VARIANCE OF ML-ESTIMATES

Given a random sample $x = (x_1,..,x_n)$ from a normal distribution, $N(\mu,\sigma^2)$, consider the simplest case of *maximum-likelihood estimation* of μ with σ^2 known:

$$L(\mu \mid x,\sigma^2) = \ln\left(\prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}\right) = \sum_{i=1}^n \left[\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$

$$= -n\ln\left(\sigma\sqrt{2\pi}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2$$

So by solving the *first-order condition* for μ we obtain:

$$0 = \frac{dL}{d\mu} = \frac{1}{\sigma^2}\sum_{i=1}^n (x_i-\mu) = \frac{1}{\sigma^2}\sum_{i=1}^n x_i - \frac{n}{\sigma^2}\mu$$

$$\Rightarrow \sum_{i=1}^n x_i - n\mu = 0 \quad \Rightarrow \quad \boxed{\hat{\mu} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}_n}$$

Hence the *variance* of this estimate is given by $\mathrm{var}(\hat{\mu}) = \sigma^2 / n$. But by taking the second derivative of L with respect to μ, notice that

$$\frac{d^2L}{d\mu^2} = -\frac{n}{\sigma^2} \quad \Rightarrow \quad \boxed{\mathrm{var}(\hat{\mu}) = \left(-\frac{d^2L}{d\mu^2}\right)^{-1}}$$

More generally, it is *always* true that:

> *The variance of any maximum-likelihood estimator is given by the inverse of (the expected value) of minus the second derivative of L evaluated at its maximum.*