Programming Up-to-Congruence

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ZOMBIE

A functional programming language with a dependent type system intended for “lightweight” verification

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The ZOMBIE programming language

Goal: FP++

- Functional programming enhanced by reasoning in constructive logic
- Full-spectrum dependent types (for uniformity)
- Erasable arguments (for efficient compilation)
- Simple semantics for indexed types and dependently-typed pattern matching
The **Zombie** programming language

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- Functional programming enhanced by reasoning in constructive logic
- Full-spectrum dependent types (for uniformity)
- Erasable arguments (for efficient compilation)
- Simple semantics for indexed types and dependently-typed pattern matching
- **Proof automation based on congruence closure**
ZOMBIE: A language, in two parts

- Programmatic fragment: nontermination allowed (similar to ML and Haskell)

\[
\text{prog} \quad \text{div} : \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Nat} \\
\text{rec} \quad \text{div} \ n \ m = \text{if } n < m \ \text{then} \ 0 \ \text{else} \ 1 + \text{div} \ (n - m) \ m
\]
Zombie: A language, in two parts

1. Programmatic fragment: nontermination allowed (similar to ML and Haskell)

    prog div : Nat → Nat → Nat
    rec div n m = if n < m then 0 else 1 + div (n - m) m

2. Logical fragment: all programs must terminate (similar to Coq and Agda)

    log add : Nat → Nat → Nat
    ind add x y = case x [eq] of
      Zero → y          -- eq : x = Zero
      Suc x’ → add x’ [ord eq] y    -- eq : x = Suc x’, used for ind
ZOMBIE: A language, in two parts

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2. Logical fragment: all programs must terminate (similar to Coq and Agda)

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add x y = case x [eq] of
  Zero   → y          -- eq : x = Zero
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```

Uniformity: Both fragments use the same syntax, have the same (call-by-value) operational semantics.
Dependent types in ZOMBIE

The logical fragment can reason about the programmatic fragment.

\[
\text{log } \text{div62 : div 6 2 = 3 }
\text{div62 = join}
\]

Here, \text{join} proves that two terms are equal because they reduce to the same value.
Dependent types in ZOMBIE

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log div62 : div 6 2 = 3
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Here, `join` proves that two terms are equal because they reduce to the same value.

Type checking `join` is undecidable, so includes an overridable timeout—the programmer is in control.
Restricted $\beta$-equality

The type checker reduces terms *only* when directed by the programmer (e.g. while type checking `join`).
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ZOMBIE does not include $\beta$-convertibility in definitional equality!

In a context with

\[
\begin{align*}
f & : \text{Vec Bool 3} \to \text{Nat} \\
x & : \text{Vec Bool} (\text{div 6 2})
\end{align*}
\]

the expression $f\ x$ does not type check because $\text{div 6 2}$ is not automatically equal to 3.
Restricted $\beta$-equality

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**ZOMBIE** does not include $\beta$-convertibility in *definitional equality*!

In a context with

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  f &: \text{Vec Bool } 3 \rightarrow \text{Nat} \\
  x &: \text{Vec Bool } (\text{div } 6 \ 2)
\end{align*}
\]

the expression $f \ x$ does **not** type check because $\text{div } 6 \ 2$ is **not** automatically equal to $3$.

In other words, $\beta$-conversion is only available for *propositional* equality.

