Dependent types and program equivalence

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Doing dependent types wrong without going wrong

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What are dependent types?

Types that depend on elements of other types.

- **Examples:**
  - `vec n` – type of lists of length in
  - Generalized tries
  - PADS
  - Type of ASTs that represent well-typed code

- **Statically enforce expressive program properties**
  - BST ops preserve BST invariants
  - CompCert compiler
## Two sorts in practice today

<table>
<thead>
<tr>
<th>Pure everywhere</th>
<th>Pure types only</th>
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<tbody>
<tr>
<td>Types indexed by actual computations, everything is pure (terminating)</td>
<td>Types indexed by a pure language, separate from impure computations</td>
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<tr>
<td>• Decidable type checking</td>
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<td>• Easy to connect type system to actual computation</td>
<td>• Expressive computation language, including nontermination, state &amp; control</td>
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<td>• Uniform reasoning independent of phase</td>
<td>effects, etc</td>
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<td>• Total correctness</td>
<td></td>
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<tr>
<td>• Not really a programming language</td>
<td>• Index language may have minimal similarity to computation language, both in</td>
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<td></td>
<td>syntax and semantics</td>
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<td></td>
<td>• &quot;Partial&quot; Correctness</td>
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<td>Examples: Coq, Epigram, Agda2</td>
<td>Examples: DML, ATS, Ømega, Haskell</td>
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Let’s do it wrong…

- What about languages that are impure everywhere?
  - Deliberately allow nonterminating terms in types
  - Type:Type [Cardelli 86], Cayenne [Augustsson 98]
- What does a type soundness proof for such a language look like?
  - Note: type checking undecidable

Advantages
- Linguistic uniformity, reasoning does not depend on phase
- Programming language, not a logic

Disadvantages
- How to type check?
- Partial correctness
What else do we want?

- **Syntactic type soundness proof**
  - Easily extensible

- **Strong eliminators**
  - "If x = true then int else bool"
  - Important for expressivity, refinements, etc.

- **Call-by-value language**
  - If we have an impure language, we must fix the evaluation order
  - CBV has better treatment of control effects

- **"Modular" metatheory**
  - Program equivalence is hard. Let's not commit to a particular definition.
"Pure everywhere" type system - PTS

- No distinction between types, terms, kinds
  
es, τ, k ::= x | λx.e | e e' | (x:τ₁) → τ₂ | * | □
  
  | T | C | case e { Ci xi ⇒ ei }

- One set of formation rules
  
  \[ \Gamma \vdash e : \tau \]

- Conversion rule uses beta-equivalence
  
  \[ \Gamma \vdash e : \tau_1 \quad \Gamma \vdash \tau_2 : s \quad \tau_1 \sim \tau_2 \]
  
  \[ \Gamma \vdash e : \tau_2 \]

- Term equivalence is fixed by type system (and defined to be the same as type equivalence).
New vision

- Syntactic distinction between terms, types, and kinds
  
  \[ k ::= \ast | (x:\tau) \rightarrow \ast \]
  
  \[ \tau ::= (x:\tau_i) \rightarrow \tau_2 \mid T \mid \tau \ e \mid \text{case } e \ {T' e' \rangle \ {\{ {C_i x_i \Rightarrow \tau_i} \}} \]
  
  \[ e ::= x \mid \text{fun } f (x) = e \mid e \ e' \mid C \ e \mid \text{case } e \ {\{ {C_i x_i \Rightarrow e_i} \}} \]

- Key syntactic changes
  - Term language includes non-termination
  - Curry-style, no types in expressions

- Convenient simplifications
  - Datatypes have one index, data constructors have one argument (unit/products in paper)
  - No polymorphism, no higher-kindled types (future work)
Parameterized term equivalence

- Given an "equivalence context"
  \[ \Delta ::= \Delta, e_1 = e_2 \]
- Assume the existence of program equivalence predicate
  \[ \text{isEq} (\Delta, e_1, e_2) \]
- Equality is untyped
  - No guarantee that \( e_1 \) and \( e_2 \) have the same type
  - No assumptions about the types of the free variables
- Rules do not use substitution, add to equivalence context instead
Type system

- Two sorts of judgments
  - Equality for type, contexts, and kinds
    \[ \Delta \vdash \tau \equiv \tau' \]
  - Formation for contexts, kinds, types and terms
    \[ \Gamma \vdash e : \tau \]
- All judgments derivable from an inconsistent context
  - `incon (\Delta)` if there exist pure terms \( C_i w_i \) and \( C_j w_j \) such that
    \( \text{isEq} (\Delta, C_i w_i, C_j w_j) \) and \( C_i \neq C_j \)
- Pure terms
  - \( w ::= x \mid \text{fun} f(x) = e \mid C w \)
Typing rules (excerpt)

$$\Gamma \vdash e : \tau \quad \Gamma^* \vdash \tau \equiv \tau' \quad \Gamma \vdash \tau' : *$$

$$\Gamma \vdash e : \tau'$$

$$\vdash \Gamma \text{ incon} (\Gamma^*)$$

$$\Gamma \vdash e : \tau$$

$$\Gamma \vdash e_1 : (x:\tau_1) \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1$$

$$\Gamma^*, x \equiv e_2 \vdash \tau_2 \equiv \tau \quad \Gamma \vdash \tau : *$$

