.*, $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ is an increasing function of j for each a, d_G, d_O, j. Then by Lemma 5, $x_n^*(i,a,c_G,c_O)$ is monotone decreasing in i. Hence, by Lemma 6, $i_n^{m+1}(a,c_G,c_O) = i_n^m(a+1,c_G,c_O)$ for $m = 1, 2, \ldots$.

[4] If $x_n^*(a,i,c_G,c_O) < m\ \forall i$, then let $i_n^m(a,c_G,c_O)$ be equal to the smallest demand state.

Now, we show that $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$ is an increasing function of i. Fix a, c_G and c_O. We have the following cases:

Case 1: $i > i_n^1(a,c_G,c_O)$

By Lemma 4 and Lemma 6:

$$i > i_n^1(a,c_G,c_O) \geq i_n^2(a,c_G,c_O) = i_n^1(a+1,c_G,c_O)$$

So by Lemma 4, $x_n^*(a,i,c_G,c_O) = x_n^*(a+1,i,c_G,c_O) = 0$. Hence, by (5) and (6):

$$\begin{aligned} & V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O) \\ = \; & W_n(a,i,c_G,c_O,0) - W_n(a+1,i,c_G,c_O,0) \\ = \; & (J(a,i,c_O) - J(a+1,i,c_O)) \\ + \; & \sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij}(V_{n-1}(a,j,d_G,d_O) \\ & -V_{n-1}(a+1,j,d_G,d_O)) \end{aligned} \tag{22}$$

Case 2: $i_n^m(a,c_G,c_O) \geq i > i_n^{m+1}(a,c_G,c_O)$, where $m \geq 1$.

By Lemma 4, $x_n^*(a,i,c_G,c_O) = m$ and hence by Theorem 1, $x_n^*(a+1,i,c_G,c_O) = m-1$. So by (5) and (6) and some cancellation of terms, we get:

$$\begin{aligned} & V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O) \\ = \; & W_n(a,i,c_G,c_O,m) - W_n(a+1,i,c_G,c_O,m-1) \\ = \; & \alpha(n)c_G \end{aligned} \tag{23}$$

By (22) and (23):

$$\begin{aligned} & V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O) \\ = & \begin{cases} \alpha(n)c_G & \text{if } i \leq i_n^1(a,c_G,c_O) \\ (J(a,i,c_O) - J(a+1,i,c_O)) & \\ +\sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij}(V_{n-1}(a,j,d_G,d_O) & \\ -V_{n-1}(a+1,j,d_G,d_O)) & \text{if } i > i_n^1(a,c_G,c_O) \end{cases} \end{aligned}$$

The expression for $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$ for $i > i_n^1(a,c_G,c_O)$ is an increasing function of i by Property 3, induction hypothesis and Condition 1. Thus, to show that $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$ is increasing in i, it is sufficient to show that for $i > i_n^1(a,c_G,c_O)$:

$$\begin{aligned} & (J(a,i,c_O) - J(a+1,i,c_O)) \\ & +\sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij}(V_{n-1}(a,j,d_G,d_O) \\ & -V_{n-1}(a+1,j,d_G,d_O)) \geq \alpha(n)c_G \end{aligned}$$

which is equivalent to

$$\begin{aligned} & J(a,i,c_O) + \sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij} V_{n-1}(a,j,d_G,d_O) \\ & \geq \alpha(n)c_G + J(a+1,i,c_O) + \sum_j Q_{ij} V_{n-1}(a+1,j,d_G,d_O) \end{aligned}$$

that is, $W_n(a,i,c_G,c_O,0) \geq W_n(a,i,c_G,c_O,1)$, which is true because $x_n^*(a,i,c_G,c_O) = 0$ for $i > i_n^1(a,c_G,c_O)$.

The result follows. ■

From the above lemmas, we get the desired monotonicity of $x_n^*(i,a,c_G,c_O)$.

Proof of Theorem 2: Fix n, a, c_G and c_O. By Lemma 7, $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ is an increasing function of j for each d_G, d_O. The result follows by Lemma 5. ■

Proof of Theorem 3: The proof is very similar to the proof of Theorem 2 and hence omitted. ■

Appendix B
Proofs of results in Section II-D

Proof of Theorem 4: We show that the Maximum Independent Set (MIS) problem is a special case of Primary-SCTM. Consider the following special case of Primary-SCTM: $M = 1$, $T = 1$. At each node, the primary's demand is always 0, and the prices of type G and O contracts are fixed, equal to $\frac{1}{2}$ and 1 respectively. Thus, it is optimal never to sell a type G contract.

The Primary-SCTM problem reduces to that of finding a maximum independent set of nodes (at which to sell type O contracts). The result follows, since the MIS problem is NP-Hard [10]. ■

Proof of Lemma 1: Let $N^t_{e,j}$ be the number of type j contracts ($j \in \{G,O\}$) sold by a policy P in slot t on channel e. We make the following key observations:

(1) The revenue of any policy depends only on the *number* of type G and O contracts it sells on each channel, in each slot, independent of which nodes it sells them at.

That is, the revenue of the policy P is completely determined by:

$$\{N^t_{e,G}, N^t_{e,O} : e = 1,\ldots,M; t = 1,\ldots,T\}$$

This follows from the fact that the prices of both types of contracts and the usage status are the same at all nodes on each channel.

(2) For every policy, on each channel, at any time, the total number of type G and type O contracts currently leased is at most equal to $|I(S)|$.

That is, for the above policy P, for every slot t:

$$\sum_{\tau=1}^{t} N^\tau_{e,G} + N^t_{e,O} \leq |I(S)|, \; e = 1,\ldots,M \tag{24}$$

This follows from the fact that $I(S)$ is a maximum independent set.

Now, let P be an optimal policy. Consider a policy $f \in \mathcal{F}$, which initially finds a maximum independent set $I(S)$. Also, whenever P sells a contract, f sells the same type of contract on the same channel at a node in $I(S)$ at which no contract has been sold on this channel. More precisely, number the nodes in $I(S)$ from 1 to $|I(S)|$. In slot t, on channel e, policy f sells type G contracts at the nodes $\sum_{\tau=1}^{t-1} N^\tau_{e,G}+1$ to $\sum_{\tau=1}^{t} N^\tau_{e,G}$ and type O contracts at the nodes $\sum_{\tau=1}^{t} N^\tau_{e,G}+1$ to $\sum_{\tau=1}^{t} N^\tau_{e,G}+N^t_{e,O}$. It can be checked that on each channel e, (a) for policy f, two or more contracts never stand leased at the same node and (b) by (24), in each slot t, f finds enough nodes in $I(S)$ to sell contracts at.

Now, by observation (1), the revenue of f is the same as that of P, and therefore f is optimal. ■