Dynamic Contract Trading in Spectrum Markets

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***Abstract*—We address the question of optimal trading of bandwidth (service) contracts in wireless spectrum markets, for the primary as well as the secondary spectrum providers. We propose a structured spectrum market and consider two basic types of spectrum contracts that can help attain desired flexibilities and trade-offs in terms of service quality, spectrum usage efficiency and pricing: long-term guaranteed-bandwidth contracts, and short-term opportunistic-access contracts. A primary provider (seller) and a secondary provider (buyer) creates and maintains a portfolio composed of an appropriate mix of these two types of contracts. The optimal contract trading question in this context amounts to how the spectrum contract portfolio of a seller (buyer) in the spectrum market should be dynamically adjusted, so as to maximize return (minimize cost) subject to meeting the bandwidth demands of its own subscribers. In this paper, we formulate the optimal contract trading question as a stochastic dynamic programming problem, and obtain structural properties of the optimal dynamic trading strategy that takes into account the current market prices of the contracts and the subscriber demand process in the decision-making. We evaluate and study the optimal dynamic trading strategy numerically, and compare it with a static portfolio optimization strategy where the key trading decision is made in advance, based on the steady-state statistics of the price and subscriber demand processes.**

I. Introduction

The number of users of the wireless spectrum, as well as the demand for bandwidth per user, has been growing at an enormous pace in recent years. Since spectrum is limited, its effective management is vitally important to meet this growing demand. The spectrum available for public use can be broadly categorized into the *unlicensed* and *licensed* zones. In the unlicensed part of the spectrum, any wireless device is allowed to transmit. To use the licensed part, however, license must be obtained from appropriate government authority – the Federal Communications Commission (FCC) in the United States, for example – for the exclusive right to transmit in a certain block of the spectrum over the license time period, typically for a fee. The need for bringing market-based reform in spectrum trading, with the goal of ensuring efficient use of spectrum and fairness in allocation and pricing of bandwidth, is being increasingly recognized by both economists and engineers [4], [8], [17], [18], [19], [28]. The literature on the economics of spectrum allocation has so far mostly focused on the debate of spectrum commons [13], [17], [19] and spectrum auction mechanism design [11], [20], [26], [27]. Spectrum sharing games and/or pricing issues have been considered in [5], [7], [9], [16], [23]. A clear design of the spectrum market structure, precise definition of spectrum contracts, or how the different contracts can be optimally traded in a dynamic market environment is yet to emerge. This is the space in which we contribute in this paper.

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We consider a spectrum market where the license holders (referred to as *primary providers* henceforth) can potentially sell to the *secondary providers*[1] the spectrum they have licensed from the FCC but do not envision using in near future. Primary providers may either be providers of TV broadcasts, or large providers of wireless service who operate nationwide. Secondary providers are relatively smaller, but larger in number, and can be geographically limited providers, whose access to spectrum occurs through the bandwidth (service) contracts that they buy from primary providers. Providers in both categories have their subscriber (TV or mobile communication subscriber) bases whom they need to serve using the spectrum they respectively license from the FCC or buy in the spectrum market. This spectrum market structure is motivated by, and closely resembles, secondary financial markets used for trading of financial instruments (such as stocks, bonds) among investment banks, hedge-funds etc. Like in secondary financial markets, we allow trading in spectrum markets, not only of the raw spectrum (bandwidth), but also of the different kinds of service *contracts* derived from the use of spectrum. A question that is key to the efficient operation of the spectrum market is how the players in the market – the *primary* and the *secondary* providers – should trade spectrum (bandwidth/service) contracts dynamically, based on time-varying demand patterns arising from their subscribers, to maximize their returns while satisfying their subscriber base. This is the central focus of this paper.

We formulate and evaluate the solutions for the *spectrum contract trading* problem for the primary and the secondary providers. We consider two basic forms of contracts that are used for selling/ buying spectral resources: i) *Guaranteed-bandwidth (Type-G)* contracts, and (ii) *Opportunistic-access (Type-O)* contracts. Under the Type-G contracts, a secondary provider purchases a guaranteed amount of bandwidth (in units of frequency bands or sub-bands) for a specified duration of time (typically a "long term") from a primary provider, and pays a fixed fee (either as a lump-sum or as a periodic payment through the duration of the contract) irrespective of how much it uses this bandwidth. If after selling the contract, the primary is unable to provide the promised bandwidth (this may for example happen when the primary is forced to use a band it has sold due to an unexpected rise in its subscriber demand), the primary financially compensates the secondary for contractual violation. On the other hand, Type-O contracts are short-term (one time unit in our model), and a secondary which buys a Type-O contract pays only for the amount of bandwidth it actually uses on the corresponding band. The primary does not provide any guarantee on a Type-O contract and may use the channel sold as a Type-O contract without incurring any penalty. Thus, a Type-O contract provides the secondary the

[1] Note that our notion of "primary" and "secondary" spectrum providers must be distinguished from similar terms often associated with users (subscribers in our case) in the spectrum allocation literature.

right to use the channel if the primary is not using it.

The *spectrum contract trading* problem that we formulate and solve allows the primary (secondary) provider to dynamically adjust its *spectrum contract portfolio*, i.e, choose how much of each type of contract to sell (buy) at any time, so as to maximize (minimize) its profit (cost) subject to satisfying its own subscriber demand that varies with time, and given the current market prices of Type-G and Type-O contracts which also vary with time. The exact nature of the spectrum contract trading (selling/buying) question will depend on whether it is considered from the perspective of the primary provider (seller) or the secondary provider (buyer). We therefore separately address the *Primary's Spectrum Contract Trading (Primary-SCT)* problem (Section II) and the *Secondary's Spectrum Contract Trading (Secondary-SCT)* problem (Section III). We formulate each problem as a finite horizon stochastic dynamic program whose computation time is polynomial in the input size. We *prove* several structural properties of the optimum solutions. For example, we show that the optimal number of Type-G contracts, for both primary and secondary providers, are monotone (increasing or decreasing) functions of the subscribers' demands and the contract prices. These structural results provide more insight into the problems, and allow us to develop faster algorithms for solving the dynamic programs. Finally, using numerical evaluations, we investigate properties of the optimal solutions and demonstrate that the revenues they earn substantially outperform static spectrum portfolio optimization strategies that determine the portfolio based on the steady-state statistics of the contract price and subscriber demand processes (Section IV).

Although the spectrum contract trading problem has been motivated by analogues in financial markets, the actual questions posed and the techniques used to answer them turn out to be quite different owing to the nature of the specific commodity, that is RF spectrum, under consideration. First, both the primary and the secondary must decide their trading strategies considering their subscriber demand which changes with time. For example, a primary (or secondary) can not simply decide to sell (buy) a large number of Type-G contracts at any given time at which their market prices are high (low). This is because a primary will need to pay a hefty penalty if it can not deliver the promised bandwidth owing to an increase in its subscriber demand, and the secondary will need to pay for the contract even if it does not use the corresponding bands owing to a decrease in its subscriber demand. The portfolio optimization literature in finance does not usually address the demand satisfaction constraint. Next, spectrum usage must satisfy certain temporal and spatial constraints that are perhaps unique. Specifically, a frequency band can not be simultaneously successfully used at neighboring locations (without causing significant interference), but can be simultaneously successfully used at geographically disparate locations. Thus, the spectrum trading solution for the primary provider must also take into account spatial constraints for spectrum reuse, and therefore the computation of the optimal trading strategy requires a joint optimization across all locations. We prove a surprising *separation theorem* in this context: when the same signal is broadcast at all locations, the Primary-SCT problem can be solved separately for each location and the individual optimal solutions can subsequently be combined so as to optimally satisfy the global reuse constraints, and obtain the same revenue as the solution of a computationally prohibitive joint optimization across locations (Section II).

The question we address in this paper also differs significantly from existing related work in the Economics and Operations Research literature. In the *inventory problem* [24], [25], a firm maintains an inventory of some good to meet customer demand, which is uncertain. The firm needs to decide the amount to purchase in every slot of a finite or infinite horizon. There is a tradeoff between purchasing and storing costs of the inventory and the cost of not satisfying customers. This is somewhat related to our model, in which a secondary provider needs to decide the number of Type-G and Type-O contracts to buy in every time slot to meet its subscriber demand. However, contracts in our model have a different nature from goods in the inventory model: *e.g.*, Type-G contracts, once bought, can be used in every subsequent time slot to satisfy subscriber demand, whereas goods in an inventory can be used only once to satisfy customer demand. This aspect of Type-G contracts is loosely related to production capacity: once a firm installs capacity, it can be used to manufacture goods in all subsequent time periods. In *capacity expansion* problems [6], [14], a firm needs to optimally decide the volumes, times, and locations of production plants; the tradeoff is that if capacity falls short of demand, the demand cannot be met; on the other hand, if capacity exceeds demand, the excess capacity is wasted. However, our model differs in several aspects from the capacity expansion problem: *e.g.*, (i) there is no counterpart of Type-O contracts in the capacity expansion model, (ii) Type-G contracts can be bought on the spot, whereas capacity installation typically needs to be planned in advance. Finally, spatial reuse constraints being spectrum-specific, are not considered in either inventory or capacity expansion models.