$$\Gamma \vdash e_1 \ e_2 : \tau$$
Typing rules for case

\[
\begin{align*}
C : (x:\sigma) \rightarrow T & \quad u \in \Sigma_0 \quad \Gamma \vdash e : \sigma \\
\Gamma^*, \ x \simeq e & \vdash Tu \equiv \tau \quad \Gamma \vdash \tau : * \\
\hline
\Gamma \vdash C e : \tau
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e : Tu & \quad \text{CtrOf}(T) = \overline{C_i}^{i \in 1..n} \\
\Gamma \vdash \tau : * & \quad C_i : (x_i:\tau_i) \rightarrow Tu_i \in \Sigma_0 \quad \overline{C_i}^{i \in 1..n} \\
\Gamma, x_i : \tau_i, u \simeq u_i, e \simeq C_i x_i \vdash e_i : \tau^{i \in 1..n} \\
\hline
\Gamma \vdash \text{case } e \text{ of } \{ C_i x_i \Rightarrow e_i \}^{i \in 1..n} : \tau
\end{align*}
\]
Type equivalence (excerpt)

\[
\frac{\text{incon}(\Delta)}{\Delta \vdash \tau \equiv \tau'}
\]

\[
\frac{\Delta \vdash \tau \equiv \tau' \quad \text{isEq}(\Delta, e, e')} {\Delta \vdash \tau \ e \equiv \tau' \ e'}
\]

\[
isEq(\Delta, e, C_j \ w) \quad C_j \in \overline{C_i}^{i \in 1..n} \\
C_j : (x_j : \sigma_j) \rightarrow T \quad u_j \in \Sigma_0 \\
isEq((\Delta, w \equiv x_j), u, u_j) \\
\Delta, w \equiv x_j, e \equiv C_j \ x_j \vdash \tau_j \equiv \tau
\]

\[
\Delta \vdash \text{case } e \langle T \ u \rangle \text{ of } \{ \overline{C_i \ x_i \Rightarrow \tau_i}^{i \in 1..n} \} \equiv \tau
\]
Questions to answer

- What properties of isEq must hold to show preservation & progress?
- What instantiations of isEq satisfy these properties?
Necessary assumptions about isEq

- Is an equivalence class
- Contains evaluation: \( e \mapsto e' \) implies \( \text{isEq}(\Delta, e, e') \)
- Constructors are injective for pure arguments
  - \( \text{isEq}(\Delta, Cw, Cw') \) implies \( \text{isEq}(\Delta, w, w') \)
- Empty context is consistent
  - \( C \neq C' \) implies \( \text{isEq}(., Cw, C'w') \) does not hold
- Closed under pure substitution
  - \( \text{isEq}(\Delta, e, e') \) implies \( \text{isEq}(\Delta\{w/x\}, e\{w/x\}, e'\{w/x\}) \)
- Preserved under contextual operations
  - \( \text{isEq}((\Delta, e = e', \Delta'), e_1, e_2) \) and \( \text{isEq}(\Delta, e, e') \) implies \( \text{isEq}(\Delta\Delta', e_1, e_2) \)
  - \( \text{isEq}(\Delta\Delta'', e_1, e_2) \) implies \( \text{isEq}(\Delta\Delta'\Delta'', e_1, e_2) \)
  - \( \text{isEq}(\Delta, e_1, e_2) \) and \( \Delta = \Delta' \) implies \( \text{isEq}(\Delta', e_1, e_2) \)
What satisfies these properties?

- Compare normal forms (ignoring $\Delta$)
  - Only types STLC terms
- Contextual equivalence (ignoring $\Delta$)
  - Only types STLC terms
- RST-closure of evaluation, constructor injectivity, and equivalence assumptions
- CBV Contextual equivalence modulo $\Delta$
- Some strange equalities that identify nonterminating terms with terminating terms
  - Safe to conclude $\text{isEq(let } x = \text{loop in 3, 3)}$ as long as we don’t conclude $\text{isEq(let } x = \text{loop in 3, loop)}$
  - Safe to say $\text{isEq(loop,3)}$ as long as we don’t say $\text{isEq(loop, 4)}$
What about decidable type checking?

- All instantiations of `isEq` are undecidable
  - Must contain evaluation relation
- Decidable approximations are type safe, but don’t satisfy preservation
  - Any types system that checks strictly fewer terms than a safe type system is safe
- Preservation important for compiler transformations
  - Want to know that inlining always produces safe code
  - Not really an issue: Decidable doesn't mean tractable
What about termination analysis?

- Like most type systems, only get "partial correctness" results:
  - \( \Sigma x : t. \ P(x) \) means “If this expression terminates, then it produces a value of type \( t \) such that \( P \) holds”
  - Implications (\( P_1 \rightarrow P_2 \)) may be bogus
- Termination analysis produces total correctness
- Termination/stage analysis is an optimization
  - permits proof erasure in CBV language
Future work

- Add polymorphism, higher-order types
  - Keep curry-style system for simple specification of isEq
- Annotated external language to aid type checking
  - Similar to ICC* [Barras and Bernardo]
  - Terms contain type annotations, but equality defined for erased terms
  - Type checking still undecidable but closer to an algorithm
- Add control/state effects to computations
  - Should we limit domain of isEq?
  - Non-termination ok in types, but exceptions are not?
- Can we provide type/termination information to strengthen equivalence?
Conclusions – What have we achieved?

- Uniform design
  - Same reasoning for compile time as run time
  - Not easy for CBV!

- Simple design
  - Program equivalence isolated from type system
  - Proved all metatheory in Coq in ~2 weeks (OTT + LNgen)

- General design
  - Program equivalence not nailed down
  - Lots of examples that satisfy preservation, not just type soundness