MAXIMUM LIKELIHOOD ESTIMATION

1. General Linear Model Estimation

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0,\sigma^2 V) \Rightarrow \boxed{y \sim N(X\beta, \sigma^2 V)}$$

$$\Rightarrow L(\beta,\sigma^2 \mid y, X, V) = \ln\phi(y \mid X, V, \beta, \sigma^2) \ \ (V \ known)$$

$$= -\tfrac{n}{2}\ln(2\pi) - \tfrac{1}{2}\ln(\sigma^2) - \tfrac{1}{2}\ln|V| - \tfrac{1}{2\sigma^2}(y - X\beta)'V^{-1}(y - X\beta)$$

$$\Rightarrow \boxed{\begin{aligned} \hat{\beta} &= (X'V^{-1}X)^{-1}X'V^{-1}y \\ \hat{\sigma}^2 &= \tfrac{1}{n}(y - X\hat{\beta})'V^{-1}(y - X\hat{\beta}) \end{aligned}}$$

2. SAR-Model Estimation

$$y = X\beta + u,\ u = \rho W u + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

$$\Rightarrow u = (I_n - \rho W)^{-1}\varepsilon = B_\rho^{-1}\varepsilon \Rightarrow \boxed{y \sim N\left(X\beta, \sigma^2 (B_\rho' B_\rho)^{-1}\right)}$$

$$\Rightarrow \boxed{\begin{aligned} \hat{\beta}_\rho &= (X'B_\rho' B_\rho X)^{-1} X'B_\rho' B_\rho y \\ \hat{\sigma}_\rho^2 &= \tfrac{1}{n}(y - X\hat{\beta}_\rho)'B_\rho' B_\rho (y - X\hat{\beta}_\rho) \end{aligned}}$$

Solve for ρ by substitution into the **Concentrated Likelihood**:

$$\boxed{\max_\rho L(\rho \mid y, X) = -\tfrac{n}{2}\left(\ln(2\pi) + 1\right) + \ln|B_\rho| - \tfrac{n}{2}\ln(\hat{\sigma}_\rho^2)}$$

2. SL-Model Estimation

$$y = \lambda W y + X\beta + \varepsilon, \ \ \varepsilon \sim N(0, \sigma^2 I_n)$$

$$\Rightarrow \ B_\lambda y = (I_n - \lambda W) y = X\beta + \varepsilon$$

$$\Rightarrow y = B_\lambda^{-1} X\beta + B_\lambda^{-1}\varepsilon \ \Rightarrow \ \boxed{y \sim N\left(B_\lambda^{-1} X\beta, \sigma^2 (B_\lambda' B_\lambda)^{-1}\right)}$$

$$\Rightarrow \quad \boxed{\begin{array}{l} \hat{\beta}_\lambda = (X'X)^{-1} X'B_\lambda y \\ \hat{\sigma}_\lambda^2 = \frac{1}{n}(B_\rho y - X\hat{\beta}_\lambda)'(B_\rho y - X\hat{\beta}_\lambda) \end{array}}$$

Solve for λ by substitution into **Concentrated Likelihood**:

$$\boxed{\max_\lambda L(\lambda \mid y, X) = -\tfrac{n}{2}\left(\ln(2\pi) + 1\right) + \ln|B_\lambda| - \tfrac{n}{2}\ln(\hat{\sigma}_\lambda^2)}$$