Recently in [15], the authors considered a spectrum market with two types of spectrum contracts– one that provides guaranteed bandwidth, possibly at a higher price, and the other that provides an uncertain amount of bandwidth. A buyer needs to decide the optimal numbers of the two types of contracts to buy, so as to minimize cost subject to constraints on bandwidth shortage. However, in [15], contract trading and optimization are done in a single slot at a time (which makes it a "static" optimization question), unlike in our paper, which considers trading and "dynamic" optimization over a horizon of multiple slots.

II. The Primary's Spectrum Contract Trading (SCT) Problem

In this section we pose and address *Primary-SCT*, the spectrum contract trading question from a primary provider's perspective. We first formulate the problem when a primary provider owns channels in a single region (Section II-A), solve it using a stochastic dynamic program (Section II-B), and identify the structural properties of the optimal solution (Section II-C). Later we formulate and solve the trading problem when the primary owns channels in multiple locations, considering the spatial reuse of channels across different locations (Section II-D).

A. SCT in a single region

We now define the Primary-SCT problem for a primary provider that owns M orthogonal frequency bands (channels) in a single region, which it sells as Type-G or Type-O contracts to secondary providers. We assume that each channel corresponds to one unit of bandwidth and at most one contract – either Type-G or Type-O – can stand leased on a channel at any time. We also assume that the spectrum market has *infinite liquidity*: there is a large number of buyers, and hence the primary provider can sell any or all of the channels it owns anytime and in any combination of Type-G and Type-O contracts.

We assume that time is slotted. Trading of bandwidth is done between primary and secondary providers separately in each of successive time windows of duration T slots each. Henceforth, we focus on the trading and optimization in a single window or time horizon of T time slots. At the beginning of each slot t, the primary determines the number of channels $x_G(t)$ and $x_O(t)$ to be sold as Type-G and Type-O contracts respectively. A Type-G ("long term") contract that is sold at the beginning of any slot $t = 1, \ldots, T$ lasts till the end of the horizon. T therefore represents the maximum duration of a Type-G contract. Type-O contracts last for a single slot from the time they are negotiated.

The prices of both types of contracts (i.e, the prices at which they can be bought/ sold in the spectrum market) vary randomly with time and are determined "by the market", possibly depending on the current supply-demand balance in the market and other factors. The "per-slot" market prices for Type-G and Type-O contracts at time t are denoted by $c_G(t)$ and $c_O(t)$ respectively. When a Type-G contract is sold at slot t, it remains active for $T - t + 1$ slots (that is, until the end of the optimization horizon), and therefore fetches a revenue of $(T - t + 1)c_G(t)$ [2]. We assume that the process $\{c_G(t)\}$ (respectively, $\{c_O(t)\}$) constitutes a Discrete time Markov chain (DTMC) with a finite number of states and transition probability $H^G_{c,d}$ (respectively, $H^O_{c,d}$) from state c to d. For simplicity, we assume that the DTMCs $\{c_G(t)\}$ and $\{c_O(t)\}$ are independent of each other, although our results readily extend to the case when the joint process $\{c_G(t), c_O(t)\}$ is a DTMC.

Each primary provider is associated with a randomly time-varying *demand* process, $\{i(t)\}$ which corresponds to its subscriber demand (of TV channel subscribers or wireless service subscribers, for example) that it must satisfy. We assume that the process $\{i(t)\}$ constitutes a DTMC with a finite number of states and transition probability Q_{ij} from state i to j, that is independent of the price process; each demand state lies in $[0, M]$ and corresponds to an integral amount of bandwidth consumption in subscriber demand.

We assume that the transition probabilities $\{H^G_{c,d}\}$, $\{H^O_{c,d}\}$ and $\{Q_{ij}\}$ are known to the primary provider. They can be estimated from the history of the price and demand processes.

The contract trading is done at the beginning of time slot t, and $(x_G(t), x_O(t))$ are determined after the market prices $c_G(t)$, $c_O(t)$ and demand levels $i(t)$ are known. Let $(a_G(t), x_O(t))$ denote the spectrum contract portfolio held by the primary during time slot t, *i.e.* the number of Type-G and Type-O contracts that stand leased. Since Type-G contracts last till the end of the time horizon, we have:

$$a_G(t) = \sum_{t' \le t} x_G(t') \tag{1}$$

The bandwidth not leased as Type-G contracts or used to satisfy the demand is sold as Type-O contracts. Thus, at any time t:

$$x_O(t) = K(a_G(t), i(t)) := \max\{0, M - a_G(t) - i(t)\}. \tag{2}$$

However, for all slots, t, for which $a_G(t) + i(t) > M$, the primary will have to use channels already sold under Type-G contracts to satisfy its subscriber demand, due to unavailability of additional bandwidth. In this case, the primary incurs a penalty, $Y(a_G(t), i(t))$, for breaching Type-G contracts. The penalty is proportional to the number of such channels the provider uses for satisfying its subscriber demand. Thus,

$$Y(a_G(t), i(t)) \quad = \quad \beta \max\{0, a_G(t) + i(t) - M\}, \tag{3}$$

where β is the proportionality constant. We make the natural assumption that the penalty is hefty; in particular, β is greater than or equal to the maximum possible price of a Type-O contract.

The Primary-SCT problem then is to choose the primary's trading strategy $((x_G(t), x_O(t))$, $t = 1, \ldots T$, *so as to maximize its expected revenue, expressed as*

$$\boldsymbol{E}\left(\sum_{t=1}^{T}((T-t+1)c_G(t)x_G(t) + c_O(t)x_O(t) - Y(a_G(t), i(t)))\right), \tag{4}$$

subject to relations (1)-(3). The optimum strategy must be causal in that for each $t \in \{1, \ldots T\}$, $(x_G(t), x_O(t))$ *must be chosen by time t. Note that at time t,* $\{i(t'), c_G(t'), c_O(t') : t' = 1, \ldots, t\}$ *are known, but* $\{i(t'), c_G(t'), c_O(t') : t' = t+1, \ldots, T\}$ *are* not known *to the primary provider. From (1) and (2),* $x_O(t)$ *is a function of* $\{x_G(t') : t' = 1, \ldots, t\}$ *and the current demand* $i(t)$. *Therefore, the Primary-SCT problem as defined above reduces to finding the optimal* $(x_G(t), t = 1, \ldots, T)$.

Note that the revenue function in (4) ignores any revenue earned from the primary's subscribers. Since the subscriber demand process $i(t)$ is unaffected by the trading decisions, such revenue adds a constant offset to the revenue in (4), and therefore does not influence the optimal spectrum trading decisions.

Generalizations:

1) For a Type-O contract, the secondary provider pays the primary only for the amount of bandwidth it uses. Thus, the expected revenue earned by a primary on selling such a contract equals the secondary's expected usage of such a channel times the market price of such a contract. We can incorporate this by considering the revenue from a Type-O contract in slot t as $\kappa c_O(t)$, where κ is the secondary's expected usage of such a channel. The formulation and the results extend to this case.

2) Our formulation and results can be extended to consider the case that $i(t)$ is only an estimate of the demand in slot

[2] All our results readily generalize to the case in which a Type-G contract that is sold at slot t fetches a revenue of $\alpha(T - t + 1)c_G(t)$, where $\alpha(n)$ is any (deterministic) increasing function of n and captures the increase in value of a Type-G contract with the number of slots for which it remains active.

t, and the estimation error in each slot is an independent, identically distributed random variable whose distribution is known to the primary. Then, $x_O(t)$ must be selected so that $M - x_O(t) - a_G(t)$ is greater than or equal to the actual demand with a desired probability. Thus, $x_O(t)$ will be a function, $K(a_G(t), i(t))$, of $(a_G(t), i(t))$, which may be different from that in (2), but can nevertheless be determined from the knowledge of the distribution of the estimation error. Also, in this case, the lack of exact knowledge of the demand will force the primary to use part or whole of the bandwidth it has sold as Type-O contracts to satisfy its demand. This will not incur any penalty for the primary owing to the nature of the contract, but will reduce the secondary's expected usage κ of each channel sold as a Type-O contract, and thereby reduce the expected amount $\kappa c_O(t)$ the secondary pays the primary for each such channel.

3) For clarity of exposition, we assumed integral demands $i(t)$. However, in practice, the demands may be fractional. For example, when a set of subscribers intermittently access the Internet on a channel, a fraction of the bandwidth on a channel is used every slot. In this case, a Type-G or Type-O contract may be sold on the channel (while incurring a penalty proportional to the fraction used on the channel for the former). All our results apply without change in this case.

B. Polynomial-time optimal trading

We show that the Primary-SCT problem defined in Section II-A can be solved as a stochastic dynamic program (SDP) [21]. A *policy* [21] is a rule, which specifies the decision $(x_G(t))$ at each slot t, as a function of the demands and prices and past decisions. Now, since the demand and prices are Markovian, the statistics of the future evolution of the system from slot t onwards are completely determined by the vector $(a_G(t-1), i(t), c_G(t), c_O(t))$, which we call the *state* at slot t, and the primary's decisions $\{x_G(t') : t' = t, \dots, T\}$ under the policy being used. Now, in general, a policy may determine $x_G(t)$ at slot t based on all past states and actions. However, a well-known result (Theorem 4.4.2 in [21]) shows that there exists an optimal policy which specifies the optimal $x_G(t)$ at any slot t only as a (deterministic) function of the current state and t [3]. We next compute such an optimal policy by solving a SDP.

