BIAS OF REGRESSION ESTIMATOR FOR RHO

To establish that the regression estimator, $\hat{\rho}$, of $\rho$ is biased, suppose that the actual residual values, $u$, in the relation,

$$u = \rho W u + \varepsilon$$

are observable. Then we can estimate $\rho$ by a (no-intercept) regression to obtain,

$$\hat{\rho} = \frac{u'W'u}{u'W'Wu}$$

But from (1) it then follows that

$$\hat{\rho} = \frac{u'W' (\rho W u + \varepsilon)}{u'W'Wu} = \frac{\rho \cdot u'W'Wu}{u'W'Wu} + \frac{u'W' \varepsilon}{u'W'Wu}$$

$$= \rho + \frac{u'W' \varepsilon}{u'W'Wu} = \rho + \frac{(Wu)' \varepsilon}{\|Wu\|^2} = \rho + \frac{(Wu)' \varepsilon}{\|Wu\| \|\varepsilon\|}$$

$$= \rho + \text{corr}(Wu, \varepsilon) \left( \frac{\|\varepsilon\|}{\|Wu\|} \right)$$

Now if $\rho > 0$ then it is clear from (1) that $\text{corr}(u, \varepsilon) > 0$ and $\text{corr}(u, Wu) > 0$, so that one must almost surely have $\text{corr}(Wu, \varepsilon) > 0$ [which can in fact be shown]. Hence by the positivity of the ratio, $\|\varepsilon\| / \|Wu\|$, it follows that that $E(\|\varepsilon\| / \|Wu\|) > 0$, so that:

$$E(\hat{\rho}) = \rho + \text{corr}(Wu, \varepsilon) E \left( \frac{\|\varepsilon\|}{\|Wu\|} \right) > \rho$$

Thus $\hat{\rho}$ is biased upwards. Moreover, this bias does not disappear even for large sample sizes, so that this result also shows that $\hat{\rho}$ is not even a consistent estimator of $\rho$.

Finally, since $\hat{\rho}$ tends to make $\rho$ look too large (for $\rho > 0$), it tends to inflate the significance of spatial autocorrelation (the worst of all possible cases!)
But Moran’s $I$ tends to *mitigate* this effect. To see this, note first that

\[
I = \frac{u'Wu}{\|u\|^2} = \frac{u'Wu(\|Wu\|^2)}{\|Wu\|^2 (\|u\|^2)} = \hat{\rho} \left( \frac{\|Wu\|^2}{\|u\|^2} \right)
\]

Next recall that *row normalized* matrices, $W$, act to *average* the components of $u$, and thus tend to reduce their variance, so that

\[
E\left( \|Wu\|^2 \right) < E\left( \|u\|^2 \right)
\]

This in turn can be shown to imply that $E\left( \|Wu\|^2 / \|u\|^2 \right) < 1$, and hence that

\[
E(I) = E \left[ \hat{\rho} \left( \frac{\|Wu\|^2}{\|u\|^2} \right) \right] < E(\hat{\rho})
\]

Hence Moran’s $I$ tends to *reduce* the inflated values of $\hat{\rho}$, and thus provides a *more stable* indicator of spatial autocorrelation.