Quality Sensitive Price Competition in Spectrum Oligopoly

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***Abstract*—We investigate a spectrum oligopoly where primary users allow secondary access in lieu of financial remuneration. Transmission qualities of the licensed bands fluctuate randomly. Each primary needs to select the price of its channel with the knowledge of its own channel state but not that of its competitors. Secondaries choose among the channels available on sale based on their states and prices. We formulate the price selection as a non-cooperative game and prove that a symmetric Nash equilibrium (NE) strategy profile exists uniquely. We explicitly compute this strategy profile and analytically and numerically evaluate its efficiency. Our structural results provide certain key insights about the unique symmetric NE.**

I. Introduction

Recent investigations augur that demand for mobile broadband – driven by the large scale proliferation of wireless industry – will surpass the availability of wireless spectrum in imminent future. Yet, as recent measurements suggest, the licensed bands remain largely under-utilized. A reasonable conjecture therefore is that unlicensed access of idle (but licensed) spectrum bands, commonly referred to as secondary spectrum access, would avert the impending crisis. Recently, FCC has legalized the access of TV white space spectrum, and the advent of cognitive radios together with the design of a plethora of sophisticated algorithms have enabled intelligent selection of bands. Large-scale secondary spectrum access can not however be realized only through the availability of the enabling technology and the regulatory progress: secondary access must also be rendered profitable for the license holders. Accordingly, we investigate a spectrum oligopoly [14] where license holders (hitherto referred to as primaries) allow unlicensed users (hitherto referred to as secondaries), in lieu of financial remuneration, access to the channels (licensed bands) that are not in use. Different channels offer different transmission rates to the secondaries depending on their states which evolve randomly and reflect the usage levels of the primaries as also transmission quality fluctuations owing to fading. Each primary quotes a price for the channel that it offers and secondaries select among the available channels depending on the states and the prices quoted. Thus, if a primary quotes a high price, it will earn a large profit if it sells its channel, but may not be able to sell at all; on the other hand a low price will enhance the probability of a sale but also fetch lower profits in the event of a sale.

Price selection in oligopolies may be modeled as a non-cooperative game and has naturally been extensively investigated in economics, implementing *Bertrand Game* [14] and its modifications [1], [11], [18]. However, all the above papers ignore the uncertainty of competition which distinguishes spectrum markets from standard oligopolies: a primary knows the state of its channel but does not know those of its competitors before deciding the price for its channel. Pricing in communication services have been explored to a great extent ([2] prsenets a brief overview). References [5], [12], [13], [16], [17], [22] have analyzed price competition among spectrum providers. References [16], [17] modeled price competition among multiple players. But all the above papers suffer from drawbacks: first, they did not assay uncertainty of states of channels of competitors ; second, most of them did not explicitly determine a Nash Equilibrium (NE) (exceptions are [13], [16]). On the other hand, the papers that consider uncertainty of competition, namely [6]–[10], assume that the commodity on sale can be in one of two states: available or otherwise. This assumption does not capture different transmission qualities offered by the available channels. The consideration of the latter significantly complicates the analysis of the game. A primary may now need to employ different pricing strategies for different states, while in the former case a single pricing strategy will suffice as a price need not be quoted for an unavailable commodity. Our investigation seeks to contribute in this space.

We have modeled the price selection as a game with primaries as the players (Section II) and seek a NE pricing strategy. We consider that the preference of the secondaries can be captured by a penalty function which associates a penalty value to each channel that is available for sale depending on its state and price quoted. Given the state of a channel, there is a one-to-one correspondence between the price quoted and the penalty perceived by a secondary. Thus, the strategy for selection of a price for a channel in a given state may be equivalently represented as a strategy for selection of penalty that the channel offers to a secondary. Since prices and therefore the penalties take real values, the strategy set of the players are continuous; also the payoff functions for the primaries turn out to be discontinuous. Thus, classical results do not guarantee the existence, let alone the uniqueness, of a NE. In addition, existing literature does not provide algorithms for computing a NE does not exist unlike when the strategy set is finite [15]. Starting from a general set of strategy profiles for the primaries which allows for selecting penalties using arbitrary probability distributions, we show that for a large class of penalty functions, there exists a unique symmetric NE strategy profile, which we explicitly compute (Section III). Our analysis reveals several interesting insights about the structure of the symmetric NE. First, we learn that if a channel in state i provides a higher transmission rate

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to a secondary than that in state j, then the symmetric NE strategy profile selects the penalties for i, j respectively from ranges $[L_i, U_i], [L_j, U_j]$ where $U_i \leq L_j$. Thus, a secondary will always prefer a channel in state i to a channel in state j considering both the prices and the states. This negates the intuition that prices ought to be selected for the states so as to render them equally preferable to a secondary - symmetric NE strategy profiles in fact price the channels so as to retain the preference order provided by the states. The analysis also reveals that the unique symmetric NE strategy profile consists "nice" probability distributions in that they are continuous and strictly increasing; the former rules out pure strategy symmetric NEs and the latter ensures that the support sets are contiguous. Finally, utilizing the explicit computation algorithm for the symmetric NE strategies, we analytically and numerically investigate the reduction in expected profit suffered under the unique symmetric NE pricing strategies as compared to the maximum possible value allowing for collusion among primaries (Section IV).

All the proofs are deferred to the Appendix.

II. System Model

We consider a spectrum market with n primaries and m secondaries. We will initially consider the case that the primaries know m, later generalize our results for random, apriori unknown m. Each primary has access to a channel which can be in states $0, 1, \ldots, n+1$, where state i provides a lower transmission rate to a secondary than state j if $i < j$ and state 0 arises when the channel is not available for sale and provides 0 transmission rate. Different channels constitute disjoint frequency bands leased by the primaries. A channel is in state $i \geq 1$ w.p. q_i and in state 0 w.p. $1 - q$ where $q = \sum_{i=1}^{n} q_i$, independent of the states of other channels. If a primary quotes a price p for a channel in state i, then the channel offers a penalty $g_i(p)$ to a secondary. Each $g_i(\cdot)$ is continuous, strictly increasing in its argument, and therefore invertible. We denote $f_i(\cdot)$ as the inverse of $g_i(\cdot)$; clearly $f_i(\cdot)$ is continuous and strictly increasing in its argument as well. No secondary buys a channel whose penalty is higher than v, and as the name suggests a secondary prefers a channel with a lower penalty (a secondary's preference depends entirely on the penalty). Thus, we must have $g_i(p) > g_j(p)$ and $f_i(x) < f_j(x)$ for each x, p and $i < j$. Each primary also incurs a transition cost $c > 0$ for an available channel, and therefore never selects a price lower than c. We assume that

$$\frac{f_j(y) - c}{f_k(y) - c} < \frac{f_j(x) - c}{f_k(x) - c} \quad \text{for all } x > y > g_j(c), j < k \tag{1}$$

A large class of penalty functions $g_i(\cdot)$ satisfy the above property required of the corresponding inverses, e.g., $g_i(p) = \zeta(p - h(i)), g_i(p) = \zeta(x/h(i))$ where $\zeta(\cdot), h(\cdot)$ are continuous, strictly increasing functions, $g_i(p) = p^r - h(i), g_i(p) = p^r h(i), g_i(p) = \exp(p) - h(i), g_i(p) = \log(p) - h(i)$ for $r > 0$ and a strictly increasing, continuous $h(\cdot)$. In addition, $g_i(\cdot)$ such that the inverses are of the form $f_i(x) = h_1(x) + h_2(i), f_i(x) = h_1(x)h_2(i)$, where $h_1(\cdot), h_2(\cdot)$ are continuous and strictly increasing satisfy the above assumption.