For a given t, let $n = T - t + 1$ be the number of slots remaining until the end of the horizon, and $V_n(a, i, c_G, c_O)$ denote the maximum possible revenue from the remaining n slots, under any policy, when the current state is $(a_G(t-1), i(t), c_G(t), c_O(t)) = (a, i, c_G, c_O)$. In particular, note that $V_T(0, i, c_G, c_O)$ is the maximum possible value of the expected revenue in (4) under any policy when $i(1) = i$, $c_G(1) = c_G$ and $c_O(1) = c_O$. The function $V_n(.)$ is called the *value function* [21]. We have:

$$V_n(a, i, c_G, c_O) = \max_{0 \le x \le M-a} W_n(a, i, c_G, c_O, x), \quad (5)$$

where $W_n(a, i, c_G, c_O, x) = nc_G x + J(x + a, i, c_O)$

$$+ \sum_{d_G} \sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij} V_{n-1}(a + x, j, d_G, d_O), \text{ and} \quad (6)$$

$$J(a_G(t), i(t), c_O(t)) = c_O(t)K(a_G(t), i(t)) - Y(a_G(t), i(t)), \quad (7)$$

and the maximum in (5) is over integer values of x in $[0, M - a]$. Equation (5) is called *Bellman's optimality equation* [21] and holds because, by definition of $V_{n-1}(.)$, $W_n(a, i, c_G, c_O, x)$ defined by (6) is the maximum possible expected revenue when n slots remain until the end of the horizon and $x_G(t) = x$ is chosen. Note that the first two terms in (6) account for the revenue earned in slot t from the sale of Type-G and Type-O contracts minus the penalty paid. The last term in (6) is the maximum expected revenue from slot $t + 1$ onwards. The summations over d_G, d_O and j take the expectation of the revenue over the prices of Type-G and Type-O contracts and the demand respectively in slot $t+1$. We get (5) by taking the maximum over all permissible values of x. Denote the (largest) x that maximizes $W_n(a, i, c_G, c_O, x)$ by $x^*_n(a, i, c_G, c_O)$. The function $x^*_n(.)$ provides the optimal solution to the Primary-SCT problem.

Now, the value function and optimal policy can be found from (5) using *backward induction* [21], which proceeds as follows. Note that $V_0(.) = 0$. Thus, $W_1(.)$ can be computed using (6), and $V_1(.)$ and $x^*_1(.)$ using (5), and similarly, $W_2(.), V_2(.), x^*_2(.), \dots W_n(.), V_n(.), x^*_n(.)$ can be successively computed. This backward induction consumes $O((N_G N_O M^2)^2 T)$ time, where N_G (respectively, N_O) is the number of states in the Markov Chain $\{c_G(t)\}$ (respectively, $\{c_O(t)\}$)– the computation time is therefore polynomial in the input size.

Remark 1: Note that we consider a finite horizon formulation. An alternative would be to consider an infinite horizon formulation, in which a Type-G contract is valid for T slots from the time of sale (instead of until the end of horizon), where T is some finite constant. But in this case, at a given slot t, the state would include $(y^G_1(t), \dots, y^G_T(t))$, where $y^G_j(t)$ is the number of Type-G contracts that are valid for j slots more. Thus, the size of the state space is $O(M^T)$, which is exponential in T. Hence, we do not consider an infinite horizon formulation in our analysis. However, based on the insights that we get from the analysis of the finite horizon formulation, we design a heuristic for the infinite horizon formulation and investigate its performance via simulations (see Section IV).

C. Properties of the optimal solution

We analytically prove a number of structural properties of the optimal policy, which provide insight into the nature of the optimal solution. Our results are quite general in that they hold not only for the $K(.), Y(.)$ functions defined in (2), (3), but also for any functions that satisfy the following properties (which are of course satisfied by those in (2), (3)). This loose requirement allows our results to extend to the generalizations described at the end of Section II-A.

Property 1: $K(a, i)$ decreases in a and $Y(a, i)$ increases in a for each i. Hence, by (7), for each i and c_O, $J(a, i, c_O)$ decreases in a.

Property 2: The $K(.), Y(.)$ functions are such that $J(a, i, c_O)$ is concave [4] in a for fixed i, c_O.

[3] Such a policy is called a *deterministic Markov* policy [21].

[4] A function $f(k)$ with domain being a subset of the integers is concave [3] if $f(k + 2) - f(k + 1) \le f(k + 1) - f(k)$ for all k [22]. If the inequality is reversed, $f(.)$ is convex.

Property 3: The $K(.), Y(.)$ functions are such that, for each a, $J(a, i, c_O) - J(a+1, i, c_O)$ is an increasing function of i.

We next state a technical assumption on the statistics of the demand and price processes that we need for our proofs.

Assumption 1: If X_i is the demand in the next slot given that the present demand is i, or, if X_i is the price of a Type-G (respectively, Type-O) contract in the next slot given that the present price is i, then for $i \leq i'$, $X_i \leq_{st} X_{i'}$ (X_i is *stochastically smaller* [22] than $X_{i'}$), *i.e.*, for each $b \in R$, $Pr(X_i > b) \leq Pr(X_{i'} > b)$.

Intuitively, this assumption says that the primary's demand and the prices do not fluctuate very rapidly, and the demand (or price) in the next slot is more likely to be high when the current demand (or price) is high as opposed to when the current demand (or price) is low.

We are now ready to state the structural properties of the optimum trading policy. We defer the proofs of these properties until Appendix A.

The first property identifies the relation between $x_n^*(a, i, c_G, c_O)$ and a:

Theorem 1: For each n, i, c_G, c_O,

$$x_n^*(a+1, i, c_G, c_O) = \max(x_n^*(a, i, c_G, c_O) - 1, 0). \quad (8)$$

Intuitively, this theorem suggests that for each n, i, c_G, c_O, there exists an optimal portfolio level of Type-G contracts, $a_G^*(t)$, such that if $a_G(t-1) = a$, then $x_G(t)$ should be chosen so as to make $a_G(t) = a_G^*(t)$. That is, the optimal $x_G(t) = a_G^*(t) - a$ (if the latter is non-negative).

Also, due to Theorem 1, for each n, i, c_G and c_O, it is sufficient to find $x_n^*(a, i, c_G, c_O)$ only for $a = 0$ while performing backward induction, and $x_n^*(a, i, c_G, c_O)$ for other a can be deduced using (8). This reduces the overall computation time by a factor of M: the optimal policy can now be computed in $O((N_G N_O)^2 M^3 T)$ time.

The next two results identify the nature of the dependence between $x_n^*(a, i, c_G, c_O)$ and the demand i and prices c_G, c_O.

Theorem 2: For each n, a, c_G and c_O, $x_n^*(a, i, c_G, c_O)$ is monotone decreasing in i.

Theorem 2 confirms the intuition that when the primary's demand is high, it should sell fewer Type-G contracts so as to reserve bandwidth to meet its demand and vice versa. At the same time, note that this result is not obvious– when the demand is lower, more free bandwidth is available, which can be sold as Type-G *or as Type-O contracts*. Clearly, the number of Type-G versus Type-O contracts sold would influence the states reached in the future and the revenue earned. Theorem 2 asserts that the primary should sell at least as many Type-G contracts as before (that is, as for the high demand state), while possibly also increasing the number of Type-O contracts to sell.

Theorem 3: $x_n^*(a, i, c_G, c_O)$ is monotone increasing in c_G for fixed n, a, i, c_O and monotone decreasing in c_O for fixed n, a, i, c_G.

Theorem 3 confirms the intuition that the primary should preferentially sell the type of contract (G or O) with a "high" price.

Remark 2: Theorems 2 and 3 can be used to speed up the computation of the optimal policy using the *monotone backward induction* algorithm [21]. Similarly, in Theorem 10 (in Appendix A), we prove that the value function is concave in a for fixed n, i, c_G, c_O, which can be used to speed up the computation of $x_n^*(.)$ from the value function since the maximizer in (5) can be found in $O(\log M)$ time using a binary search like algorithm [10]. In both cases, the worst case asymptotic running time remains the same, although substantial savings in computation can be obtained in practice.

D. SCT across multiple locations

We now consider spectrum contract trading across multiple locations from a primary provider's point of view. Wireless transmissions suffer from the fundamental limitation that the same channel can not be successfully used for simultaneous transmissions at neighboring locations, but can support simultaneous transmissions at geographically disparate locations. Thus, a primary provider can not sell contracts in the same channel at neighboring locations, but can do so at far off locations. Hence, the spectrum contract trading problem at different locations is inherently coupled, and must be optimized jointly. We now extend the problem formulation to consider the case of multiple locations, taking into account possible interference relationships between adjacent regions.

We model the overall region under consideration using an undirected graph $\mathcal{G}$ with the set of nodes S. Each node represents a certain area at some location in the overall region. There is an edge between two nodes if and only if transmissions at the corresponding locations on the same channel interfere with each other. A primary provider owns M channels throughout the region. At any time slot, at a given node and on a given channel, (a) either a Type-G contract can be sold, (b) a Type-O contract can be sold or (c) no contract can be sold, subject to the constraint that at no point in time, a contract can stand leased at neighbors on the same channel. That is, on each channel, the set of nodes at which a contract stands leased constitutes an *independent set* [29].

