Homework 11 Solutions

Problem 1 Solution: Both algorithms work if some edges have negative weight edges. Their correctness is not affected by the negative weight edges. In Kruskal's algorithm the safe edge added to A (subset of a MST) is always a least weight edge in the graph that connects two distinct components. So, if there are negative weight edges they will not affect the evolution of the algorithm. Similarly, in Prim's algorithm set A forms a single tree. The safe edge added to A is always a least-weight edge connecting the tree to a vertex not in the tree. Again, negative weights do not affect. Thus, the light edge does not distinguish between positive or negative edge weights.

Problem 2 Solution: We will prove that a minimum weight spanning tree under the normal definition, is also a spanning tree with the minimum value of the largest edge weight.

Consider an MST T, suppose there exists a spanning tree T' such that the largest edge weight is smaller than the largest edge weight in T. Call the corresponding edges, e' and e respectively. Remove e from T. This breaks T into two connected components. There must exist an edge e'' in T' that connects these components. Clearly, $w(e'') \leq w(e') < w(e)$. Thus the tree T'' obtained from T by replacing the edge e with e'' has weight $w(T'') = w(T) + w(e'') - w(e) < w(T)$, which is a contradiction.

We can use Prim's algorithm to solve the problem in time $O(V logV + E)$.

Problem 3 Solution: Every tree of V vertices has $V - 1$ edges. Since all edges in our case have equal weights, any spanning tree T will also be an MST. If the weight of an edge is a, then total weight $w(T) = a|V - 1|$. Thus, using BFS we can obtain a breadth-first tree that will be an MST. This costs $O(V + E)$.

Problem 4 Solution: Proceed by contradiction. Let T be some minimum spanning tree of graph G not containing edge $e = (u, v)$. Since T is a spanning tree, there is a path from v to u. For any edge (x, y) on the path from v to u, take the edge (x, y) out of the tree and add the edge (u, v). Since the weight of (u, v) is less than or equal to the weight of all edges including the edge (x, y), the new tree T' has weight less than or equal to T. So, T' is a minimum spanning tree which is a contradiction. QED

Problem 5 Solution: Consider cycle C in G as the path $< v_0, v_1, \ldots, v_k >$ where $v_0 = v_k$. For every pair of vertices (v_i, v_j) (wlog assume $i < j$) there are 2 paths connecting them. These are $P_1 = (v_i, v_{i+1}, \ldots, v_j)$ and $P_2 = (v_j, v_{j+1}, \ldots, v_0, v_1, \ldots, v_i)$. Removing e from the graph, will still keep one path connecting every pair of vertices.

Notice that T is also a spanning tree in the new graph G', since G and G' contain the same vertices. Suppose that T is not a minimum spanning tree in G'. Notice also that G and G' differ by the edge e. There must be a tree T' in G' that is a minimum spanning tree with weight less than T and containing the edge e. Remove the edge e from T' and add an edge (x, y) from the cycle that is not already in T'. This new tree T'' has weight less than T' since the edge (x, y) has weight less than e (since e is maximum in the cycle). Then, T' is not an MST. QED