If primary i quotes a price p for its channel then its profit(payoff) is

$$\begin{cases} p - c & \text{if the primary sells its channel} \\ 0 & \text{otherwise} \end{cases}$$

Note that if Y is the number of channels offered for sale for which the penalties are upper bounded by v, then those with $\min(Y, m)$ lowest penalties are sold since secondaries select channels in increasing order of penalties. The ties among channels with identical penalties are broken randomly and symmetrically among the primaries.

Each primary selects the penalty for its channel with the knowledge of the state of the channel, but without knowing the states of the other channels; a primary however knows $l, m, n, q_1, \ldots, q_n$. Note that the choice of the penalty uniquely determines the price since there is a one-to-one correspondence between the two given the state of a channel. Primary i chooses its penalty using an arbitrary probability distribution function (d.f.) [1] $\psi_{i,j}(.)$ when its channel is in state $j \geq 1$. If $j = 0$ (i.e., the channel is unavailable), i chooses a penalty of $v + 1$: this is equivalent to considering that such a channel is not offered for sale as no secondary buys a channel whose penalty exceeds v. For $j \geq 1$, each primary selects its price so as to maximize its expected profit. Thus, if $m \geq l$, primaries select the highest penalty for each state $1, \ldots, n$, since all available channels will be sold. So, we consider $m < l$. $S_i = (\psi_{i,1}, ..., \psi_{i,n})$ denotes the strategy of primary i, and $(S_1, ..., S_l)$ denotes the strategy profile of all primaries (players).

Definition 1. *S_{-i} denotes the strategy profile of primaries other than i. $E\{u_{i,j}(\psi_{i,j}, S_{-i})\}$ denotes the expected profit when primary i's channel is in state j and it uses strategy $\psi_{i,j}(\cdot)$ and other primaries use strategy S_{-i}.*

Definition 2. *A* Nash equilibrium $(S_1, \ldots, S_n)$ *is a strategy profile such that no primary can improve its expected profit by unilaterally deviating from its strategy [14]. So, with $S_i = (\psi_{i,1}, ..., \psi_{i,n})$, $(S_1, \ldots, S_n)$, is a Nash equilibrium (NE) if for each primary i and channel state j*

$$E\{u_{i,j}(\psi_{i,j}, S_{-i})\} \geq E\{u_{i,j}(\tilde{\psi}_{i,j}, S_{-i})\} \ \forall \ \tilde{\psi}_{i,j}. \tag{2}$$

A NE $(S_1, \ldots, S_n)$ is a symmetric NE *if $S_i = S_j$ for all i, j.*

The above game is a symmetric one since primaries have the same action sets, payoff functions and their channels are statistically identical. We therefore consider only symmetric NEs[2]. Clearly, for any symmetric NE, we can represent the strategy of any primary as $S = (\psi_1(.), \psi_2(.), ..., \psi_n(.))$ where we drop the index corresponding to the primary.

[1] Recall the definition of distribution function (d.f) of a random variable X is the function $G(x) = P(X \leq x) \quad x \in \Re$ [3]

[2] For a symmetric game, an asymmetric NE is rarely realized. For example. for two players, if (S_1, S_2) is a NE, (S_2, S_1) is also a NE. The realization of such a NE is possible only when each player knows whether the other uses S_1 or S_2. This complication is somewhat alleviated for a symmetric NE as all players play the same strategy; this complication is eliminated only when there is a unique symmetric NE. Note that, there are plethora of examples of symmetric games, which have multiple NEs. We prove that that there is a unique symmetric NE for the game we consider.

Let $\phi_j(x)$ denote the expected profit of a primary whose channel is in state j and who selects a penalty x and $r(x)$ denote the probability that a channel quoted at penalty x is sold. Note that the dependence of $\phi_j(x), r(x)$ on the strategy profile of the primaries is not explicitly indicated to ensure notational simplicity. Also, note that $r(x)$ does not depend on the state of the channel since secondaries select the channels based only on the penalties. Next,

$$\phi_j(x) = (f_j(x) - c)r(x). \tag{3}$$

(recall that the inverse of the penalty function $g_j(\cdot)$, $f_j(\cdot)$, provides the price that corresponds to penalty x and channel state j).

Definition 3. *A* best response *penalty for a channel in state $j \geq 1$ is x if and only if*

$$\phi_j(x) = \sup_{y \in \Re} \phi_j(y).$$

Let $u_{j,max} = \phi_j(x)$ for a best response x for state j, $j \geq 1$ i.e., $u_{j,max}$ is the maximum expected profit that a primary earns under NE strategy profile, when its channel is in state j, $j \geq 1$.

III. A SYMMETRIC NE: EXISTENCE, UNIQUENESS AND COMPUTATION

First, we identify key structural properties of a symmetric NE (should it exist). Next we show that the above properties leads to a unique strategy profile which we explicitly compute - thus the symmetric NE is unique should it exist. We finally prove that the strategy profile resulting from the structural properties above is indeed a symmetric NE thereby establishing existence.

A. Structure of a symmetric NE

We start with by providing some important properties that any symmetric NE $(\psi_1(\cdot), \ldots, \psi_n(\cdot))$ must satisfy.

Theorem 1. *$\psi_i(.), i \in \{1, .., n\}$ is a continuous probability distribution.*

The above theorem rules out any pure strategy symmetric NE.

Definition 4. *We denote the lower and upper endpoints of the support set[3] of $\psi_i(.)$ as L_i and U_i respectively i.e.*

$$L_i = \inf\{x : \psi_i(x) > 0\}$$

$$U_i = \inf\{x : \psi_i(x) = 1\}$$

We next show that the support sets are ordered in increasing order of the state indices.