A primary provider needs to satisfy its subscriber demand which is also subject to certain reuse constraints. We consider the case where the subscribers of a primary provider require broadcast transmissions. This, for example, happens when the primary is a TV transmitter that broadcasts signals across all locations over different channels. At any given slot t, the primary needs to broadcast over a certain number, say $i(t)$, channels which randomly varies with time depending on subscriber demands. Whenever the primary broadcasts on a channel, the broadcast reaches all nodes, and thus the channel can not be used by the secondaries at any node. Hence, if the primary has sold a Type-G contract on the channel at any node it incurs a penalty of β at the node. Thus, at slot t, $i(t)$ represents the primary's demand at each node. Note that the set of nodes at which the primary uses a given channel for demand satisfaction does not constitute an independent set (as opposed to the set of nodes at which contracts stand leased). Also, the primary's usage status on any given channel at any given time (*i.e.*, whether or not the primary is using the channel for subscriber demand satisfaction) is the same across all nodes.

The durations of Type-G and Type-O contracts are as described in Section II-A. We assume that at any slot t, Type-G (respectively, Type-O) contracts have equal prices $c_G(t)$ (respectively, $c_O(t)$) at all nodes. The processes

$\{i(t)\}, \{c_G(t)\}, \{c_O(t)\}$ evolve as per independent DTMCs as stated in Section II-A.

The spectrum contract trading problem across multiple locations for a primary (Primary-SCTM) is to optimally choose at each slot t, the type of contract to sell (if any) at each location on each channel so as to maximize the total expected revenue from all nodes over a finite horizon of T slots.

Theorem 4: Primary-SCTM is NP-Hard.
The proof is deferred until Appendix B.

We now characterize the optimal solution of the Primary-SCTM problem.

Lemma 1: Consider the class of policies $\mathcal{F}$, such that a policy $f \in \mathcal{F}$ operates as follows. At the beginning of the horizon, it finds a maximum independent set, $I(S)$, in $\mathcal{G}$. Then, in each slot, it sells contracts only at nodes in $I(S)$. There exists a policy in $\mathcal{F}$ that optimally solves the Primary-SCTM problem.
The proof is deferred until Appendix B.

We refer to a policy in $\mathcal{F}$, which at each node in $I(S)$, sells contracts according to the optimal solution of the Primary-SCT problem with demand and price processes $\{i(t), c_G(t), c_O(t)\}$ as a *Separation Policy*.

Theorem 5 (Separation Theorem): A Separation Policy optimally solves the Primary-SCTM problem.

Proof of Theorem 5: By Lemma 1, we can restrict our search for an optimal policy to the policies in $\mathcal{F}$. Now, the total revenue of a policy in $\mathcal{F}$ is the sum of the revenues at the nodes in $I(S)$. Clearly, the total revenue is maximized if the stochastic dynamic program for the single node case is executed at each node. Note that this solution satisfies the interference constraints since $I(S)$ is an independent set. ■

Note that the optimum solution at any node can be computed in polynomial time using the SDP presented in Section II-A. However, computation of a maximum size independent set is an NP-hard problem [12]. This computation therefore seems to be the basis of the NP-hardness of Primary-SCTM. Also, the following theorem, which is a direct consequence of Theorem 5, shows that Primary-SCTM can be approximated in polynomial time within a factor of μ if the maximum independent set problem can be approximated in polynomial time within a factor of μ.

Theorem 6 (Approximate Separation Theorem): Consider a μ-separation policy that differs from a separation policy in that it sells contracts as per the single node optimum solution, at each node of an independent set whose size is at least $\frac{1}{\mu}$ times that of a maximum independent set. This policy's expected revenue is at least $\frac{1}{\mu}$ times the optimal expected revenue.

However, in a graph with N nodes, the maximum size independent set problem can not in general be approximated to within a factor of $O(N^{\epsilon})$ for some $\epsilon > 0$ in polynomial time unless $P = NP$ [1]. Nevertheless, *polynomial time approximation algorithms* (PTAS) i.e., algorithms that compute an independent set whose size is within $(1-\epsilon)$ of the maximum size independent set, for any given $\epsilon > 0$, using a computation time of $O(N^{1/\epsilon})$ are known in important special cases, e.g., when the degree of each node is upper-bounded [2] (this happens in our case when the number of locations each location interferes with is upper-bounded). Thus, in view of Theorem 6, for any given $\epsilon > 0$, the Primary-SCTM problem can be approximated within a factor of $1-\epsilon$ using a computation time of $O(N^{1/\epsilon})$ in such graphs.

III. Secondary's Spectrum Contract Trading Problem

In this section we pose and address *Secondary-SCT*, the spectrum contract trading question from a secondary provider's (buyer's) perspective. First note that the Secondary-SCT problem need not consider the interference constraints for channels since the secondary provider buys the spectrum bands that are offered in the market (presumably in a manner that satisfies the reuse constraints), and also because they are usually localized (i.e., operate in small regions). Thus, the secondary's spectrum trading decisions in different regions can be separately optimized. So henceforth in this section, we restrict ourselves to the case of a single location.

A. Formulation

We consider an arbitrary secondary provider that is interested in buying contracts in the secondary spectrum market. Our assumptions regarding the optimization horizon T, the durations of Type-G and Type-O contracts and their price processes $(c_G(t), c_O(t))$ remain the same as in Section II-A. Let $\tilde{i}(t)$ denote the subscriber demand of the provider at time t– it is a DTMC similar to $\{i(t)\}$ in Section II-A, but with transition probabilities P_{ij} in place of Q_{ij}.

The secondary decides the number of Type-G and Type-O contracts it will buy (from primary providers) at slot t, $(\tilde{x}_G(t), \tilde{x}_O(t))$, after it learns the market prices $c_G(t)$ and $c_O(t)$ and the demand level $\tilde{i}(t)$ at t. We continue to assume that the market has infinite liquidity, which now implies that the market has a lot of sellers (i.e., primary providers), and hence the secondary can buy as many contracts of any type by paying their market price. Let $(\tilde{a}_G(t), \tilde{x}_O(t))$ denote the spectrum contract portfolio held by the secondary during slot t, where $\tilde{a}_G(t)$ denotes the number of Type-G contracts that the secondary has leased out until time t. Then we have

$$\tilde{a}_G(t) \;=\; \sum_{t' \leq t} \tilde{x}_G(t'). \tag{9}$$

The secondary provider's spectrum trading goal is to meet its time-varying subscriber demand in every time slot at the minimum cost, by choosing an appropriate portfolio of Type-G and Type-O contracts, $\{(\tilde{a}_G(t), \tilde{x}_O(t))\}$, adjusted dynamically.

Note that there are uncertainties on how much bandwidth the secondary actually ends up getting from each contract at a time t during its duration, since a Type-O contract only allows the secondary the right to use the channel when the owner (primary) is not using it, and there is a non-zero probability of contract violation for a Type-G contract by the primary due to its subscriber demand level plus the number of Type-G contracts sold exceeding its total owned spectrum (see the Primary-SCT formulation in Section II). Due to this, the subscriber demand $\tilde{i}(t)$ can be met only in statistical terms, e.g., in expectation, or with a certain probability, by any spectrum contract portfolio. (We assume that statistics on such contract violations are available (possibly from historical data) to the buyers, and can be incorporated in the corresponding contract trading decision.) We generalize this notion by associating with each value of subscriber demand δ, a *demand satisfaction set* $\mathcal{F}_\delta$ within which a spectrum contract portfolio $(\tilde{a}_G, \tilde{x}_O)$ must lie for meeting the demand level δ satisfactorily. A portfolio $(\tilde{a}_G(t), \tilde{x}_O(t))$ is said to be *demand-satisfactory* at time t if it can meet the demand level at time t satisfactorily, i.e., if $(\tilde{a}_G(t), \tilde{x}_O(t)) \in \mathcal{F}_{\tilde{i}(t)}$.

Thus, the Secondary-SCT problem is to minimize the expected contract trading cost subject to the spectrum contract portfolio being demand-satisfactory at all times t. The objective is thus to minimize

$$\boldsymbol{E}\left(\sum_{t=1}^{T}\left((T-t+1)c_G(t)\tilde{x}_G(t) + c_O(t)\tilde{x}_O(t)\right)\right), \qquad (10)$$

subject to (9) and

$$(\tilde{a}_G(t), \tilde{x}_O(t)) \quad \in \quad \mathcal{F}_{\tilde{i}(t)}, \ \forall t, \qquad (11)$$

and such that for each $t \in \{1, \ldots T\}$, $(\tilde{x}_G(t), \tilde{x}_O(t))$ *must be chosen by time t. Note that at time t,* $\{\tilde{i}(t'), c_G(t'), c_O(t') : t' = 1, \ldots, t\}$ *are known, but* $\{\tilde{i}(t'), c_G(t'), c_O(t') : t' = t+1, \ldots, T\}$ *are* not *known.*

We assume that the sets $\mathcal{F}_\delta$ for different δ are given. Typically, we will have $\mathcal{F}_{\delta'} \subseteq \mathcal{F}_\delta$ for $\delta \leq \delta'$. Also, we make the natural assumption that if $(\tilde{a}_G, \tilde{x}_O) \in \mathcal{F}_\delta$ for some δ, then $(\tilde{a}_G, \tilde{x}'_O) \in \mathcal{F}_\delta \ \forall \tilde{x}'_O \geq \tilde{x}_O$. Accordingly, let $L(\tilde{a}_G(t), \tilde{i}(t))$ be the minimum number of Type-O contracts $\tilde{x}_O$ required for a portfolio $(\tilde{a}_G(t), \tilde{x}_O)$ to be in $\mathcal{F}_{\tilde{i}(t)}$, for a given $(\tilde{a}_G(t), \tilde{i}(t))$. It is easy to see that for a given $(\tilde{a}_G(t), \tilde{i}(t))$, it is optimal to select $\tilde{x}_O = L(\tilde{a}_G(t), \tilde{i}(t))$ (not more).

