Dependent Lambda Encoding with Self Types

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It is well known that natural numbers can be encoded as lambda terms using Church encoding [2] or Scott encoding (reported in [4]). So operations such as plus, multiplication can be performed by beta-reduction on lambda terms. Other inductive data structures such as trees, lists, etc. (II, chapter 11 in [6]) can also be represented in a similar fashion.

Church-encoded data can be typed in system F [5]. But this approach is rarely adopted in dependent type systems. As summarized by Werner [8], it is inefficient to define certain operation on Church-encoded data, e.g. the predecessor function; the induction principle is not derivable and \( 0 \neq 1 \) cannot be proved. Thus we are led to the consideration of extending the Calculus of Construction (CoC) [3] with inductive datatypes [7].

In CoC à la Curry, we define \( \text{Nat} := \forall X.(X \to X) \to X \to X \). One can obtain a notion of indexed iterator by defining \( \text{lt} := \lambda x.\lambda f.\lambda a.x f a \) and \( \text{lt} : \forall X.\Pi x : \text{Nat}.(X \to X) \to X \to X \). Thus we have \( \text{lt} \bar{n} =_\beta \lambda f.\lambda a.\bar{n} f a =_\beta \lambda f.\lambda a.f(f(...(f a)...)) \).

An indexed iterator is nice, but one may want to know if we can obtain a finer version, namely, the induction principle \( \text{Id} \) such that:

\[
\text{Id} : \forall P : \text{Nat} \to \ast. \Pi x : \text{Nat} \to P((\text{S} y)) \to P 0 \to P \bar{n}.
\]

Let us try to construct such an \( \text{Id} \). First observe the following beta equalities:

\[
\begin{align*}
\text{Id} \bar{0} &= _\beta \lambda f.\lambda a.a \\
\text{Id} \bar{n} &= _\beta \lambda f.\lambda a.f\bar{n} - I(\cdots f \bar{1}(f \bar{0} a)).
\end{align*}
\]

with \( f : \Pi y : \text{Nat} \to P((\text{S} y)), a : P \bar{0} \).

So the above equalities suggest \( \text{Id} := \lambda x.\lambda f.\lambda a.x f a \), with a different notion of lambda numerals, i.e.

\[
\begin{align*}
\bar{0} &= \lambda s.\lambda z.z \\
\bar{n} &= \lambda s.\lambda z.s \bar{n} - I(\bar{n} - I s z).
\end{align*}
\]

Now let us try to type these lambda numerals. It is reasonable to assign \( s : \Pi y : \text{Nat} \to P((\text{S} y)) \) and \( z : P 0 \). Thus we have the following typing relation:

\[
\begin{align*}
\bar{0} : \Pi y : \text{Nat} \to P((\text{S} y)) \to P 0 \to P \bar{0} \\
1 : \Pi y : \text{Nat} \to P((\text{S} y)) \to P 0 \to P 1 \\
\bar{n} : \Pi y : \text{Nat} \to P((\text{S} y)) \to P 0 \to P \bar{n}
\end{align*}
\]

So we are led to define

\[
\text{Nat} := \Pi y : \text{Nat} \to P((\text{S} y)) \to P 0 \to P \bar{n} \text{ for any } \bar{n}.
\]

Two problems arise with this scheme of encoding. The first problem involves mutual recursion. The definitions of \( \text{Nat} \) contains \( \text{Nat} \) and \( \text{S} 0 \), while the type of \( \text{S} \) is \( \text{Nat} \to \text{Nat} \) and the type of \( 0 \) is \( \text{Nat} \). This problem can be addressed by adopting mutually recursive definitions. The second problem is about quantification. We want to define a type \( \text{Nat} \) for any \( \bar{n} \), but right now what we really have is one \( \text{Nat} \) for each numerals \( \bar{n} \). We aims to solve this problem by introducing a new type construct \( \iota x.T \) called \textit{self type}. The idea is that the \( \iota x.T \) allows \( T \) to refer, via bound variable \( x \), to the term which the self type is typing. Thus we define

\[
\text{Nat} := \iota x.\Pi y : \text{Nat} \to P((\text{S} y)) \to P 0 \to P \bar{n} \text{ x } \text{Nat} \text{ for any } \bar{n}.
\]

The self type can only be instantiated/generalized by its own subject, so we add the following two rules and the judgement:

\[
\begin{align*}
\Gamma \vdash t : [t/x]T & \quad \text{SelfGen} \quad \Gamma \vdash t : \iota x.T & \quad \text{SelfInst} \\
\Gamma \vdash \bar{n} : \Pi y : \text{Nat} \to P((\text{S} y)) & \quad \bar{n} : \Pi y : \text{Nat} \to P((\text{S} y)) \to P 0 \to P \bar{n}
\end{align*}
\]

In this talk, we will introduce a type system called \textit{Selfstar} with mutually recursive definitions, self types, and \( \ast : \ast \). We will see how standard Church- and Scott-encoded datatype can be presented in \textit{Selfstar}.
References