\[
f \ (x \ |ightarrow \ \text{[Vec Bool } \sim \text{div62}])
\]
Isn’t type checking without $\beta$ awful?
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Yes.
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Yes. And our simple semantics for dependently-typed pattern matching makes it worse.

```
log npluszero : (n : Nat) → (n + 0 = n)
ind npluszero n =
  case n [eq] of
  Zero → (join : 0 + 0 = 0)
    |> [~eq + 0 = ~eq]  -- explicit type coercion
    -- eq : 0 = n
  Suc m →
    let ih = npluszero m [ord eq] in
    (join : (Suc m) + 0 = Suc (m + 0))
    |> [(Suc m) + 0 = Suc ~ih]  -- ih : m + 0 = m
    |> [~eq + 0 = ~eq]  -- eq : Suc m = n
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\text{ind} \ npluszero \ n = \begin{align*}
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\text{Zero} & \rightarrow (\text{join} : 0 + 0 = 0) \\
& \quad | (\text{eq} + 0 = \text{eq}) & \quad \text{-- explicit type coercion} \\
& \quad | (\text{Suc} \ m + 0 = \text{Suc} ~ \text{ih}) & \quad \text{-- ih} : m + 0 = m \\
\text{Suc} \ m & \rightarrow (\text{let} \ ih = npluszero \ m [\text{ord} \ \text{eq}] \ in) \\
& \quad (\text{join} : (\text{Suc} \ m) + 0 = \text{Suc} (m + 0)) \\
& \quad | (\text{Suc} \ m + 0 = \text{Suc} ~ \text{ih}) & \quad \text{-- ih} : m + 0 = m \\
& \quad | (\text{eq} + 0 = \text{eq}) & \quad \text{-- eq} : \text{Suc} \ m = n
\end{align*}
\]

But we can do better.
Better

What if the type checker could determine those coercions automatically?

\[
\begin{align*}
\text{log npluszero} & \colon (n : \text{Nat}) \rightarrow (n + 0 = n) \\
\text{ind npluszero} n = \\
\quad \text{case } n [\text{eq}] \text{ of} \\
\quad \quad \text{Zero } \rightarrow (\text{join} : 0 + 0 = 0) \\
\quad \quad \quad \quad \text{-- coercion by eq inferred} \\
\quad \quad \text{Suc } m \rightarrow \\
\quad \quad \quad \quad \text{let } \text{ih} = \text{npluszero } m \ [\text{ord eq}] \text{ in} \\
\quad \quad \quad \quad (\text{join} : (\text{Suc } m) + 0 = \text{Suc } (m + 0)) \\
\quad \quad \quad \quad \text{-- coercion by eq and } \text{ih} \text{ inferred}
\end{align*}
\]

\text{i.e. automatically coerce type } 0 + 0 = 0 \text{ to type } n + 0 = n \text{ in contexts where } \text{eq: } n = 0 \text{ is assumed.}
What if the type checker could determine those coercions automatically?

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&\text{Suc } m \rightarrow \\
&\quad \text{let } \text{ih} = \text{npluszero } m \text{ [ord eq] in} \\
&\quad (\text{join : } (\text{Suc } m) + 0 = \text{Suc (m + 0)}) \\
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Capture this idea with a relation:

\[\text{eq: } n = 0 \vdash (0 + 0 = 0) \equiv (n + 0 = n)\]
Opportunity: Congruence Closure

The relation that we need is the *congruence closure* of equations in the context.

\[
\frac{x : a = b \in \Gamma}{\Gamma \vdash a = b} \quad \frac{\Gamma \vdash a = b}{\Gamma \vdash \{a/x\}c = \{b/x\}c}
\]

\[
\frac{\Gamma \vdash a = a}{\Gamma \vdash a = b} \quad \frac{\Gamma \vdash b = a}{\Gamma \vdash a = c}
\]

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\frac{\Gamma \vdash a = b \quad \Gamma \vdash b = c}{\Gamma \vdash a = c}
\]

Efficient algorithms for deciding this relation exist [Nieuwenhuis and Oliveras, 2007].

Note, extending this relation with β-conversion makes it undecidable.
What we have done

Designed and implemented a concise surface language for ZOMBIE programmers

- Specification via bidirectional type system
  \[ \Gamma \vdash a \Rightarrow A \quad \text{and} \quad \Gamma \vdash a \Leftarrow A \]

- Type checking is up-to Congruence Closure

\[
\frac{\Gamma \vdash a \Rightarrow A \quad \Gamma \vdash A = B} {\Gamma \vdash a \Rightarrow B} \quad \frac{\Gamma \vdash a \Leftarrow A \quad \Gamma \vdash A = B} {\Gamma \vdash a \Leftarrow B}
\]

- Elaborates to explicitly-typed core language, previously proven sound
  [POPL ’14][MSFP’12]
ZOMBIE-style Congruence Closure

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2. Makes use of assumptions that are equivalent to equalities

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x : A \in \Gamma \quad \Gamma \vdash A = (a = b) \\
\Gamma \vdash a = b
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3. Supports injectivity of type (and data) constructors

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\Gamma \vdash ((x : A_1) \to B_1) = ((x : A_2) \to B_2) \\
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\Gamma \vdash A_1 = A_2
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4. Works up-to-erasure

\[
|a| = |b| \quad \Gamma \vdash a : A \quad \Gamma \vdash b : B \\
\Gamma \vdash a = b
\]
**ZOMBIE-style Congruence Closure**

1. Only includes well-typed terms
2. Makes use of assumptions that are *equivalent* to equalities
   \[
   x : A \in \Gamma \quad \Gamma \vdash \ A = (a = b) \\
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   |a| = |b| \quad \Gamma \vdash a : A \quad \Gamma \vdash b : B \\
   \frac{}{\Gamma \vdash a = b}
   \]
5. and generates proof terms in the core language
Properties of Elaboration

- **Elaboration is sound**
  If elaboration succeeds, it produces a well-typed core language term.
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  If a term type checks according to the surface language specification, then elaboration will succeed.
Properties of Elaboration

- **Elaboration is sound**
  If elaboration succeeds, it produces a well-typed core language term.

- **Elaboration is complete**
  If a term type checks according to the surface language specification, then elaboration will succeed.

- **Elaboration doesn’t change the semantics**
  If elaboration succeeds, it produces a core language term that differs from the source term only in irrelevant information (type annotations, type coercions, erasable arguments).
Extensions
Proof inference

Congruence closure can also supply proofs of equality

