

Network Devolution and the Growth of Sensory Lacunae in Sensor Networks

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Abstract. Battery lifetimes in wireless sensor networks are dictated by usage patterns and the elected transmission power. As batteries fail there is an inevitable devolution of the network characterized by the growth of sensory lacunae or dead spots in the sensor field and eventually a breakdown in connectivity between the surviving nodes of the network. Sharp limit theorems characterizing the time at which these phenomena make their appearance are derived. These results provide explicit fundamental tradeoffs between transmission power, node density, and battery design and suggest how efficient choices may be made.

1 Introduction

Recent advances in sensor technology, low-power RF design and portable computing (cf. Chandrakasan, *et al* [1], Clare, *et al* [2], and Dong, *et al* [3]) have enabled the development of densely distributed, wireless micro-sensor networks. Applications of such sensor networks include their deployment in battlefields, disaster stricken areas, environment monitoring systems and space exploration. A main feature of such networks is the untethered nature of the sensors a consequence of which is that the battery power at each sensor becomes the primary resource constraint. These networks hence exhibit a particularly transient nature with ongoing node failures due to battery exhaustion causing a continual devolution of the network with a concomitant degradation of functionality. The projected “lifetime” of these networks hence plays an important role in their deployment.

Previous work on dense sensor networks has concentrated on the critical power required for asymptotic connectivity (cf. Gupta and Kumar [7] and Xue and Kumar [13]). These investigations assume a dense uniform distribution of the nodes to answer the fundamental question: What is the transmission radius (or number of neighbors) required at each sensor to maintain network connectivity? Similar questions are considered by Shakkottai, *et al* [11] when a certain fraction of nodes placed on a grid are functional at any given time. These results are concerned with establishing initial connectivity or coverage. There is much less known, however, about how such a network devolves as nodes degrade and fail over time, primarily due to limited battery power at the nodes.

Battery lifetimes in wireless sensor networks are dictated by usage patterns and the transmission power at each node. As batteries fail there is an inevitable devolution of the network characterized by the growth of sensory lacunae or dead spots in the sensor field and eventually a breakdown in connectivity between the surviving nodes of the network. We investigate fundamental attributes of these phenomena in a simple model of randomly deployed sensors where the battery lifetimes of the sensors are independent random variables with a common but arbitrary lifetime distribution parametrized by the power expenditure and the mean usage. In this somewhat sanitized but fundamental setting we derive sharp limit theorems characterizing the time at which these phenomena make their appearance. A characteristic feature is the appearance of *phase transitions* or *threshold functions*: emergent (disruptive!) phenomena appear abruptly in a sharply concentrated time span. Our results provide explicit tradeoffs between transmission power, node density, and the time to emergence of various phenomena and suggest how efficient choices may be made, while providing partial answers to the following fundamental questions:

- *When does the network fail?*
- *What is the distribution of failures in the network?*

The included examples illustrate the tradeoffs in various cases.

2 The Probabilistic Setting

We consider a sensor field comprised of a circle of unit radius in which n sensors are to be dispersed. Sensors are assumed to be dimensionless nodes equipped with both a sensing and a transmission capability. We will suppose that each sensor can sense events within a distance s from it and can communicate with any other sensor located within a distance r from it. These are not considered to be adaptable quantities, per se, but rather are assumed to be set either prior to deployment or immediately after deployment. As design parameters we will suppose that both the sensing radius $s = s_n$ and the transmission radius $r = r_n$ are suitably decaying functions of the number of sensors n .

We suppose that the sensors are deployed randomly in the unit circle. More precisely, the sensor locations X_1, \dots, X_n are assumed to be drawn independently from the uniform distribution in the unit circle. Each sensor is able to transmit information to and receive information from sensors within a distance r_n of it. The sensor locations induce a “metric” random graph $\mathcal{G}_{n,r}$ whose vertices $i \in \{1, \dots, n\}$ are indexed by node locations X_i . A pair of vertices (i, j) forms an edge of the graph if, and only if, $|X_i - X_j| < r_n$. If (i, j) is an edge of the graph we say that the vertices i and j are *adjacent* or *communicating*. The graph will be almost surely connected (for large n) if the transmission radius r_n is sufficiently large. A known result asserts indeed that $\sqrt{\frac{1}{n} \log n}$ is a threshold function for the radius at which network connectivity appears abruptly (cf. Gupta and Kumar [7] for the result in the current framework; for a modest tightening of the results and extensions see Venkatesh [12]; the classical result is due to Erdős and Renyi [4]). We will be concerned mainly with the situation when the graph is initially connected.