For example, suppose the secondary seeks to meet the current demand level in expectation. Due to the uncertain amount of bandwidth available on Type-G and Type-O contracts, suppose the expected amount of bandwidth obtained from a Type-G contract is γ $(0 < \gamma \leq 1)$. Also, η Type-O contracts are required, on average, to meet one unit of demand, where η is a positive integer. For simplicity, assume that the product $\gamma\eta$ is an integer. Then:

$$L(\tilde{a}_G(t), \tilde{i}(t)) = \max\left\{\eta(\tilde{i}(t) - \gamma\tilde{a}_G(t)), 0\right\} \qquad (12)$$

Remarks: 1) Note that in (10), we do not consider the revenue earned from the penalties paid by the primary due to Type-G contract violations. Such penalties lead to a net decrease in the price of a Type-G contract, and their effects can be incorporated by considering the price process of Type-G contracts as $\{\tilde{c}_G(t)\}$, where $\tilde{c}_G(t) = c_G(t) - \kappa(t)$, where $\kappa(t)$ is i.i.d and independent of $\{c_G(t)\}$. Subsequent formulations and analysis do not change owing to the above modification.
2) Like for the Primary-SCT problem, our results can be extended to the case where the secondary knows only an estimate of $\tilde{i}(t)$ at the beginning of time slot t.
3) Like for the Primary-SCT problem, the cost function in (10) ignores any revenue earned from the secondary's subscribers. Since the subscriber demand process $\tilde{i}(t)$ is unaffected by the trading decisions, such revenue adds a constant offset to the cost in (10), and therefore does not influence the optimal spectrum trading decisions.

B. Analysis

We formulate the secondary's problem as a stochastic dynamic program (SDP) and prove a number of structural properties of the optimal solution. The formulation and analysis are very similar to that for the primary; hence we only provide a brief outline.

Let $(\tilde{a}_G(t-1), \tilde{i}(t), c_G(t), c_O(t))$ be the state at the beginning of slot t, $n = T - t + 1$ and $V_n(a, i, c_G, c_O)$ denote the value function, *i.e.*, the minimum possible cost over the remaining slots, starting from slot t. In particular, note that $V_T(0, i, c_G, c_O)$ is the minimum possible value of the expected cost in (10) under any policy when $\tilde{i}(1) = i$, $c_G(1) = c_G$ and $c_O(1) = c_O$. Then the optimality equation is given by:

$$V_n(a, i, c_G, c_O) = \min_x W_n(a, i, c_G, c_O, x) \qquad (13)$$

where

$$W_n(a, i, c_G, c_O, x) = nc_G x + c_O L(x + a, i) + \sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j P_{ij} V_{n-1}(a + x, j, d_G, d_O) \qquad (14)$$

and the minimum in (13) is over nonnegative integer values of x. Denote the (smallest) x that minimizes $W_n(a, i, c_G, c_O, x)$ by $\tilde{x}^*_n(a, i, c_G, c_O)$. The value function and optimal policy can be found from (13) using *backward induction* [21] in $O((N_G N_O D^2)^2 T)$ time, where D is the number of states in the Markov Chain $\{\tilde{i}(t)\}$.

We now identify the structure of the optimal trading strategy $\{\tilde{x}^*_n(a, i, c_G, c_O), n = 1, \ldots, T\}$ for the following properties of the $L(.)$ function, which are analogous to Properties 1, 2 and 3 of the $J(.)$ function for the Primary-SCT problem. (i) For each i, $L(a, i)$ decreases in a, (ii) $L(a, i)$ is convex in a for fixed i, (iii) For each a, $L(a, i) - L(a + 1, i)$ is an increasing function of i. It can be checked that these properties are true for the function $L(.)$ in (12). We also assume that the price and demand processes satisfy Assumption 1.

We have the following structural results, which closely parallel Theorems 1 to 3. The proofs are similar to those of Theorems 1 to 3, and hence omitted.

Theorem 7: For each n, i, c_G, c_O, $\tilde{x}^*_n(a + 1, i, c_G, c_O) = \max(\tilde{x}^*_n(a, i, c_G, c_O) - 1, 0)$.

Theorem 8: For each n, a, c_G and c_O, $\tilde{x}^*_n(i, a, c_G, c_O)$ is monotone increasing in i.

Theorem 9: $\tilde{x}^*_n(a, i, c_G, c_O)$ is monotone decreasing in c_G for fixed n, a, i, c_O and monotone increasing in c_O for fixed n, a, i, c_G.

IV. Numerical Studies

We next study the properties of the optimal trading strategy using numerical investigations, and explore how the expected revenue varies as a function of key system parameters. Due to the similarity in the results for Primary-SCT and Secondary-SCT, we only present our results for the former. We consider $M = 20$ channels, penalty parameter $\beta = 3.0$ and a birth-death demand process with 21 states and integral state values $\{0, 1, \ldots, 20\}$. The price process $c_G(t)$ ($c_O(t)$, respectively) is again a birth-death process that varies between 1.0 and 4.0 (1.0 and 2.0, respectively) with a total of 10 uniformly-spaced states. For both the demand and price processes, we assume that the forward and backward transition probabilities equal p (a parameter).

In Theorems 2 and 3, we have established the monotonicity properties of the optimal solution $x^*_n(a, i, c_G, c_O)$ with respect to the demand level i and prices c_G, c_O. Recall that $n = T -$

$t+1$ at slot t, and represents the duration of a Type-G contract made at slot t. Now, our numerical evaluations suggest that the optimal solution $x_n^*(.)$ is decreasing in n, and when n is close to T, $x_n^*(.)$ is zero (see Figure 1). Thus, the primary prefers Type-G contracts towards the end of the optimization horizon, and Type-O towards the beginning. This is because when n is close to T, Type-G contracts are very long-term, and hence likely to incur hefty penalties since demand and prices may be difficult to predict long-term.

The two plots in Figure 2 show the variation in the primary's average (expected) revenue per slot with respect to p and T. For these results, the initial state for the demand and price processes are chosen according to the steady state distributions of these processes. The average revenue obtained from the optimal dynamic trading strategy is compared with that of an optimal *static* strategy. In the latter strategy, the number of Type-G contracts is chosen only once (optimally, based on the steady state distribution of the demand and price processes), at the very beginning of the time horizon; the number of Type-O contracts made is adjusted dynamically to the amount of "free bandwidth" available at any slot (i.e., the number of channels minus the sum of the demand and Type-G contracts made). We observe that the average revenue for the optimal static strategy is invariant to changes in p or T – this happens because the initial states for the demand and price processes follow their steady state distributions, which in our case is uniform and does not depend on p or T. We observe that the optimal dynamic contract trading strategy significantly outperforms the optimal static strategy, demonstrating the benefits of dynamic choice of the number of Type-G contracts. Note that if the static strategy buys a Type-G contract, it must buy one that is really long-term (i.e., one that lasts for the entire T slots), whereas the dynamic strategy can choose the duration of Type-G contracts it buys by deciding when they are purchased, based on its demand and prices of the contracts that evolve dynamically. The figures also show that the primary's average revenue per slot under dynamic choice increases with an increase in p and T (for the same value of the other parameters). Note that a larger p (respectively, larger T) implies larger temporal variation in the prices (respectively, a longer optimization horizon), giving the primary more opportunities in which the price of a Type-G contract is high and the primary can "lock in" a good price for a contract. From the bottom plot in Figure 2, we also observe that the average per-slot revenue shows diminishing returns as T increases, and appears to stabilize eventually (at a faster rate for a larger p). This is intuitive since the revenue earned per unit time is upper bounded, and also because very long-term Type-G contracts offer small returns.