Theorem 2. *$U_i \leq L_j$, if $j < i$*

We finally rule out any "gaps" inside the support sets and between the support sets for different $\psi_i(\cdot)$, $i = 1, .., n$. This also establishes that $\psi_i(\cdot)$ is strictly increasing in $[L_i, U_i]$.

[3]The support set of a probability distribution is the smallest closed set such that the probability of its complement is 0.

Theorem 3. *The support set of $\psi_i(.), i = 1, .., n$ is $[L_i, U_i]$ and $U_i = L_{i-1}$ for $i = 2, .., n$, $U_1 = v$.*

Remark: The structure of the symmetric NE identified in Theorems 1 to 3 provide several interesting insights:

- Theorem 2 implies that the primaries select the highest penalties for the worst states. The primaries therefore do not strive to render all states equally preferable to the secondaries through price selection.
- Theorems 1 and 3 reveal that the symmetric NE strategy profile consists of "well-behaved" distribution functions.

B. Computation and Uniqueness of a Symmetric NE

We now show that the structural properties of a symmetric NE identified in Theorems 1, 2, 3 are satisfied by a unique strategy profile, which we explicitly compute. This proves the uniqueness of a symmetric NE subject to existence. We start with the following definitions.

$$w(x) = \sum_{i=m}^{l-1} \binom{l-1}{i} x^i (1-x)^{l-i-1} \tag{4}$$

$$w_i = w(\sum_{j=i}^{n} q_j) \quad \text{for} i = 1, ..., n \quad \text{and} w_{n+1} = 0 \tag{5}$$

Clearly, for $x \in [0, 1]$, $w(x)$ is the probability of at least m successes out of l-1 independent Bernoulli trials, each of which occurs with probability x. Note that $w(\cdot)$ is continuous and strictly increasing in $[0, 1]$ [21], so its inverse exists. Note that $w_i > w_j$ if $i < j, i, j \in \{1, ..., n\}$ as w_i is the success probability of at least m successes out of (l-1) independent Bernoulli Events, where each of which occurs with probability $\sum_{j=i}^{n} q_j$.

Lemma 1. *For $1 \leq i \leq n$,*

$$\begin{aligned} u_{i,max} &= p_i - c \\ \textit{where, } p_i &= c + (f_1(v) - c)(1 - w_1) \\ &\quad + \sum_{j=1}^{i-1} (f_{j+1}(L_j) - f_j(L_j))(1 - w_{j+1}) \quad (6) \\ \textit{and } L_i &= g_i(\frac{p_i - c}{1 - w_{i+1}} + c) \quad (7) \end{aligned}$$

Using (6) and (7), $u_{i,max}, L_i$ can be computed recursively starting from $i = 1$. Note that as $1 - w_i > 0, \forall i \in \{1, \ldots, n\}$, thus, $p_i - c > 0$. Hence, from the definition of L_k (7), it is evident that

$$f_k(L_k) > c \tag{8}$$

Expressions of L_i and p_i are used in the following lemma to determine the unique $\psi_i(\cdot)$, if it exists

Lemma 2. *A symmetric NE strategy profile $(\psi_1(\cdot), \ldots, \psi_n(\cdot))$ comprises of:*

$$\begin{aligned} \psi_i(x) =& 0, \textit{if } x < L_i \\ & \frac{1}{q_i}(w^{-1}(\frac{f_i(x) - p_i}{f_i(x) - c}) - \sum_{j=i+1}^{n} q_j), \textit{if } L_{i-1} \geq x \geq L_i \\ & 1, \textit{if } x > L_{i-1} \quad (9) \end{aligned}$$

where $L_i, i = 1,..,n$ are as defined in (7) and $L_0 = v$.

Next lemma will ensure that $\psi_i(\cdot)$ as defined in lemma 2 is indeed a d.f.

Lemma 3. *$\psi_i(\cdot)$ as defined in Lemma 2 is a strictly increasing and continuous distribution function.*

C. Existence of a symmetric NE

In this section, We prove that symmetric strategy profile identified in previous section is indeed a NE strategy profile.

Theorem 4. *$(\psi_1(\cdot), \ldots, \psi_n(\cdot))\, j = 1,..,n$ as defined in lemma 2 is a symmetric NE.*

a) Remark: Note that all our results readily generalize to allow for random number of secondaries (M) with probability mass functions (p.m.f.) $\Pr(M = m) = \gamma_m$. A primary does not have the exact realization of number of secondaries, but it knows the p.m.f. . We only have to redefine $w(x)$ as-

$$\sum_{k=0}^{\max(M)} \gamma_k \sum_{i=k}^{l-1} \binom{l-1}{i} x^i (1-x)^{l-1-i} \tag{10}$$

and $w_{n+1} = \gamma_0$.

IV. Performance evaluation of the symmetric NE

Definition 5. *Let R_{NE} denote the total expected profit at Nash equilibrium. Then,*

$$R_{NE} = l.\sum_{i=1}^{n}(q_i.(p_i - c)) \tag{11}$$

Lemma 4. *Let $c_j = g_j(c), j = 1,..,n$.*

1) *If $m \geq (l-1)(\sum_{j=1}^{n} q_j + \epsilon)$ for some $\epsilon > 0$, then $R_{NE} \to l.\sum_{j=1}^{n} q_j.(f_j(v) - c)$ as $l \to \infty$.*
2) *If $(l-1)(\sum_{j=i-1}^{n} q_j - \epsilon) \geq m \geq (l-1)(\sum_{j=i}^{n} q_j + \epsilon), i \in \{2,..,n\}$, for some $\epsilon > 0$, then $R_{NE} \to l.\sum_{j=i}^{n} q_j.(f_j(c_{i-1}) - c)$ as $l \to \infty$.*
3) *If $m \leq (l-1)(q_n - \epsilon)$ for some $\epsilon > 0$, then $R_{NE} \to 0$ as $l \to \infty$.*

Note that, for $j > i$, $c_j < c_i$ (as $g_j(c) < g_i(c)$), thus, asymptotically R_{NE} decreases as m decreases. This is expected as competition increases with decrease in m, and thus prices are chosen progressively closer to the lower limit, that of the transition cost, c.

Definition 6. *Let R_{OPT} be the maximum expected profit earned through collusive selection of prices by the primaries.* Efficiency *η is defined as $\dfrac{R_{NE}}{R_{OPT}}$.*

Efficiency is a measure of the reduction in the expected profit owing to competition. Asymptotic behavior of η is characterized by the following lemma:

Lemma 5.
1) *If $m \geq (l-1)(\sum_{j=1}^{n} q_j + \epsilon)$ for some $\epsilon > 0$, then $\eta \to 1$ as $l \to \infty$.*
2) *If $m \leq (l-1)(q_n - \epsilon)$ for some $\epsilon > 0$, then $\eta \to 0$ as $l \to \infty$.*

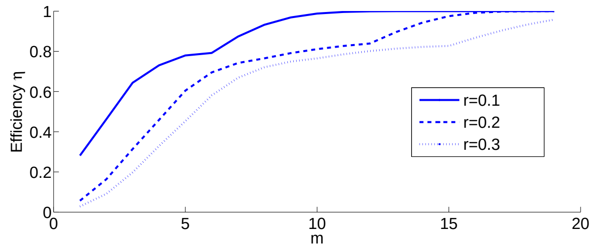

Fig. 1: Efficiency versus m for three different sets of values of probabilities for $l = 20$ and $n = 3$, $q_1 = q_2 = q_3 = r$, $v = 100, c = 1$, $g_i(x) = x^{10} - i^7$.