Each sensor is equipped with a battery which has a finite lifetime determined by the usage patterns of the sensor and the selected transmission radius. Let T_1, \dots, T_n denote the random lifetimes of the batteries of the n sensors. We will suppose that these lifetimes are independent random variables with a common distribution $\mathbf{P}\{T_i \leq t\} = F(t) = F_r(t)$. It will be convenient to also introduce the notation $1 - F(t) = G(t) = G_r(t)$ for the probability that the battery lifetime exceeds t . As indicated in the notation, the lifetime distribution is considered to be parametrized by the transmission radius supported by the battery; larger choices of r permit a smaller deployment of nodes while guaranteeing connectivity but also deplete individual batteries faster; smaller choices of r will deplete batteries slower but require a larger numbers of nodes to maintain connectivity with a concomitant increase in the likelihood of node extinction by a given time. While this is the main parametrization that we will consider in this paper, the notation, analysis and results extend smoothly to the situation where the lifetime distribution is parametrized additionally by a node-dependent, possibly random usage parameter α_i so that $F(t) = F_{r,\alpha_i}(t)$.

3 The Emergence of Isolated Nodes and the Growth of Sensory Lacunae

We say that a node i of the graph $\mathcal{G}_{n,r}$ is *isolated at time t* if there are no live nodes adjacent to it.¹ In other words, vertex i is isolated at time t if $T_j \leq t$ for every vertex j adjacent to i . An isolated node implies a lacuna or hole in the sensor coverage at that point in the network. Once a node is isolated, events detected by that sensor cannot be communicated to the rest of the network. What can be said about the distribution of isolated nodes and their evolution in time?

Let $L_i(t)$ be the event that node i is isolated at time t and write $N(t)$ for the number of isolated nodes at time t . Our main result asserts a sharp limit theorem for $N(t)$ in a suitable range.

Theorem 1. *For any fixed $\lambda > 0$ suppose $r = r_n$ and $t = t_n$ vary with n such that $r_n \sqrt{n} / \log n \rightarrow 0$ and $ne^{-nr_n^2 G_{r_n}(t_n)} \rightarrow \lambda$ as $n \rightarrow \infty$. Then, for every fixed non-negative integer m , $\mathbf{P}\{N(t_n) = m\} \rightarrow e^{-\lambda} \lambda^m / m!$ as $n \rightarrow \infty$.*

In other words, $N(t)$ is asymptotically Poisson in a suitable range of time. Before we turn to the implications of this theorem, we provide a skein of the main ideas in the proof. For technical details we refer the interested reader to the complete papers [9, 10].

¹ A variety of other definitions may also be entertained within this framework. For instance, we may require for isolation that the central node also be extinguished. Another variant can focus on live nodes that are isolated. The analysis extends gracefully to all these settings though we will not present the variations on the theme here. For details and extensions we refer the reader to [9, 10].

SKETCH OF PROOF: For the nonce suppress the subscript n for notational clarity. Consider any node i . Let A denote the area of the intersection of the unit circle with the circle of radius r centered at the node. If i is in the interior of the unit circle, i.e., $|X_i| \leq 1 - r$, then A is identically πr^2 . If i is in the boundary of the unit circle, i.e., $1 - r < |X_i| \leq 1$, then $A < \pi r^2$. The probability that i lies in the interior is $\pi(1 - r)^2/\pi = (1 - r)^2 = 1 + \mathcal{O}(r)$ whence the probability that i lies in the boundary is $1 - (1 - r)^2 = \mathcal{O}(r)$. If r is suitably small this suggests that the contribution of the boundary may be insignificant and, indeed, with $r = r_n$ as given in the theorem, with some diligence it can be shown that the boundary contribution is sub-dominant. Now condition on i being in the interior of the unit circle. The probability that any given node j is adjacent to i is then simply $\pi r^2/\pi = r^2$. As the nodes are placed independently, the probability that i has k nodes adjacent to it is given by the binomial $\binom{n-1}{k}(r^2)^k(1 - r^2)^{n-1-k}$ and the probability that they are all extinguished by time t is $F(t)^k$. It follows that the (conditional) probability that i is isolated at time t is given by