Now, consider an alternative contract trading model as described in Remark 1, in which there is an infinite horizon, and a Type-G contract is valid for T slots from the point of sale, where T is some finite constant. As explained in Remark 1, computation of the optimal policy using stochastic dynamic programming requires an exponential state space formulation in this case. So now, we design a heuristic for this case based on the insights that the analysis of the finite horizon formulation provided. Suppose there are $M = 20$ channels, and the demand and price processes are birth-death processes with the parameters in the first paragraph of this section and

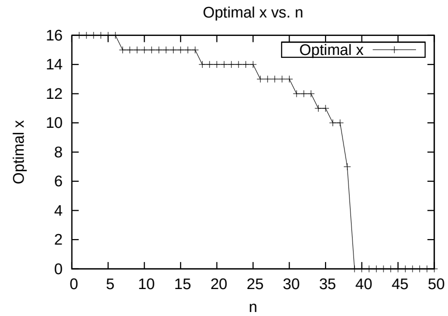

Fig. 1. $x_n^*(a, i, c_G, c_O)$ versus n for $a = 0$, $i = 4$, $c_G = 2.0, c_O = 1.0$ and $T = 50$.

with $p = 0.4$. Also, let $T = 20$. Let $a(t-1)$, $c_G(t)$, $c_O(t)$ and $i(t)$ be the number of Type-G contracts that stand leased at the beginning of slot t, and the prices of Type-G and Type-O contracts and the demand in slot t respectively. Also, at the beginning of slot t, if a Type-G contract stands leased on channel j, then let $r_j(t-1)$ be the number of slots until its expiry and let $r_j(t-1)$ be 0 otherwise. Since the average duration until expiry of a Type-G contract at an arbitrary slot is $T/2$ slots, $\frac{\sum_{j=1}^{M} r_j(t-1)}{(T/2)}$ is the "number of Type-G contracts of average duration" that stand leased at the beginning of slot t, and is the analog of a in Theorem 1. Based on this fact, and the insights into the structure of the optimal policy provided by Theorems 1, 2 and 3, we consider the following heuristic for selecting the number of Type-G contracts to sell. At the beginning of slot t, the primary sells:

$$\max\left(\min\left(\frac{3M}{4} - \frac{\sum_{j=1}^{M} r_j(t-1)}{(T/2)} - i(t) + q(c_G(t) - c_O(t)),\right.\right.$$
$$\left.\left. M - a(t-1)\right), 0\right)$$

Type-G contracts, where q is a parameter. Fig. 3 plots the average per-slot revenue achieved by this heuristic versus q for different values of the penalty parameter β. Now, note that for the parameter values used, the expected demand $i(t)$ is 10 channels, and the expected prices $c_G(t)$ and $c_O(t)$ are 2.5 and 1.5 respectively. So if the demand and prices were to be constant at their expected values, the maximum average per-slot revenue that any policy can achieve is 25 (the optimal policy in this case always sells the free $M - i(t) = 10$ channels as Type-G contracts). Since computing the optimal per-slot revenue when the demand and prices are dynamic requires exponential time as explained in Remark 1, we use the above value of 25 as a rough benchmark for evaluating the performance of the heuristic. Fig. 3 shows that for an appropriate choice of the parameter q, the heuristic achieves an average per-slot revenue close to 25 and hence it performs well. Also, consistent with intuition, the revenue is higher for lower values of the penalty parameter β.

V. Conclusions

We proposed two types of spectrum contracts (Type-G and Type-O) aimed at achieving the desired tradeoffs between service quality, spectrum usage efficiency and pricing, and formulated the problem of selection of an optimal portfolio of

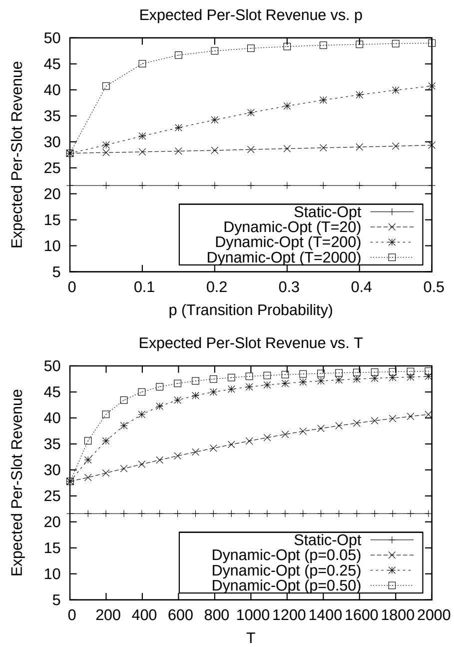

Fig. 2. The top plot shows the average per-slot revenue vs transition probability p. The bottom plot shows the average per-slot revenue vs time horizon T.

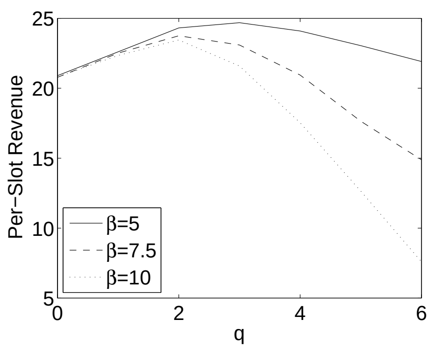

Fig. 3. The figure shows the average per-slot revenue of the heuristic versus q for different values of the penalty parameter β. The simulations were run for 10^6 slots.

Type-G and Type-O contracts for both primary and secondary providers as a stochastic dynamic programming problem. We provided a polynomial-time algorithm for this problem and analytically proved several structural properties of the optimal solution. These properties provide several insights into the optimal solution, which we used, in particular, to design a heuristic for the infinite horizon case and showed via simulations that it performs well in practice.

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APPENDIX

A. Proofs of results in Section II-C

Notation: Let R denote the set of real numbers.

Let X_i be as in Assumption 1. Recall that Q_{ij}, H^G_{ij} and H^O_{ij} are the transition probabilities of the demand and the prices of Type-G and Type-O contracts respectively. So, if X_i represents the demand, price of a Type-G contract or price of a Type-O contract respectively in the next slot given that the present demand, price of a Type-G contract or price of a Type-O contract equals i, then for a function $f(.)$, $E(f(X_i))$ equals $\sum_j Q_{ij}f(j)$, $\sum_j H^G_{ij}f(j)$ and $\sum_j H^O_{ij}f(j)$ respectively. The assumption $X_i \leq_{st} X_{i'}$ for $i \leq i'$ in Assumption 1 is equivalent to the following condition [22]:

Condition 1: For every increasing function $f(i)$,

$$E(f(X_i)) \leq E(f(X_{i'})) \ \forall i \leq i'$$

i.e., $\sum_j Q_{ij}f(j)$, $\sum_j H^G_{ij}f(j)$ and $\sum_j H^O_{ij}f(j)$ are increasing functions of i.

Note that in the summations in Condition 1, as well as in those in the rest of this section, the summation is over all possible states of the respective Markov Chain.

1) Proof of Theorem 1: We first prove that the value function is concave in a (Theorem 10). Then, using Theorem 10, we prove Theorem 1. We start with a simple lemma, which is used in the proof of Theorem 10.

Lemma 2: For fixed i, c_G, c_O, $V_n(a, i, c_G, c_O)$ decreases in a.

Proof: We prove the result by induction. Let $V_0(a, i, c_G, c_O) = 0$. Then the claim is true for $n = 0$. Suppose $V_{n-1}(a, i, c_G, c_O)$ decreases in a for each i, c_G, c_O. Now, let $a_1 \geq 1$ and $x^*_n(a_1, i, c_G, c_O) = x_1$ for some x_1. Then, by (5):

$$V_n(a_1, i, c_G, c_O) = W_n(a_1, i, c_G, c_O, x_1) \tag{15}$$

Now,

$$\begin{aligned}
& V_n(a_1 - 1, i, c_G, c_O) \\
\geq\ & W_n(a_1 - 1, i, c_G, c_O, x_1) \text{ (by (5))} \\
=\ & nc_Gx_1 + J(x_1 + a_1 - 1, i, c_O) \\
& + \sum_{d_G}\sum_{d_O} H^G_{c_Gd_G}H^O_{c_Od_O}\sum_j Q_{ij}V_{n-1}(a_1 + x_1 - 1, j, d_G, d_O) \\
\geq\ & nc_Gx_1 + J(x_1 + a_1, i, c_O) \\
& + \sum_{d_G}\sum_{d_O} H^G_{c_Gd_G}H^O_{c_Od_O}\sum_j Q_{ij}V_{n-1}(a_1 + x_1, j, d_G, d_O) \\
& \text{(by induction hypothesis and Property 1)} \\
=\ & W_n(a_1, i, c_G, c_O, x_1) \\
=\ & V_n(a_1, i, c_G, c_O) \ \text{ (by (15))}
\end{aligned}$$

The result follows. ■

Theorem 10: For each n, $V_n(a, i, c_G, c_O)$ is concave in a for fixed i, c_G, c_O.

Proof: We prove the result by induction. $V_0(a, i, c_G, c_O)$ is concave in a since it is equal to 0. Suppose $V_{n-1}(a, i, c_G, c_O)$ is concave in a for fixed i, c_G, c_O. Recall that $V_{n-1}(a, i, c_G, c_O)$ is defined for integer values of a. Now, for fixed i, c_G and c_O, define $\tilde{V}_{n-1}(a, i, c_G, c_O)$ for a real as the function obtained by linearly interpolating $V_{n-1}(a, i, c_G, c_O)$ between each pair of adjacent integers a_0 and $a_0 + 1$. Similarly, define $\tilde{J}(a, i, c_O)$.

Now, $J(x + a, i, c_O)$ (respectively, $V_{n-1}(x + a, i, c_G, c_O)$) is concave decreasing in $x + a$ for fixed i, c_O (respectively, for fixed i, c_G, c_O) by Properties 1 and 2 (respectively, by Lemma 2 and induction hypothesis). Hence, we get:

Property 4: $\tilde{J}(x + a, i, c_O)$ (respectively, $\tilde{V}_{n-1}(x + a, i, c_G, c_O)$) is concave decreasing in $x + a$ for fixed i, c_O (respectively, for fixed i, c_G, c_O).