The lemma does not characterize the asymptotic limits for η for $m \in [(l-1)\sum_{j=1}^{n} q_j, (l-1)q_n]$. However, our numerical computation reveals that η increases from 0 to 1 with increase in m (figure 1) - the variation is largely monotonic barring for a few discrepancies owing to n, m being finite. Intuitively, demand increases with increase in m; thus primaries set their penalties close to the highest possible value for all states which leads to higher efficiency. On the other hand, when m decrease, competition becomes intense and primaries chooses prices close to c and expected profits under the symmetric NE decreases: R_{NE} is very small as lemma 4 reveals. But, if primaries collude, primaries can judiciously offer only the channels of highest possible states to the secondaries to gain a large profit. Hence, the decrease in R_{OPT} with decrease in m is slower, which leads to lower efficiency for low m.

Similarly, when q_is increases, competition becomes intense and primaries chooses price closer to c, hence R_{NE} decrease. But, when primaries collude, they still sell at highest possible penalty for a channel and hence η decrease. On the other hand, when, q_is decrease, primaries set their prices closer to highest possible values for all states and thus, η increase.

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V. Appendix

First, we introduce some terminologies and observation that we will use throughout this section.

Definition 7. *Let X_m be the mth smallest offered penalty offered by primaries $i = 2, \dots, l$, and let $F(\cdot)$ denote the distribution function of X_m.*

For a symmetric strategy profile, $F(\cdot)$ would remain the same if we had considered any $l-1$ primaries rather than $2, \dots, l$.

Observation 1. *Any point $y \leq g_j(c)$ can not be a best response (definition 3) for channel state j.*

The observation is evident as the profit $\phi_j(\cdot)$ of a primary is non-positive if the selected penalty is $\leq g_j(c)$. But, $\phi_j(x) > 0$ for $g_j(c) < x \leq v$ as $0 < \sum_{i=1}^n q_i < 1$.

A. Proof of Section III-A

Proof of Theorem 1: Suppose, $\psi_j(\cdot)$ has a jump[4] at x, then all primaries select x as their penalties with positive probability whenever their channel states are j. As, no primary selects a penalty other than a best response with positive probability, thus, x has to be a best response to primaries whenever their channel state is j. Hence, Observation 1 entails that $x > g_j(c)$. We will complete the proof by showing that primary 1 can attain better expected profit by choosing penalty just below x, when channel state is j.
We have,

$$r(x) = P(X_m > x) + r(x|X_m = x)P(X_m = x) \tag{12}$$

[4]A d.f. $G(x)$ is said to have a jump at y of size a, if $G(y) - G(y-) = b$, where $G(y-) = \lim_{x \uparrow y} G(x)$

For every $\epsilon > 0$, it is worthy to note the following relation

$$r(x-\epsilon) \geq P(X_m > x) + P(X_m = x) \tag{13}$$

Thus, from (12) and (13), for every $\epsilon > 0$,

$$\begin{aligned} r(x-\epsilon) - r(x) &\geq P(X_m = x) \cdot (1 - r(x|X_m = x)) \\ &= \gamma \end{aligned} \tag{14}$$

where, $\gamma = P(X_m = x)(1 - r(x|X_m = x))$. Note that $r(x|X_m = x) < 1$ and $P(X_m = x) > 0$, due to symmetry and $l > m$. So, $\gamma > 0$.
Now, expected profit to primary 1 for channel state j at penalty $x - \epsilon$ is:

$$\begin{aligned} \phi_j(x-\epsilon) &= (f_j(x-\epsilon) - c)r(x-\epsilon) \quad (\text{from}(3)) \\ &\geq (f_j(x-\epsilon) - c)(r(x) + \gamma) \end{aligned} \tag{15}$$

Let, $\delta = f_j(x) - f_j(x-\epsilon)$, so from (15)

$$\begin{aligned} \phi_j(x-\epsilon) &\geq (f_j(x) - \delta - c)(r(x) + \gamma) \\ &= (f_j(x) - c)(r(x) + \gamma) - \delta(r(x) + \gamma) \end{aligned} \tag{16}$$

$f_j(\cdot)$ is continuous and strictly increasing ($f_j(x) > c$),$\gamma > 0$ and independent of ϵ, so $\exists \epsilon > 0$, such that

$$\phi_j(x-\epsilon) > (f_j(x) - c)r(x) = \phi_j(x)$$

which contradicts that x is a best response and hence, the result follows. □

Now, we will show the following lemma and observation, which will facilitate our later analysis.

Lemma 6. *$F(\cdot)$ is continuous in $[c_{min}, v]$ and if $\sum_{j=1}^n \psi_j(y) > \sum_{j=1}^n \psi_j(x)$, then $F(y) > F(x)$*

where, $c_{min} = \min_{i \in \{1,..,n\}} g_i(c)$

Proof. Suppose $a \in [c_{min}, v]$. At any time slot, the event that primary 1 selects penalty less than or equal to a and state of a channel is $i \geq 1$, occurs with probability $q_i \cdot \psi_i(a)$. Hence, the event that primary 1 offers penalty less than or equal to a occurs with probability $\sum_{i=1}^n q_i \cdot \psi_i(a)$. Thus,

$$F(a) = P(X_m \leq a) = w(\sum_{i=1}^n \psi_i(a)) \quad (\text{Recall}(4))$$

Continuity of $F(\cdot)$ follows from the fact that $\psi_i(\cdot), i = 1, \dots, n$ are continuous (Theorem 1).
Now $l < m$, and $\sum_{i=1}^n q_i < 1$, thus $F(\cdot)$increases if $\sum_{i=1}^n \psi_i(.), i \in \{1,..,n\}$ increases (as $\psi_i(.)$ is d.f. so it is non-decreasing). □

Lemma 6 implies that $P(X_m = x) = 0$ for each x and thus $r(x) = 1 - F(x)$. Hence,

$$\phi_j(x) = (f_j(x) - c)(1 - F(x)) \tag{17}$$

Observation 2. *Every element in the support set of $\psi_i(\cdot)$ is a best response; thus, so are L_i, U_i.*

Proof. Suppose there exists a point z in the support set of $\psi_i(\cdot)$, which is not a best response. Therefore, primary 1 plays at z with probability 0 when channel state is i.
Now, one of the following two cases must arise.

