MARGINAL IMPACT ANALYSES FOR SPATIAL LAG MODELS

Recall that in an *OLS regression* model the conditional expectation of the dependent variable, Y_i, is given for each observation, $i = 1,..,n$, by:

(1) $$\mu_i(x_{i1},..,x_{ik}) \equiv E(Y_i \,|\, x_{i1},..,x_{ik}) = \beta_0 + \sum_{r=1}^{k} \beta_r x_{ir}$$

Hence the *marginal impact* of each explanatory variable, x_{ir}, $r = 1,..,k$ on this conditional expectation is given simply by β_r, i.e.,

(2) $$\boxed{\frac{\partial \mu_i}{\partial x_{ir}} = \beta_r \;\;,\;\; r = 1,..,k}$$

This is exactly the same for *Spatial Autoregressive* (*SAR*) models since all spatial dependencies are in the residuals – which have zero expectation and thus continue to yield (1). But in a *Spatial Lag* (*SL*) model, the situation is much more complex. Given the *SL model*:

(3) $$Y_i = \lambda \sum_{j \neq i} w_{ij} Y_j + \beta_0 + \sum_{r=1}^{k} \beta_r x_{ir} + \varepsilon_i \;\;,\;\; i = 1,..,n$$

it is not even clear how to calculate such conditional expectations. But if we write the model in matrix form as

(4) $$Y = \lambda W Y + X\beta + \varepsilon$$

and calculate the *reduced form*:

(5) $$(I_n - \lambda W)Y = X\beta + \varepsilon \;\Rightarrow\; Y = (I_n - \lambda W)^{-1} X\beta + (I_n - \lambda W)^{-1}\varepsilon$$

then the assumption, $E(\varepsilon \,|\, X) = 0$, implies that the *full vector* of conditional expectations for Y given all explanatory data, X, can be computed as:

(6) $$\mu(X) \equiv E(Y \,|\, X) = (I_n - \lambda W)^{-1} X\beta + (I_n - W)^{-1} E(\varepsilon \,|\, X)$$

$$= (I_n - \lambda W)^{-1} X\beta$$

Hence if we write X in *column form* as $X = (x_0, x_1,..,x_r,..,x_k)$ [with x_0 denoting the *unit vector*, 1_n] and let

(7) $$M_{(\lambda,W)} = (I_n - \lambda W)^{-1}$$

denote the *spatial multiplier matrix* defined by (λ,W), then it follows that (7) can be rewritten as:

(8) $$\mu(X) = M_{(\lambda,W)}\, X\beta = M_{(\lambda,W)} \sum\nolimits_{r=0}^{k} \beta_r\, x_r = \sum\nolimits_{r=0}^{k} \beta_r\, M_{(\lambda,W)}\, x_r$$

Hence the *conditional expectation* of each Y_i given X is seen to be defined by the i^{th} row of (8), i.e., by

(9) $$\mu_i(X) = \sum\nolimits_{r=0}^{k} \beta_r\, M_{(\lambda,W)}(i,\cdot)\, x_r$$

where $M_{(\lambda,W)}(i,\cdot)$ denotes the i^{th} row of the spatial multiplier matrix. Notice in particular that μ_i now depends on *all* explanatory data, and not just the values, $x_{(i,\cdot)} = (x_{ir} : r = 1,..,k)$ directly associated with Y_i in (3). Hence, unlike OLS or SAR, there can be *indirect* impacts on Y_i from changes in j-attributes, $x_{(j,\cdot)}$ for samples $j \neq i$. However, for simplicity we shall continue to focus only on the *direct impacts* of $x_{(i,\cdot)}$ on Y_i. In particular, we can use (9) to calculate the marginal impact of x_{ir} on $\mu_i(X)$ as follows:

(10) $$\tfrac{\partial}{\partial x_{ir}} \mu_i(X) = \sum\nolimits_{s=0}^{k} \beta_s \tfrac{\partial}{\partial x_{ir}} [M_{(\lambda,W)}(i,\cdot)\, x_s]$$

$$= \beta_r \tfrac{\partial}{\partial x_{ir}} [M_{(\lambda,W)}(i,\cdot)\, x_r] = \beta_r \tfrac{\partial}{\partial x_{ir}} \sum\nolimits_{j=1}^{n} M_{(\lambda,W)}(i,j) x_{jr}$$

But since x_{ir} appears only in the i^{th} term of the last summation in (10), it follows that the *direct marginal impact* of each x_{ir} on the conditional expectation of Y_i is now given by

(11) $$\boxed{\tfrac{\partial}{\partial x_{ir}} \mu_i(X) = \beta_r\, M_{(\lambda,W)}(i,i) \quad , \quad r = 1,..,k}$$

Hence we see that for *SL* models, the direct marginal impact of x_{ir} on μ_i involves not only β_r but also a *spatial multiplier*, $M_{(\lambda,W)}(i,i)$, that summarizes the spatial interactions between Y_i and all other Y variables. To relate this to (2) above, notice in particular that if $\lambda = 0$ then $M_{(0,W)} = I_n$ so that, as expected, $M_{(0,W)}(i,i) = 1$, and (11) reduces to (2). Notice also from the simple product form of (11) that these effects are *separable* in the sense that *relative marginal impacts* depend only on the β coefficients, i.e., that for all $r,s = 1,..,k$

(12) $$\frac{\frac{\partial}{\partial x_{ir}}\mu_i(X)}{\frac{\partial}{\partial x_{is}}\mu_i(X)} = \frac{\beta_r}{\beta_s}$$

Hence this is another property shared with the simple case in (2).

Finally, it is important to notice that these marginal impacts vary from sample to sample, depending on the particular spatial locations of each sample. Hence one convenient way to summarize the direct marginal impacts of each explanatory variable x_r is to define an *average spatial multiplier* , $m_{(\lambda,W)}$, by:

(13) $$m_{(\lambda,W)} = \tfrac{1}{n}\sum_{i=1}^{n} M_{(\lambda,W)}(i,i) = \tfrac{1}{n}\,tr[M_{(\lambda,W)}]$$

[where for any matrix, $A=(a_{ij})$ the *trace* of A is the sum of its diagonal elements, $tr(A)=\sum_i a_{ii}$]. In terms of this average multiplier, the *average direct marginal impact* of the r^{th} explanatory variable is then given by

(14) $$\boxed{\overline{\tfrac{\partial}{\partial x_r}\mu(X)} = \beta_r\, m_{(\lambda,W)}}$$

For further development of such impact measures in the context of spatial regression models, see Section 2.7 in LeSage, James and R. Kelley Pace, (2009) *Introduction to Spatial Econometrics*, Chapman-Hall.