$$\sum_{k=0}^{n-1} \binom{n-1}{k} (r^2)^k (1 - r^2)^{n-1-k} F(t)^k = (r^2 F(t) + 1 - r^2)^{n-1} = (1 - r^2 G(t))^{n-1}$$

as could also have been directly deduced. Now observe that $nr^4 G(t)^2 \rightarrow 0$ for the range of $r = r_n$ and $t = t_n$ given in the theorem. It follows that the right-hand side is asymptotic to $e^{-nr^2 G(t)} \sim \lambda/n$. Remove the conditioning by taking expectations and as the boundary condition is sub-dominant obtain that the probability that node i is isolated at time t is asymptotic to $\mathbf{P}(L_i(t)) \sim \lambda/n$.

Now, for each fixed positive integer k , let S_k denote the sum of all conjunctions of the events L_i taken k at a time, that is to say,

$$S_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} \mathbf{P}(L_{i_1}(t) \cap L_{i_2}(t) \cap \dots \cap L_{i_k}(t)).$$

The relevance of the S_k to our problem is seen through the inclusion-exclusion formula,

$$\mathbf{P}\{N(t) = m\} = \sum_{k=0}^{n-m} (-1)^k \binom{m+k}{m} S_{m+k},$$

so that it will suffice to estimate the S_k .

Now fix k and observe that for specified nodes i_1, \dots, i_k , the probability that the circles of radius r at each of the locations X_{i_j} are mutually non-overlapping is $\geq (1 - 4r^2)(1 - 8r^2) \dots (1 - (k-1)4r^2) = 1 + \mathcal{O}(r^2)$. It is plausible now, and is indeed the case, that this is the dominant term though the details are delicate and complicated by pervasive dependencies and boundary effects; we eschew the rigorous considerations here. Conditioned on mutually non-overlapping circles at the nodes, the probability that each of i_1, \dots, i_k is isolated is then given by $(1 - kr^2 G(t))^{n-k}$ which in turn is asymptotic to λ^k/n^k for the stated conditions. As this is the dominant contribution to the required probability, it follows by symmetry that $S_k = \binom{n}{k} \mathbf{P}(L_1(t) \cap \dots \cap L_k(t)) \sim \frac{n^k}{k!} \cdot \frac{\lambda^k}{n^k} = \frac{\lambda^k}{k!}$. Brun's sieve (cf. Bonferroni's inequalities in Feller [5]) now shows that $N(t)$ tends to the Poisson distribution with parameter λ to finish off the proof. \square

The probability that there are no isolated nodes tends to $e^{-\lambda}$ whence the probability that there are one or more isolated nodes tends to $1 - e^{-\lambda}$. The choice of the positive λ will determine which of the two situations is likely to prevail: a small λ makes it unlikely that any nodes are isolated in the time frame of interest while a large λ makes it rather likely that nodes will become isolated. For a given parametric family of distributions $G(t) = G_{r_n}(t)$, the condition $ne^{-nr_n^2 G_{r_n}(t_n)} \rightarrow \lambda$ now requires that $G_{r_n}(t_n) = \frac{\log(n/\lambda)}{nr_n^2} + o\left(\frac{1}{nr_n^2}\right)$ and this in turn determines the critical value of time $t = t_n$ at which there is a phase transition and isolated nodes begin to appear. Some illustrative examples may help fix the idea.