Case I: $\exists$ a neighborhood [20] of radius $\delta > 0$ around z, such that no point in this neighborhood is a best response. Neighborhood of radius $\delta > 0$ of z is an open set (theorem 2.19 of [20]). Hence, we can eliminate that neighborhood and can attain a smaller closed set, such that its complement has probability zero under $\psi_i(\cdot)$, which is against the definition of support set.
Case II: For every $\epsilon > 0$, $\exists y \in (z-\epsilon, z+\epsilon)$, such that y is a best response. Then, we must have a sequence $z_k, k = 1, 2, \ldots$ such that each z_k is a best response, and $\lim_{k\to\infty} z_k = z$ [20]. But profit to primary 1 for channel state i at each of z_k is $(f_i(z_k) - c)(1 - F(z_k))$. Now, from continuity of $f_i(\cdot)$ and $F(\cdot)$ (lemma 6)-

$$\begin{aligned}\lim_{k\to\infty} \phi_i(z_k) &= (f_i(z_k) - c)(1 - F(z_k)) \\ &= (f_i(z) - c)(1 - F(z)) = \phi_i(z) \end{aligned} \tag{18}$$

As each of $z_k, k = 1, 2, \ldots$ is a best response, $u_{i,max} = \phi_i(z_k), k = 1, 2, \ldots$. Hence, from (18) $u_{i,max} = \phi_i(z)$ and z is a best response. We can conclude the result by noting that U_i, L_i (Definition 4) are in the support set of $\psi_i(\cdot)$. $\square$

Proof of Theorem 2: From Observation 2 it is sufficient to show that for any x, y such that $c_{min} < x < y \leq v$, if x is a best response when the state of the channel is j, then y can not be a best response when the state of the channel is i for $i > j$. If not consider $y > x$ such that x, y are the best responses when channel states are respectively j, i. Now, from Observation 1 $f_i(y) > c, f_j(x) > c$. Also,

$$\begin{aligned} u_{i,max} &= (f_i(y) - c)(1 - F(y)) && (19) \\ \phi_j(y) &= (f_j(y) - c)(1 - F(y)) \\ &= u_{i,max}\cdot\frac{f_j(y) - c}{f_i(y) - c} \quad (\text{from}(19)) \\ u_{j,max} &\geq u_{i,max}\cdot\frac{f_j(y) - c}{f_i(y) - c} && (20) \end{aligned}$$

Next,

$$\begin{aligned} u_{j,max} &= (f_j(x) - c)(1 - F(x)) \\ \phi_i(x) &= (f_i(x) - c)(1 - F(x)) \\ &= u_{j,max}\cdot\frac{f_i(x) - c}{f_j(x) - c} && (21) \end{aligned}$$

Using (20) and (21), we obtain-

$$\phi_i(x) \geq u_{i,max}\cdot\frac{(f_j(y) - c)(f_i(x) - c)}{(f_i(y) - c)(f_j(x) - c)} \tag{22}$$

But, then, since $y > x$, $i > j$, (1) implies that $\phi_i(x) > u_{i,max}$ which contradicts the definitions of $u_{i,max}$ and $\phi_i(x)$. $\square$

Proof Of Theorem 3: Suppose the statement is not true. But, it follows from Theorem 2 that there exists an interval $(x, y) \subseteq [L_n, v]$, such that no primary offers penalty in the interval (x, y) with positive probability. So, we must have $\tilde{a}$ such that

$$\tilde{a} = \inf\{b \leq x : \psi_j(b) = \psi_j(x), \forall j\}$$

By definition of $\tilde{a}$, $\tilde{a}$ is a best response for at least one state i. But, as primaries do not offer penalty in the range $(\tilde{a}, y)$, so from (17), $\phi_i(z) > \phi_i(\tilde{a})$ for each $z \in (\tilde{a}, y)$. This is because $F(y) = F(\tilde{a})$ and $f_i(\tilde{a}) < f_i(z)$. Thus, $\tilde{a}$ can not be a best response for state i. $\square$

B. Proofs of Section III-B

proof of Lemma 1: We first prove (6) using induction, (7) follows from (6).

From theorem 3, $\psi_i(\cdot)$'s support set is $[L_i, L_{i-1}], i = \{2, ..., n\}$ and $[L_1, v]$ for $i = 1$. As v is the best response for channel state 1,

$$u_{1,max} = (f_1(v) - c)(1 - w(\sum_{i=1}^{n} q_i)) = p_1 - c \tag{23}$$

Thus, (6) holds for $i = 1$. Let, (6) be true for $i = t < n$. We have to show that (6) is satisfied for $i = t + 1$ assuming that it is true for $i = t$. Thus,by induction hypothesis,

$$\begin{aligned} u_{t,max} &= p_t - c = (f_1(v) - c)(1 - w_1) \\ &+ \sum_{j=1}^{j=t-1} (f_{j+1}(L_j) - f_j(L_j))(1 - w_{j+1}) \end{aligned} \tag{24}$$

Now, L_t is a best response for state t, and

$$\phi_t(L_t) = (f_t(L_t) - c)(1 - w_{t+1}) = p_t - c \tag{25}$$

Now, as L_t is also a best response for state t+1,

$$\phi_{t+1}(L_t) = (f_{t+1}(L_t) - c)(1 - w_{t+1}) = u_{t+1,max} \tag{26}$$

Using (25), (24) in (26) we obtain-

$$\begin{aligned} u_{t+1,max} &= (f_1(v) - c)(1 - w_1) \\ &+ \sum_{j=1}^{j=t-1} (f_{j+1}(L_j) - f_j(L_j))(1 - w_{j+1}) \\ &+ (f_{t+1}(L_t) - f_t(L_t))(1 - w_{t+1}) \\ &= (f_1(v) - c)(1 - w_1) \\ &+ \sum_{j=1}^{j=t} (f_{j+1}(L_j) - f_j(L_j))(1 - w_{j+1}) \\ &= p_{t+1} - c \end{aligned}$$

Thus, $u_{t+1,max} = p_{t+1} - c$ and it satisfies (6). Thus, (6) follows from mathematical induction.

