ASYMPTOTIC COVARIANCE

1. SAR Model

$$y = X\beta + u,\ u = \rho W u + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

$$\Rightarrow u = (I_n - \rho W)^{-1}\varepsilon = B_\rho^{-1}\varepsilon \Rightarrow \boxed{y \sim N\left(X\beta, \sigma^2 (B_\rho' B_\rho)^{-1}\right)}$$

If $G_\rho = W B_\rho^{-1}$, then the *asymptotic covariance matrix* is given by:

$$\boxed{\operatorname{cov}(\hat{\beta}, \hat{\sigma}^2, \hat{\rho}) = \sigma^4 \begin{pmatrix} \sigma^2 X' B_{\hat{\rho}}' B_{\hat{\rho}} X & 0 & 0 \\ 0 & n/2 & \sigma^2 tr(G_{\hat{\rho}}) \\ 0 & \sigma^2 tr(G_{\hat{\rho}}) & \sigma^4 tr(G_{\hat{\rho}}(G_{\hat{\rho}} + G_{\hat{\rho}}')) \end{pmatrix}^{-1}}$$

1. SP-Lag Model

$$y = \lambda W y + X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

$$\Rightarrow B_\lambda y = (I_n - \lambda W) y = X\beta + \varepsilon$$

$$\Rightarrow y = B_\lambda^{-1} X\beta + B_\lambda^{-1}\varepsilon \Rightarrow \boxed{y \sim N\left(B_\lambda^{-1} X\beta, \sigma^2 (B_\lambda' B_\lambda)^{-1}\right)}$$

If $G_\lambda = W B_\lambda^{-1}$ and $H(\beta, \lambda) = \sigma^4 tr(G_\lambda(G_\lambda + G_\lambda')) + \sigma^2 \beta' X' G_\lambda X \beta$, then the *asymptotic covariance matrix* is given by:

$$\boxed{\operatorname{cov}(\hat{\beta}, \hat{\sigma}^2, \hat{\lambda}) = \sigma^4 \begin{pmatrix} \sigma^2 X' X & 0 & \sigma^2 X' G_{\hat{\lambda}} X \\ 0 & n/2 & \sigma^2 tr(G_{\hat{\lambda}}) \\ \sigma^2 X' G_{\hat{\lambda}} X & \sigma^2 tr(G_{\hat{\lambda}}) & H(\hat{\beta}, \hat{\lambda}) \end{pmatrix}^{-1}}$$