Now, consider the function

$$\begin{aligned}
&\tilde{W}_n(a, i, c_G, c_O, x) = nc_Gx + \tilde{J}(x + a, i, c_O) \\
&+ \sum_{d_G}\sum_{d_O} H^G_{c_Gd_G}H^O_{c_Od_O}\sum_j Q_{ij}\tilde{V}_{n-1}(a + x, j, d_G, d_O)
\end{aligned} \tag{16}$$

as a function of the two real variables a, x, *i.e.* the vector (a, x).

Recall the following property of composition of functions [3]:

Property 5: Let $h : R \to R$, $g : R^k \to R$, where $k \geq 1$ and R^k denotes the k-dimensional Euclidean space. Let $f : R^k \to R$ be defined by $f(\mathbf{v}) = h(g(\mathbf{v}))$. If $h(.)$ is concave and decreasing, and $g(\mathbf{v})$ is convex in $\mathbf{v}$, then $f(\mathbf{v})$ is concave in $\mathbf{v}$.

By the fact that $a + x$ is linear and hence [3] convex in (a, x), Property 4 and Property 5, it follows that $\tilde{J}(x+a, i, c_O)$ (respectively, $\tilde{V}_{n-1}(a + x, j, d_G, d_O)$) is concave in (a, x) for fixed i, c_O (respectively, for fixed j, d_G, d_O). Also, x is clearly concave in (a, x). Hence, $\tilde{W}_n(a, i, c_G, c_O, x)$ being a nonnegative weighted linear combination of these functions, is concave in (a, x) for fixed i, c_G, c_O.

Now, define:

$$\tilde{V}_n(a, i, c_G, c_O) = \sup_{x \in R, 0 \leq x \leq M-a} \tilde{W}_n(a, i, c_G, c_O, x) \tag{17}$$

Note that $\{x : x \in R, 0 \leq x \leq M - a\}$ is a non-empty convex set. Recall the following property [3]:

Property 6: If $f(a, x)$ is concave in (a, x) and C is a convex nonempty set, then the function

$$g(a) = \sup_{x \in C} f(a, x)$$

is concave in a, provided $g(a) < \infty$ for some a.

Now, $\tilde{V}_n(a, i, c_G, c_O) < \infty$ (since the costs of Type-G and Type-O contracts are upper bounded). So by (17), Property 6 and the fact that $\tilde{W}_n(.)$ is concave in (a, x), $\tilde{V}_n(a, i, c_G, c_O)$ is concave in a for fixed i, c_G, c_O.

Now, we will show that $V_n(a, i, c_G, c_O) = \tilde{V}_n(a, i, c_G, c_O)$ for a integer, which will imply that $V_n(a, i, c_G, c_O)$ is concave.

Fix i, c_G, c_O and an integer a. Note that by (5) and (17) and since $\tilde{W}_n(.) = W_n(.)$ at integer a and x, $V_n(a, i, c_G, c_O)$ is the maximum of $\tilde{W}_n(a, i, c_G, c_O, x)$ over integer x, whereas $\tilde{V}_n(a, i, c_G, c_O)$ is the supremum over real x in the range $[0, M - a]$. Hence, to prove that $V_n(a, i, c_G, c_O) = \tilde{V}_n(a, i, c_G, c_O)$, it will suffice to show that the supremum over real x occurs at integer x.

Now, by the definition of the functions $\tilde{J}(.)$ and $\tilde{V}_{n-1}(.)$, $f(x) = \tilde{W}_n(a, i, c_G, c_O, x)$ is continuous and piecewise linear in x, with breakpoints at the integers. Also, note that the endpoints of the domain of $f(x)$, viz. 0 and $M - a$ are integers that are contained in the domain. As a result, it can be checked that the maximum of $f(x)$ must occur at an integer. This completes the proof. ■

Note that $W_n(a, i, c_G, c_O, x)$ is concave in (a, x) and $V_n(a, i, c_G, c_O)$ is the maximum of $W_n(.)$ over a *non-convex* set, namely the set of integers in $[0, M - a]$. This makes the above proof more involved, since had the maximum been over a convex set, the concavity of $V_n(a, i, c_G, c_O)$ would have simply followed from Property 6.

We are now ready to prove Theorem 1.

Proof of Theorem 1: From (6), we have:

$$W_n(a,i,c_G,c_O,x) = W_n(a+1,i,c_G,c_O,x-1) + nc_G,\ \forall x \geq 1 \quad (18)$$

Now, by optimality of $x_n^*(a,i,c_G,c_O)$:

$$W_n(a,i,c_G,c_O,x_n^*(a,i,c_G,c_O)) \geq W_n(a,i,c_G,c_O,x)\ \ \forall x \geq 1 \quad (19)$$

If $x_n^*(a,i,c_G,c_O) \geq 1$, then from (18) and (19) and some algebra, we get:

$$W_n(a+1,i,c_G,c_O,x_n^*(a,i,c_G,c_O)-1) \geq W_n(a+1,i,c_G,c_O,x-1)\ \forall x \geq 1$$

which shows that $x_n^*(a+1,i,c_G,c_O) = x_n^*(a,i,c_G,c_O) - 1$ if $x_n^*(a,i,c_G,c_O) \geq 1$.

Now, suppose $x_n^*(a,i,c_G,c_O) = 0$. By Theorem 10 and Property 2, since $V_{n-1}(a+x,j,d_G,d_O)$ and $J(x+a,i,c_O)$ are concave in x for fixed a,j,d_G,d_O,i,c_O, it follows from (6) that $W_n(a,i,c_G,c_O,x)$ is concave in x. For $x \geq 2$, we have:

$$\begin{aligned} & W_n(a+1,i,c_G,c_O,x-1) - W_n(a+1,i,c_G,c_O,0) \\ = \ & W_n(a,i,c_G,c_O,x) - W_n(a,i,c_G,c_O,1) \text{ (by (18))} \\ \leq \ & W_n(a,i,c_G,c_O,x-1) - W_n(a,i,c_G,c_O,0) \\ & \text{(by concavity)} \\ \leq \ & 0 \text{ (since } x_n^*(a,i,c_G,c_O) = 0) \end{aligned}$$

which shows that $x_n^*(a+1,i,c_G,c_O) = 0$. ■

2) Proofs of Theorems 2 and 3: The proofs of Theorems 2 and 3 are based on the concepts of submodularity and supermodularity, which we briefly review. Let $I \subseteq R$ and $X \subseteq R$ be two sets. A function $g(i,x) : I \times X \to R$ is called *supermodular* [21] if for $i^+ \geq i^-$ in I and $x^+ \geq x^-$ in X,

$$g(i^+,x^+) + g(i^-,x^-) \geq g(i^+,x^-) + g(i^-,x^+)$$

If the inequality is reversed, g is called *submodular* [21].

We will require the following key result [21].

Theorem 11: If $g(i,x)$ is supermodular (submodular) on $I \times X$, then the (largest) maximizer of $g(i,x)$ for a given i:

$$f(i) = \max\{x' : x' \in \underset{x}{\operatorname{argmax}}\, g(i,x)\}$$

is increasing (decreasing) in i.

To prove Theorem 2, we show that $W_n(a,i,c_G,c_O,x)$ is submodular in (i,x). The monotonicity of $x_n^*(a,i,c_G,c_O)$ in i then follows from Theorem 11. First, we prove some lemmas.

The following lemma provides a necessary and sufficient condition for submodularity.

Lemma 3: Let $g(i,x)$ be a function with domain being integer values of x and real values of i. $g(i,x)$ is submodular in (i,x) if and only if $g(i,x) - g(i,x+1)$ is an increasing function of i for all x.

Proof: The necessity directly follows from the definition of submodularity. We now prove sufficiency. Suppose $g(i,y) - g(i,y+1)$ is an increasing function of i for all y. For an integer $z > 0$:

$$g(i,x) - g(i,x+z) = [g(i,x) - g(i,x+1)] + \ldots + [g(i,x+z-1) - g(i,x+z)]$$

So $g(i,x) - g(i,x+z)$, being the sum of increasing functions, is increasing in i.

Hence, for $x^- < x^+$, $g(i,x^-) - g(i,x^+)$ is increasing in i. So for $i^- < i^+$:

$$g(i^-,x^-) - g(i^-,x^+) \leq g(i^+,x^-) - g(i^+,x^+)$$

Hence, $g(i,x)$ is submodular in (i,x) by definition. ■

For $m \geq 1$, define [5]

$$i_n^m(a,c_G,c_O) = \max\{i : x_n^*(a,i,c_G,c_O) \geq m\}. \quad (20)$$

Lemma 4: If $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i, then

$$i_n^1(a,c_G,c_O) \geq i_n^2(a,c_G,c_O) \geq \ldots \geq i_n^{M-a}(a,c_G,c_O)$$

Also, $x_n^*(a,i,c_G,c_O) = m$ if and only if $i_n^m(a,c_G,c_O) \geq i > i_n^{m+1}(a,c_G,c_O)$.

Proof: The result follows by definition of $i_n^m(.)$. ■

The next lemma establishes a sufficient condition for monotonicity of $x_n^*(i,a,c_G,c_O)$.

Lemma 5: Fix n. Suppose $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ is an increasing function of j for each a, d_G and d_O. Then $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i for each a,c_G and c_O.

It is important to note that the lemma requires $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ to be increasing in j *for a fixed* n, and asserts that $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i *for that* n.