EXAMPLES:

Exponential decay (1). We may model increased drain on a battery due to larger transmission radii via a power law in r_n . If we now consider a memoryless distribution for the battery lifetimes, we obtain a distribution of the form $\mathbf{P}\{T_i > t\} = G_{r_n}(t) = e^{-\alpha r_n^4 t}$ where α may represent a mean usage parameter. This suggests that the critical range for t is

$$t_n = \frac{1}{\alpha r_n^4} \left[\log(nr_n^2) - \log \log(n/\lambda) + o\left(\frac{1}{\log n}\right) \right]$$

For example, if the transmission radius is at the critical range required for connectivity, say, $r_n = \sqrt{\frac{\log n + c}{n}}$ for a suitably large positive c , then $t_n \sim n^2(c + \log \lambda)/\alpha \log^3 n$ is the critical range of time where isolated nodes begin to appear for the first time. If the transmission radius is super-critical, however, a similar analysis shows that isolated nodes crop up somewhat earlier in time. If, for instance, $r_n = \sqrt{\log^2(n)/n \log \log n}$ then the critical range of time is asymptotic to $n^2(\log \log n)^3/\alpha \log^4 n$. Thus, a super-critical transmission radius results in an earlier appearance of isolated nodes in accordance with naïve expectation though the threshold function quantifies the rôle of the battery power. The next example illustrates however that quite the reverse can occur.

Exponential decay (2). With a memoryless distribution for battery lifetimes, as above, suppose now that the lifetime dependence on power varies quadratically as would be the case in a “clean” environment. In this case the lifetime distribution is of the form $G_{r_n}(t) = e^{-\alpha r_n^2 t}$ so that the critical range for t is now

$$t_n = \frac{1}{\alpha r_n^2} \left[\log(nr_n^2) - \log \log(n/\lambda) + o\left(\frac{1}{\log n}\right) \right]$$

With initial critical connectivity $r_n = \sqrt{\frac{\log n + c}{n}}$, the threshold function for the appearance for isolated nodes is asymptotic to $t_n \sim n(c + \log \lambda)/\alpha \log^2 n$. For super-critical initial connectivity $r_n = \sqrt{\log^2(n)/n \log \log n}$ the critical range of time is now asymptotic to $n(\log \log n)^2/\alpha \log^2 n$. In this model, a *super-critical initial radius for connectivity ensures a longer period of time before the first appearance of isolated nodes and lacunae*. Thus, while a lower density of nodes is preferred when the drain is high, quite the reverse is true when the drain is low.

Regularly varying distributions. Karamata’s [8] theory of regularly varying functions yields a large range of useful heavy-tailed distributions. For example, let $\mu(r)$ be any positive monotonically decreasing function of r . If $G_{r_n}(t) \sim \mu(r_n)t^\rho$ as $t \rightarrow \infty$ for some $\rho < 0$ then G is regularly varying with exponent ρ . The critical range of t_n around which isolated nodes start to appear in the network is then $t_n \sim \left(\frac{1}{nr_n^2 \mu(r_n)} \log \frac{n}{\lambda}\right)^{1/\rho}$. The critical transmission radius for connectivity, $r_n \sim \sqrt{\log(n)/n}$, buys us little here unless $\mu(r)$ decreases to zero as $r \rightarrow 0$. If the transmission radius is supercritical, for example $r_n \sim \sqrt{\log^2(n)/n \log \log n}$, then extinction occurs much faster at $t_n \sim \left(\frac{\log \log n}{\mu(r_n) \log n}\right)^{1/\rho}$.

When the sensing radius s_n is less than the transmission radius r_n , the emergence of an isolated node also implies a sensory lacuna of radius $r_n - s_n$ centered at that node. Indeed, the elimination of all live nodes within a distance of r_n from the isolated node implies that there is no live node within a distance s_n of the circle of radius $r_n - s_n$ at the node. It follows that *under the conditions of the theorem, the number of sensory lacunae of radius $r_n - s_n$ centered at network nodes has an asymptotic Poisson distribution*. More generally, one may wish to track the evolution of the number of lacunae of radius, say $\ell = \ell_n$, centered at network nodes. If $\ell_n + s_n \leq r_n$ the result is contained within the previous theorem. If $\ell_n + s_n = r'_n > r_n$, simply replace each occurrence of r_n in the theorem by r'_n . Network connectivity is likely to have broken down well before the development of large lacunae, however. We turn to this issue next.

4 Network Devolution and the Breakdown of Connectivity

The degradation of the network due to sensor losses in time also manifests itself ultimately in a breakdown in connectivity. At the simplest level, such a breakdown occurs when a live node is isolated though connectivity may have broken down before such an occurrence. More formally, what can be said about the connectivity of the *network of survivors* at a given time t ? In particular, how long will the network of survivors remain connected in the face of continuing losses?

