# TRANSFORMED REGRESSION MODELS AND PSEUDO R-SQUARE

## 1. R-Square for Regressions with Intercept

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0,\sigma^2 I)$$

$$\Rightarrow \quad y = [1\, X_v]\begin{bmatrix}\beta_0 \\ \beta_v\end{bmatrix} + \varepsilon \quad \Rightarrow \quad \boxed{y = \beta_0 1 + X_v\beta_v + \varepsilon}$$

So if $\beta_v = 0$ then

$$y = \beta_0 1 + \varepsilon \ \Rightarrow\ E(Y) = \beta_0 1 = \mu_y 1 \ \Rightarrow\ \boxed{\beta_0 = \mu_y}$$

Hence if $\beta_v \neq 0$ then conditional variance of $y$ given $X_v$ should be *smaller* than unconditional variance of $y$, so $R^2$ is a meaningful estimate of the amount of variance "explained" by $X_v\beta_v$.

## 1. R-Square for Regressions with No Intercept

$$y = X_v\beta_v + \varepsilon, \quad \varepsilon \sim N(0,\sigma^2 I)$$

So if $\beta_v = 0$ then

$$y = \varepsilon \ \Rightarrow \boxed{E(Y) = E(\varepsilon) = 0}$$

Thus unless $\mu_y = 0$, it is *not* true that independence of $y$ from $X_v$ is equivalent to $\beta_v = 0$. So it is not clear what $R^2$ is actually measuring in cases where $\mu_y \neq 0$.

## 2. Transformed Regression for the SAR Model

$$y = X\beta + u,\ u = \rho W u + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

$$\Rightarrow u = (I_n - \rho W)^{-1}\varepsilon = B_\rho^{-1}\varepsilon$$

$$\Rightarrow\ y = X\beta + B_\rho^{-1}\varepsilon \Rightarrow \boxed{B_\rho y = B_\rho X\beta + \varepsilon}$$

This appears to be a no-intercept model. But since

$$B_\rho y\ = B_\rho[1 X_v]\begin{bmatrix}\beta_0 \\ \beta_v\end{bmatrix} + \varepsilon\ = \beta_0 B_\rho 1 + B_\rho X_v \beta_v + \varepsilon$$

it follows when $\beta_v = 0$ we must have

$$B_\rho y = \beta_0 B_\rho 1 + \varepsilon \Rightarrow\ E(B_\rho y) = \beta_0 B_\rho 1 + E(\varepsilon)$$

$$\Rightarrow\ B_\rho(\mu_y 1) = \beta_0 B_\rho 1 + 0 \Rightarrow\ \mu_y B_\rho 1 = \beta_0 B_\rho 1 \Rightarrow \boxed{\mu_y = \beta_0}$$

So in fact, this is equivalent to a standard regression with an *intercept*, and $R^2$ is perfectly meaningful.

## 3. Transformed Regression for the Spatial Lag Model

$$y = \lambda W y + X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I_n)$$

$$\Rightarrow \boxed{B_\lambda y = X\beta + \varepsilon} \quad \text{with}\ \ B_\lambda = I_n - \lambda W\,.$$

So the situation is even simpler here since the transformed model is already a standard regression *with intercept*. Only $y$ has changed.