(7) follows since $(f_i(L_i) - c)(1 - w_{i+1}) = p_i - c$ and $g_i(\cdot)$ is the inverse of $f_i(\cdot)$. $\square$

proof of Lemma 2: L_i, L_{i-1} are the end-points of the support set of $\psi_i(\cdot)$ from definition 4, and their values have been computed in lemma 1. We should have for $x < L_i$, $\psi_i(x) = 0$ and for $x > L_{i-1}$,$\psi_i(x) = 1$. From theorem 3, every point $x \in [L_i, L_{i-1}]$ is a best response for state i, and hence,

$$(f_i(x) - c)(1 - w(\sum_{j=i+1}^{n} q_j + q_i.\psi_i(x))) = u_{i,max} = p_i - c.$$

Thus, the expression for $\psi_i(\cdot)$ follows. We conclude the proof by noting that the domain and range of $w(.)$ is $[0, 1]$, and $\frac{p_i - c}{f_i(x) - c} < 1$ for $x \in [L_i, L_{i-1}]$: so $w^{-1}(.)$ is defined at $1 - \frac{p_i - c}{f_i(x) - c}$. $\square$.

Now, we will state an observation, that we will use throughout the rest of this section.

Observation 3.

$$p_i - c = (f_i(L_{i-1}) - c)(1 - w_i) \quad (27)$$

where $L_0 = v$.

Proof. From(6), for $i \geq 2$-

$$p_{i-1} - c = p_i - c - (f_i(L_{i-1}) - f_{i-1}(L_{i-1}))(1 - w_i) \quad (28)$$

From (7), (28) and the fact that $f_i(\cdot) = g_i^{-1}(\cdot)$, we obtain-

$$\begin{aligned}
&p_{i-1} - c = (f_{i-1}(L_{i-1}) - c)(1 - w_i)\\
&p_i - c - (f_i(L_{i-1}) - f_{i-1}(L_{i-1}))(1 - w_i)\\
&= (f_{i-1}(L_{i-1}) - c)(1 - w_i)\\
&p_i - c = (f_i(L_{i-1}) - c)(1 - w_i)
\end{aligned}$$

We can conclude the result for $i = 1$, by replacing v with L_0 in (6). ☐

proof of lemma 3: Note that,

$$\begin{aligned}
\psi_i(L_i) &= \frac{1}{q_i}(w^{-1}(1 - \frac{p_i - c}{f_i(L_i) - c}) - \sum_{j=i+1}^{n} q_j)\\
&= \frac{1}{q_i}(w^{-1}(w_{i+1}) - \sum_{j=i+1}^{n} q_j) \quad \text{from(7)}\\
&= 0 \quad (\text{by(5)}) \quad (29)
\end{aligned}$$

From (9) and (27), we obtain

$$\begin{aligned}
\psi_i(L_{i-1}) &= \frac{1}{q_i}(w^{-1}(1 - \frac{p_i - c}{f_i(L_{i-1} - c)}) - \sum_{j=i+1}^{n} q_j)\\
&= \frac{1}{q_i}(w^{-1}(w_i) - \sum_{j=i+1}^{n} q_j)\\
&= \frac{1}{q_i}.q_i = 1 \quad (\text{as} w_i = w(\sum_{j=i}^{n} q_j) \quad (30)
\end{aligned}$$

$w(.)$ is continuous, strictly increasing on compact set $[0, \sum_{j=1}^{n} q_j]$, so w^{-1} is also continuous (theorem 4.17 in [20]). Also, $\frac{p_i - c}{f_i(x) - c}$ is continuous for $x \geq L_i$ as $f_i(x) > c$, so $\psi_i(.)$ is continuous as it is a composition of two continuous functions. Again, $w^{-1}(.)$ is strictly increasing (as $w(\cdot)$ is strictly increasing), $1 - \frac{p_i - c}{f_i(x) - c}$ is strictly increasing (as $f_i(\cdot)$ is strictly increasing), so $\psi_i(.)$ is strictly increasing on $[L_i, L_{i-1}]$ (as it is a composition of two strictly increasing functions (theorem 4.7 in [20])) ☐.

C. Proof of Section III-C

First, it is worthy to note the following observation

Observation 4. *For* $t > s, t, s \in \{1, \ldots, n\}$

$$p_t - c = (p_s - c) \prod_{i=s}^{t-1} \frac{f_{i+1}(L_i) - c}{f_i(L_i) - c} \quad (31)$$

Proof. From Observation 3 and (7), (8), it follows that

$$p_{i-1} - c = (p_i - c) \frac{f_{i-1}(L_{i-1}) - c}{f_i(L_{i-1}) - c} \quad (32)$$

We obtain the result using recursion. ☐

proof of Theorem 4: If state of channel of primary 1 is $j \geq 1$ and it select penalty x, then its expected profit is-

$$\begin{aligned}
\phi_j(x) &= (f_j(x) - c)r(x)\\
&= (f_j(x) - c)(1 - w(\sum_{i=1}^{n} q_i.\psi_i(x))) \quad (33)
\end{aligned}$$

First, suppose $x \in [L_j, L_{j-1}]$. From (33) and (9), we obtain

$$\begin{aligned}
&(f_j(x) - c)(1 - w(\sum_{i=1}^{n} q_i \psi_i(x)))\\
&= (f_j(x) - c)(1 - w(\sum_{k=j+1}^{n} q_k + q_j \psi_j(x)))\\
&= (f_j(x) - c)(1 - w(w^{-1}(1 - \frac{p_j - c}{f_j(x) - c})))\\
&= p_j - c \quad (34)
\end{aligned}$$

It remains to show that for $x \in [L_k, L_{k-1}], k \neq j, k \in \{1, .., n\}$, profit to primary 1 is at most $p_j - c$, when channel state is in j.

Now, let $x \in [L_k, L_{k-1}]$. If primary 1 selects penalty x at channel state j, then its expected profit would be $\phi_j(x) = (f_j(x) - c)r(x)$. But,

$$\begin{aligned}
\phi_k(x) &= (f_k(x) - c)r(x)\\
&= (f_k(x) - c)(1 - w(\sum_{i=1}^{n} q_i \psi_i(x)))\\
&= p_k - c \quad (35)
\end{aligned}$$

From (8), we have $f_k(x) > c$, as $x \in [L_k, L_{k-1}]$. So, using(35)-

$$\phi_j(x) = \frac{(p_k - c)(f_j(x) - c)}{f_k(x) - c}$$

Hence,

$$\phi_j(x) - (p_j - c) = \frac{(p_k - c)(f_j(x) - c)}{f_k(x) - c} - (p_j - c) \quad (36)$$

We will show that $\phi_j(x) - (p_j - c)$ is non-positive. As, $k \neq j$, so only the following two cases are possible.