It is fruitful to think of the setting as follows. Initially, one starts with a connected metric random graph on n vertices. (Of course, we are assuming tacitly that the communication radius r_n exceeds the critical threshold $\sqrt{\log(n)/n}$ so that we have asymptotic high confidence guarantees that the network is connected.) At time t a random fraction of the nodes has expired leaving a collection of $S(t)$ survivors with the induced subgraph on those vertices. The situation may be arrived at by an equivalent probabilistic game in which random deletions of vertices (and associated edges) are performed on the original graph with each vertex removed independently from the graph with probability $F(t)$. The number of survivors $S(t)$ is hence binomially distributed with parameters n and $1 - F(t) = G(t)$. The de Moivre-Laplace

theorem tells us that $S(t)$ is concentrated around its mean value $nG(t)$. Indeed, for any $0 < \epsilon < 1/6$, we can find a positive constant c for which

$$\mathbf{P}\{|S(t) - nG(t)| > (nG(t))^{1/2+\epsilon}\} = \mathcal{O}(e^{-c[nG(t)]^{2\epsilon}}).$$

It follows that $S(t) = nG(t) + \mathcal{O}[(nG(t))^{1/2+\epsilon}]$ with asymptotic probability close to 1.

Condition on $S(t) = s$ survivors where $s = nG(t) + \zeta$ and $\zeta = \mathcal{O}[(nG(t))^{1/2+\epsilon}]$. As deletions are performed independently, the locations of the s survivors are independent of each other and uniformly distributed in the unit circle. It follows that $\sqrt{\log(s)/s}$ is a threshold function for the transmission radius to ensure survivor connectivity. More precisely, let $\omega(s)$ be any slowly growing function of s . Bear in mind that the transmission radius is still the originally set radius r_n and that $s \sim nG(t)$. We hence obtain that the survivors are disconnected with asymptotic probability approaching 1 if $r_n \leq \sqrt{(\log(s) - \omega(s))/s}$ while the survivors are connected with asymptotic probability approaching 1 if $r_n \geq \sqrt{(\log(s) + \omega(s))/s}$. Write $\nu = nG(t)$ and take expectation with respect to s to get rid of the conditioning. The concentration of the binomial allows us to focus on $s \sim nG(t)$. It follows that a threshold function for the radius is $\sqrt{\log(\nu)/\nu}$ to ensure survivor connectivity. Inverting the system we obtain the following

Theorem 2. *For any fixed real constant c suppose*

$$G_{r_n}(t_n) = \frac{\log r_n^{-2}}{nr_n^2} \left[1 + \frac{\log \log r_n^{-2}}{\log r_n^{-2}} + \frac{c}{\log r_n^{-2}} + o\left(\frac{1}{\log^2 r_n^{-2}}\right) \right]$$

then the probability that the surviving nodes in the graph at time $t = t_n$ are still connected tends asymptotically to $e^{-e^{-c}}$.

We reserve the technical details to [10, 12].

For concreteness, if the failure distribution is the memoryless distribution seen earlier with $G_{r_n}(t) = e^{-\alpha r_n^4 t}$ and r_n is initially set at just above the critical connectivity threshold, then survivor connectivity breaks sharply around $t_n = n^2 \log \log(n) / \alpha \log^3 n$. More precisely, for any $\epsilon > 0$, the probability that the survivors are connected tends to 1 if $t_n \leq (1 - \epsilon)n^2 \log \log(n) / \alpha \log^3 n$ while the probability that the survivors are disconnected tends to 1 if $t_n \geq (1 + \epsilon)n^2 \log \log(n) / \alpha \log^3 n$. Connectivity breakdown among survivors occurs somewhat later than node isolations as isolated nodes also tend to break down and will not be among the survivors.

5 Discussion

A variety of sharp asymptotic results along these lines can be shown to track both the development of sensory lacunae in the network as well as the devolution of the network architecture. The sample results of this form presented here have been presented in a clean asymptotic form because of the pleasing simplicity though error bounds can be obtained as well with a modicum of effort. The results suggest how principled tradeoffs may be effected between transmission power, node density, and battery design.

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