BIAS OF REGRESSION ESTIMATOR FOR RHO

To establish that the regression estimator, $\hat{\rho}$, of ρ is biased, suppose that the actual residual values, u, in the relation,

(1) $$u = \rho W u + \varepsilon$$

are *observable*. Then we can estimate ρ by a (no-intercept) regression to obtain,

(2) $$\boxed{\hat{\rho} = \frac{u'W'u}{u'W'Wu}}$$

But from (1) it then follows that

(3) $$\hat{\rho} = \frac{u'W'(\rho W u + \varepsilon)}{u'W'Wu} = \frac{\rho \cdot u'W'Wu}{u'W'Wu} + \frac{u'W'\varepsilon}{u'W'Wu}$$

$$= \rho + \frac{u'W'\varepsilon}{u'W'Wu} = \rho + \frac{(Wu)'\varepsilon}{\|Wu\|^2} = \rho + \frac{(Wu)'\varepsilon}{\|Wu\|\|\varepsilon\|}\left(\frac{\|\varepsilon\|}{\|Wu\|}\right)$$

$$= \rho + corr(Wu, \varepsilon)\left(\frac{\|\varepsilon\|}{\|Wu\|}\right)$$

Now if $\rho > 0$ then it is clear from (1) that $corr(u,\varepsilon) > 0$ and $corr(u, Wu) > 0$, so that one must almost surely have $corr(Wu,\varepsilon) > 0$ [which can in fact be shown]. Hence by the positivity of the ratio, $\|\varepsilon\| / \|Wu\|$, it follows that that $E\left(\|\varepsilon\| / \|Wu\|\right) > 0$, so that:

(4) $$\boxed{E(\hat{\rho}) = \rho + corr(Wu,\varepsilon)\, E\left(\frac{\|\varepsilon\|}{\|Wu\|}\right) > \rho}$$

Thus $\hat{\rho}$ is *biased upwards*. Moreover, this bias does not disappear even for large sample sizes, so that this result also shows that $\hat{\rho}$ is *not even a consistent estimator* of ρ. Finally, since $\hat{\rho}$ tends to make ρ look *too large* (for $\rho > 0$), it tends to inflate the significance of spatial autocorrelation (the *worst* of all possible cases!)

But Moran's *I* tends to *mitigate* this effect. To see this, note first that

(5) $$I = \frac{u'W'u}{\|u\|^2} = \frac{u'W'u}{\|Wu\|^2}\left(\frac{\|Wu\|^2}{\|u\|^2}\right) = \hat{\rho}\left(\frac{\|Wu\|^2}{\|u\|^2}\right)$$

Next recall that *row normalized* matrices, W, act to *average* the components of u, and thus tend to reduce their variance, so that

(6) $$E\left(\|Wu\|^2\right) < E\left(\|u\|^2\right)$$

This in turn can be shown to imply that $E\left(\|Wu\|^2 / \|u\|^2\right) < 1$, and hence that

(7) $$E(I) = E\left[\hat{\rho}\left(\frac{\|Wu\|^2}{\|u\|^2}\right)\right] < E(\hat{\rho})$$

Hence Moran's *I* tends to *reduce* the inflated values of $\hat{\rho}$, and thus provides a *more stable* indicator of spatial autocorrelation.