Case I: $k < j$

From (1) and for $i < j$, we have-

$$\frac{f_i(L_{i-1}) - c}{f_i(L_i) - c} > \frac{f_j(L_{i-1}) - c}{f_j(L_i) - c} (\text{as} L_i < L_{i-1}) \quad (37)$$

From Observation 4 and (37), we obtain-

$$p_j - c = \frac{(p_k-c)(f_j(L_{j-1})-c)}{f_k(L_k)-c}\prod_{i=k+1}^{j-1}\frac{f_i(L_{i-1})-c}{f_i(L_i)-c}$$
$$p_j - c \geq \frac{(p_k-c)(f_j(L_{j-1})-c)}{f_k(L_k)-c}\prod_{i=k+1}^{j-1}\frac{f_j(L_{i-1})-c}{f_j(L_i)-c}$$
$$= \frac{(p_k-c)(f_j(L_{j-1})-c)}{f_k(L_k)-c}.\frac{(f_j(L_k)-c)}{f_j(L_{j-1})-c}$$
$$= \frac{(p_k-c)(f_j(L_k)-c)}{f_k(L_k)-c}$$

Hence, from (36), we obtain-

$$\begin{aligned}&\phi_j(x) - (p_j - c)\\ &\leq (p_k-c)(\frac{f_j(x)-c}{f_k(x)-c} - \frac{f_j(L_k)-c}{f_k(L_k)-c})\end{aligned} \quad (38)$$

Now, $x \in [L_k, L_{k-1}]$, $j > k$. Hence, from (38) and assumption 1, we have-

$$\phi_j(x)_j(x) \leq p_j - c \quad (39)$$

Case II: $j < k$
From, Observation 4,

$$(p_k - c) = (p_j - c)\prod_{i=j}^{k-1}\frac{f_{i+1}(L_i)-c}{f_i(L_i)-c} \quad (40)$$

Now, for $i \geq j$ from (1), we have-

$$\frac{f_{i+1}(L_i)-c}{f_j(L_i)-c} < \frac{f_{i+1}(L_{i+1})-c}{f_j(L_{i+1})-c} \quad (\text{as} L_i > L_{i+1}) \quad (41)$$

Now, from (40) and using (41), we obtain-

$$p_k - c = (p_j - c)\prod_{i=j}^{k-1}\frac{f_{i+1}(L_i)-c}{f_i(L_i)-c}$$
$$= (p_j - c).\frac{f_k(L_{k-1})-c}{f_j(L_j)-c}\prod_{i=j+1}^{k-1}\frac{f_i(L_{i-1})-c}{f_i(L_i)-c}$$
$$\leq (p_j - c).\frac{f_k(L_{k-1})-c}{f_j(L_j)-c}\prod_{i=j+1}^{k-1}\frac{f_j(L_{i-1})-c}{f_j(L_i)-c}$$
$$= (p_j - c).\frac{f_k(L_{k-1})-c}{f_j(L_j)-c}\frac{f_j(L_j)-c}{f_j(L_{k-1})-c}$$
$$= (p_j - c).\frac{f_k(L_{k-1})-c}{f_j(L_{k-1})-c}$$

Thus, from (36), we obtain-

$$\begin{aligned}&\phi_j(x) - (p_j - c)\\ &\leq (p_j - c)(\frac{f_k(L_{k-1})-c}{f_j(L_{k-1})-c}.\frac{f_j(x)-c}{f_k(x)-c} - 1)\\ &\leq 0(\text{as} x \leq L_{k-1}, j < k \quad \text{and from Assumption 1})\end{aligned} \quad (42)$$

Hence, from(42), (39) and (34), every $x \in [L_j, L_{j-1}]$ is a best response to primary 1 when channel state is j and thus (9) constitute a Nash Equilibrium strategy profile. □

D. Proofs of Section IV

We will first establish part 1 and 3 of lemma 4 . Part 2 of lemma 4 is cumbersome and we defer its proof until the end of the section. Lemma 5 will readily follow from part 1 and part 3 of lemma 4.

proof of part 1 of Lemma 4: First, note that a primary can achieve profit of at most $f_i(v)-c$, when channel state is $i \geq 1$. Hence,

$$R_{NE} \leq l \cdot \sum_{i=1}^{n} q_i \cdot (f_i(v) - c) \quad (43)$$

When, primary 1 selects penalty v, at channel state $i \geq 1$, then its expected profit is $\phi_i(v) = (f_i(v) - c)(1 - w_1)$. Now, from Theorem 4 under NE strategy profile,

$$p_i - c \geq \phi_i(v) = (f_i(v) - c)(1 - w_1) \quad (44)$$

Hence,

$$R_{NE} \geq l.(\sum_{i=1}^{n} q_i.(v_i - c))(1 - w_1) \quad (45)$$

Now, let $Z_i, i = 1, .., l-1$ be the Bernoulli trials with success probabilities $\sum_{i=1}^{n} q_i$ and let $Z = \sum_{i=1}^{l-1} Z_i$, so $P(Z \geq m)$ is equal to w_1 by (5). Since $m \geq (l-1)(\sum_{i=1}^{n} q_i + \epsilon)$, $E(Z) = (l-1)\sum_{i=1}^{n} q_i$, thus, by weak law of large numbers [19], $w_1 \to 0 \quad \text{as} l \to \infty$. Hence the result follows from (45) and (43). □.

proof of Lemma part 3 of Lemma 4: Now, suppose that $m \leq (l-1)(q_n - \epsilon)$, for some $\epsilon > 0$. Let, $Z_i, i = 1, ..., l-1$ be the Bernoulli trials with success probabilities q_n and $Z = \sum_{i=1}^{l-1} Z_i$, $E(Z) = (l-1)q_n$. Hence,

$$\begin{aligned}1 - w_n \quad \leq \quad & P(Z \leq m)\\ & \leq P(Z \leq (l-1)(q_n - \epsilon))\\ & \leq P(|Z - (l-1)q_n| \geq (l-1)\epsilon)\\ & \leq 2\exp(-\frac{2(l-1)^2\epsilon^2}{l-1})\\ & \text{(from Hoeffding's Inequality [4])}\\ & = 2\exp(-2(l-1)\epsilon^2)\end{aligned} \quad (46)$$

$1 - w_i < 1 - w_j$ (if $j > i$), $f_k(L_{k-1}) > f_{k-1}(L_{k-1})$. Hence, it can be readily seen from(6) that

$$\begin{aligned}&p_i - c \leq (v_1 - c)(1 - w_n) +\\ &\sum_{k=2}^{i}(f_k(L_{k-1}) - f_{k-1}(L_{k-1}))(1 - w_n)\end{aligned} \quad (47)$$

Thus,

$$\begin{aligned}R_{NE} \leq l \cdot (1 - w_n)(\sum_{j=1}^{n} q_j.((f_1(v) - c)\\ + \sum_{k=2}^{j}(f_k(L_{k-1}) - f_{k-1}(L_{k-1}))))\end{aligned} \quad (48)$$

As $f_i(c) \leq L_i \leq v$, hence, the result follows from (46) □.