```plaintext
log npluszero : (n : Nat) → (n + 0 = n)
ind npluszero n =
case n [eq] of
  Zero →
    let _ = (join : 0 + 0 = 0) in _
  Suc m →
    let _ = npluszero m [ord eq] in
    let _ = (join : (Suc m) + 0 = Suc (m + 0)) in _
```
Extension: Unfold

Common to reduce terms as much as possible

\[ \log \ npluszero : (n : \text{Nat}) \rightarrow (n + 0 = n) \]
\[ \text{ind } npluszero \ n = \]
\[ \begin{array}{l}
\text{case } n \ [\text{eq}] \ of \\
\quad \text{Zero } \rightarrow \text{unfold } (0 + 0) \ in \ _ \\
\quad \text{Suc } m \rightarrow \\
\quad \quad \text{let } _ = \ npluszero \ m \ [\text{ord eq}] \ in \\
\quad \quad \text{unfold } ((\text{Suc } m) + 0) \ in \ _
\end{array} \]

The expression \text{unfold } a \ in \ b \ expands \ to

\[ \begin{array}{l}
\text{let } _ = (\text{join : } a = a1) \ in \\
\text{let } _ = (\text{join : } a1 = ...) \ in \\
\quad \ldots \\
\text{let } _ = (\text{join : } ... = an) \ in \\
\quad \text{b}
\end{array} \]

when \( a \leadsto a1 \leadsto \ldots \leadsto an \)
Extension: Reduction Modulo

The type checker makes use of congruence closure when reducing terms with \texttt{unfold}.

\begin{verbatim}
log npluszero : (n : Nat) → (n + 0 = n)
ind npluszero n =
  case n [eq] of
  Zero → unfold (n + 0) in _
  Suc m →
    let ih = npluszero m [ord eq] in
    unfold (n + 0) in _
\end{verbatim}

E.g., if we have \( h : n = 0 \) in the context, allow the step

\[ n + 0 \rightsquigarrow_{\text{cbv}} 0 \]
Extension: Smartjoin

Use unfold (and reduction modulo) on both sides of an equality when type checking \texttt{join}.

\begin{verbatim}
log npluszero : (n : Nat) → (n + 0 = n)
ind npluszero n =
  case n [eq] of
    Zero → smartjoin
    Suc m → let ih = npluszero m [ord eq] in
      smartjoin
\end{verbatim}
Conclusions

- Dependently-typed languages should allow nonterminating programs, but compile-time reduction is tricky.
- Restricting $\beta$-reduction allows alternative forms of automatic reasoning, specifically congruence closure.
- Congruence closure powers smart case, a simple specification of dependently-typed pattern matching.
- Proof automation is an important part of the design of dependently-typed languages, and should be backed up by specifications.
Implementation and examples available:
https://code.google.com/p/trellys/source/browse/trunk/zombie-trellys/
or Google: zombie trellys
Thanks!