Proof: By (6):

$$\begin{aligned} & W_n(a,i,c_G,c_O,x) - W_n(a,i,c_G,c_O,x+1) \\ & = -nc_G + [J(a+x,i,c_O) - J(a+x+1,i,c_O)] \\ & + \sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij}\,(V_{n-1}(a+x,j,d_G,d_O) \\ & \qquad - V_{n-1}(a+x+1,j,d_G,d_O)) \end{aligned}$$

The first term on the right hand side is constant, the second term is increasing in i by Property 3 and the third term is increasing in i since $V_{n-1}(a+x,j,d_G,d_O) - V_{n-1}(a+x+1,j,d_G,d_O)$ is increasing in j and by Condition 1.

So $W_n(a,i,c_G,c_O,x) - W_n(a,i,c_G,c_O,x+1)$ is increasing in i. Hence, by Lemma 3, $W_n(a,i,c_G,c_O,x)$ is submodular in (i,x) and so by Theorem 11, $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i. ■

The next lemma is a simple consequence of (8).

Lemma 6: Fix n. If $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i for each a,c_G,c_O, then $i_n^{m+1}(a,c_G,c_O) = i_n^m(a+1,c_G,c_O)$ for $m = 1,2,\ldots$.

Proof: Fix c_G and c_O, and let $m \geq 1$. Separately with a and with $a+1$, start with $i = M$ (the highest demand state) and keep decreasing it to the next lower state, one at a time. By (8), the maximum i at which $x_n^*(a,i,c_G,c_O) \geq m+1$ is precisely the maximum i at which $x_n^*(a+1,i,c_G,c_O) \geq m$. So $i_n^{m+1}(a,c_G,c_O) = i_n^m(a+1,c_G,c_O)$ by definition of $i_n^m(.)$. ■

Lemma 7: For each n, $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$ is an increasing function of i for each a,c_G,c_O.

Proof: We prove the claim by induction. Since $V_0(a,i,c_G,c_O) \equiv 0$, the claim is true for $n = 0$.

[5] If $x_n^*(a,i,c_G,c_O) < m\ \forall i$, then let $i_n^m(a,c_G,c_O)$ be equal to the smallest demand state.

Suppose the statement is true for $n-1$, *i.e.*, $V_{n-1}(a,j,d_G,d_O)-V_{n-1}(a+1,j,d_G,d_O)$ is an increasing function of j for each a,d_G,d_O. Then by Lemma 5, $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i. Hence, by Lemma 6, $i_n^{m+1}(a,c_G,c_O)=i_n^m(a+1,c_G,c_O)$ for $m=1,2,\ldots$.

Now, we show that $V_n(a,i,c_G,c_O)-V_n(a+1,i,c_G,c_O)$ is an increasing function of i. Fix a,c_G and c_O. We have the following cases:

Case 1: $i>i_n^1(a,c_G,c_O)$
By Lemma 4 and Lemma 6:

$$i>i_n^1(a,c_G,c_O)\geq i_n^2(a,c_G,c_O)=i_n^1(a+1,c_G,c_O)$$

So by Lemma 4, $x_n^*(a,i,c_G,c_O)=x_n^*(a+1,i,c_G,c_O)=0$. Hence, by (5) and (6):

$$\begin{aligned}
& V_n(a,i,c_G,c_O)-V_n(a+1,i,c_G,c_O)\\
= & W_n(a,i,c_G,c_O,0)-W_n(a+1,i,c_G,c_O,0)\\
= & (J(a,i,c_O)-J(a+1,i,c_O))\\
+ & \sum_{d_G}\sum_{d_O}H^G_{c_Gd_G}H^O_{c_Od_O}\sum_j Q_{ij}(V_{n-1}(a,j,d_G,d_O)\\
& -V_{n-1}(a+1,j,d_G,d_O)) \qquad (21)
\end{aligned}$$

Case 2: $i_n^m(a,c_G,c_O)\geq i>i_n^{m+1}(a,c_G,c_O)$, where $m\geq 1$.
By Lemma 4, $x_n^*(a,i,c_G,c_O)=m$ and hence by Theorem 1, $x_n^*(a+1,i,c_G,c_O)=m-1$. So by (5) and (6) and some cancellation of terms, we get:

$$\begin{aligned}
& V_n(a,i,c_G,c_O)-V_n(a+1,i,c_G,c_O)\\
= & W_n(a,i,c_G,c_O,m)-W_n(a+1,i,c_G,c_O,m-1)\\
= & nc_G \qquad (22)
\end{aligned}$$

By (21) and (22), $V_n(a,i,c_G,c_O)-V_n(a+1,i,c_G,c_O)$

$$=\begin{cases} nc_G & \text{if } i\leq i_n^1(a,c_G,c_O),\\ (J(a,i,c_O)-J(a+1,i,c_O)) & \\ +\sum_{d_G}\sum_{d_O}H^G_{c_Gd_G}H^O_{c_Od_O}\sum_j Q_{ij}(V_{n-1}(a,j,d_G,d_O) & \\ -V_{n-1}(a+1,j,d_G,d_O)) & \text{if } i>i_n^1(a,c_G,c_O).\end{cases}$$

The expression for $V_n(a,i,c_G,c_O)-V_n(a+1,i,c_G,c_O)$ for $i>i_n^1(a,c_G,c_O)$ is an increasing function of i by Property 3, induction hypothesis and Condition 1. Thus, to show that $V_n(a,i,c_G,c_O)-V_n(a+1,i,c_G,c_O)$ is increasing in i, it is sufficient to show that for $i>i_n^1(a,c_G,c_O)$:

$$\begin{aligned}
& (J(a,i,c_O)-J(a+1,i,c_O))\\
+ & \sum_{d_G}\sum_{d_O}H^G_{c_Gd_G}H^O_{c_Od_O}\sum_j Q_{ij}(V_{n-1}(a,j,d_G,d_O)\\
& -V_{n-1}(a+1,j,d_G,d_O))\geq nc_G \qquad (23)
\end{aligned}$$

By (6), (23) is equivalent to $W_n(a,i,c_G,c_O,0)\geq W_n(a,i,c_G,c_O,1)$, which is true because $x_n^*(a,i,c_G,c_O)=0$ for $i>i_n^1(a,c_G,c_O)$. The result follows. ■

From the above lemmas, we get the desired monotonicity of $x_n^*(i,a,c_G,c_O)$.

Proof of Theorem 2: Fix n, a, c_G and c_O. By Lemma 7, $V_{n-1}(a,j,d_G,d_O)-V_{n-1}(a+1,j,d_G,d_O)$ is an increasing function of j for each d_G,d_O. The result follows by Lemma 5. ■

Proof of Theorem 3: The proof is very similar to the proof of Theorem 2 and hence omitted. ■

B. Proofs of results in Section II-D

Proof of Theorem 4: We show that the Maximum Independent Set (MIS) problem is a special case of Primary-SCTM. Consider the following special case of Primary-SCTM: $M=1$, $T=1$. At each node, the primary's demand is always 0, and the prices of Type G and O contracts are fixed, equal to $\frac{1}{2}$ and 1 respectively. Thus, it is optimal never to sell a type G contract.

The Primary-SCTM problem reduces to that of finding a maximum independent set of nodes (at which to sell Type O contracts). The result follows, since the MIS problem is NP-Hard [12]. ■

Proof of Lemma 1: Let $N^t_{e,j}$ be the number of Type-j contracts ($j\in\{G,O\}$) sold by a policy P in slot t on channel e. We make the following key observations:
(1) The revenue of any policy depends only on the *number* of Type-G and Type-O contracts it sells on each channel, in each slot, independent of *which* nodes it sells them at. That is, the revenue of the policy P is completely determined by:

$$\{N^t_{e,G},N^t_{e,O}: e=1,\ldots,M; t=1,\ldots,T\}$$

This follows from the fact that on each channel, the prices of both types of contracts and the usage status (*i.e.*, whether or not the primary is using the channel for subscriber demand satisfaction) are the same at all nodes.
(2) For every policy, on each channel, at any time, the total number of Type-G and Type-O contracts currently leased is at most equal to $|I(S)|$.
That is, for the above policy P, for every slot t:

$$\sum_{\tau=1}^{t}N^\tau_{e,G}+N^t_{e,O}\leq|I(S)|,\ e=1,\ldots,M \qquad (24)$$

This follows from the fact that $I(S)$ is a maximum independent set.

Now, let P be an optimal policy. Consider a policy $f\in\mathcal{F}$, which initially finds a maximum independent set $I(S)$. Also, whenever P sells a contract, f sells the same type of contract on the same channel at a node in $I(S)$ at which no contract has been sold on this channel. More precisely, number the nodes in $I(S)$ from 1 to $|I(S)|$. In slot t, on channel e, policy f sells Type-G contracts at the nodes $\sum_{\tau=1}^{t-1}N^\tau_{e,G}+1$ to $\sum_{\tau=1}^{t}N^\tau_{e,G}$ and Type-O contracts at the nodes $\sum_{\tau=1}^{t}N^\tau_{e,G}+1$ to $\sum_{\tau=1}^{t}N^\tau_{e,G}+N^t_{e,O}$. It can be checked that on each channel e, (a) for policy f, two or more contracts never stand leased at the same node and (b) by (24), in each slot t, f finds enough nodes in $I(S)$ to sell contracts at.

Now, by observation (1), the revenue of f is the same as that of P, and therefore f is optimal. ■