Note the bound of R_{NE} (from (46) and (48)) for $m \leq (l-1)(q_n - \epsilon)$, $\epsilon > 0$,

$$R_{NE} \leq l \cdot \gamma \cdot \exp(-2\epsilon^2.(l-1)) \quad (49)$$

where $\gamma = 2\sum_{j=1}^{n} q_j.((v_1 - c) + \sum_{k=2}^{j}(f_k(L_{k-1}) - f_{k-1}(L_{k-1})))$. We will use this bound later.

From, the definition of η, it should be clear that

$$\eta \leq 1 \tag{50}$$

Now, we will show lemma 5

proof of part 1 of lemma 5: First suppose that $m \geq (l-1)(\sum_{i=1}^{n} q_i + \epsilon)$. From, definition of R_{OPT}, it is obvious that

$$R_{OPT} \leq l.(\sum_{i=1}^{n}(q_i.(v_i - c))) \tag{51}$$

Hence the result follows from part 1 of lemma 4 and (50). $\square$

proof of part 2 of Lemma 5: Now, suppose $m \leq (l-1)(q_n - \epsilon)$, for some $\epsilon > 0$.

Let, Z be the number of primaries, whose channel is in state n. Hence,

$$\begin{aligned} R_{OPT} &\geq E(\min(Z, m))(v_n - c) \\ \frac{R_{OPT}}{v_n - c} &\geq E(\min(Z, m)) \end{aligned} \tag{52}$$

Note that $E(Z) = l \cdot q_n$, $Var(Z) = l \cdot q_n(1 - q_n)$.

We introduce a new random variable Y as follows-

$$Y = \begin{cases} m, & \text{if} Z \geq m \\ 0, & \text{otherwise} \end{cases}$$

So,

$$\begin{aligned} E(\min(Z, m)) &\geq E(Y) \\ &= m.P(Z \geq m) \\ &= m.(1 - P(Z < m)) \\ &\geq m.(1 - P(Z \leq (l-1)(q_n - \epsilon))) \\ &\geq m.(1 - P(|Z - l.q_n| \geq (l-1)\epsilon) \\ &\geq m.(1 - \frac{l.q_n.(1-q_n)}{(l-1)^2.\epsilon^2}) \\ &\text{(From Chebyshev's Inequality)} \end{aligned} \tag{53}$$

Hence, from (49) and (53), we obtain-

$$\eta \leq \frac{l.\gamma.\exp(-2(l-1)\epsilon^2)}{m.(1 - \frac{l.q_n.(1-q_n)}{(l-1)^2.\epsilon^2}).(v_n - c)}$$

Thus, η tends to zero for $m \leq (l-1)(q_n - \epsilon)$,as l tends to infinity,(as $m \neq 0$). $\square$.

proof of part 3 of Lemma 4: Suppose that $(l-1)(\sum_{j=i-1}^{n} q_j - \epsilon) \geq m \geq (l-1)(\sum_{j=i}^{n} q_j + \epsilon), i \in \{2, \ldots, n\}$ for some $\epsilon > 0$. Recall from (5) that, w_i is the probability of at least m successes out of $l-1$ independent Bernoulli trials, each of which occurs with probability $\sum_{j=i}^{n} q_j$. Hence from weak law of large numbers [19]

$$\begin{aligned} w_i &\to 0 \quad \text{as} l \to \infty \\ 1 - w_i &\to 1 \quad \text{as} l \to \infty \end{aligned} \tag{54}$$

Since $w_j < w_i$, for $j > i$ (from (5)), we have for $j \geq i$

$$1 - w_j \to 1 \quad \text{as} l \to \infty \tag{55}$$

Again, as $m \leq (l-1)(\sum_{j=i-1}^{n} q_j - \epsilon)$,so, from weak law of large numbers [19], for every $\epsilon > 0$, $\exists L$, such that $1 - w_{i-1} < \epsilon$, whenever $l \geq L$. Hence,

$$\begin{aligned} 1 - w_{i-1} &\underset{l\to\infty}{\to} 0 \\ 1 - w_j &\underset{l\to\infty}{\to} 0 \quad (\text{for} j < i) \end{aligned} \tag{56}$$

Now, $c_i \leq L_i \leq v$, thus, it is evident from (6) and (56)-

$$p_j - c \underset{l\to\infty}{\to} 0 \quad (\text{for} j < i) \tag{57}$$

Hence,

$$L_{i-1} \underset{l\to\infty}{\to} c_{i-1} \tag{58}$$

Using (56) in (6), we obtain for $j \geq i$

$$\begin{aligned} p_j - c &= \sum_{k=i}^{j}(f_k(L_{k-1}) - f_{k-1}(L_{k-1}))(1 - w_k) \\ p_j - c &\underset{l\to\infty}{\to} \sum_{k=i}^{j}(f_k(L_{k-1}) - f_{k-1}(L_{k-1}))(\text{from}(55)) \end{aligned} \tag{59}$$

Next, we will evaluate the limits of $L_j, j \geq i$ as l tends to ∞.

Now, using (27), we obtain for $j \geq i$,

$$\begin{aligned} p_j - c &= (f_j(L_{j-1}) - c)(1 - w_j) \\ p_j &\underset{l\to\infty}{\to} f_j(L_{j-1}) \quad (\text{from}(55)) \end{aligned} \tag{60}$$

Again, using (7), we obtain for $j \geq i$

$$\begin{aligned} p_j - c &= (f_j(L_j) - c)(1 - w_{j+1}) \\ p_j &\underset{l\to\infty}{\to} f_j(L_j) \end{aligned} \tag{61}$$

$f_j(\cdot)$ is strictly increasing, consequently from (60) and (61), $L_j \to L_{j-1}$ (for $j \geq i$). Hence, for $j \geq i$,

$$\begin{aligned} L_j &\underset{l\to\infty}{\to} L_{i-1} \\ L_j &\underset{l\to\infty}{\to} c_{i-1} \quad (\text{from}(58)) \end{aligned} \tag{62}$$

Thus, from (62) and (59), we obtain for $j \geq i$

$$\begin{aligned} p_j - c &\underset{l\to\infty}{\to} \sum_{k=i}^{j}(f_k(c_{i-1}) - f_{k-1}(c_{i-1})) \\ &= (f_j(c_{i-i}) - f_{i-1}(c_{i-1}) \end{aligned} \tag{63}$$

Thus, from (57) and (63), we have-

$$R_{NE} \to l \cdot \sum_{j=i}^{n} q_j \cdot \sum_{k=i}^{j}(f_k(c_{i-1}) - f_{i-1}(c_{i-1})) \tag{64}$$

The result follows from the fact that $f_{i-1}(c_{i-1}) = f_{i-1}(g_{i-1}(c)) = c$. $\square$.