Problem 1 [24 points]

An ordinary deck of 52 cards is one in which each card has exactly one of 13 possible denominations: Ace, 2, ..., 10, jack, queen, and king. There are four cards of each denomination, and each of four cards with the same denomination can be of only one of four suits: Clubs, diamonds, hearts, and spades. Furthermore, no two cards with the same denomination has the same suit. So, there are 13 cards of the same suit and there is only one ace of clubs, only one ace of diamonds, only one ace of hearts, only one ace of spades, only one 2 of clubs, only one 2 of diamonds, only one 2 of hearts, only one 2 of spades, and so on so forth. Now, consider the three following questions:

(a) In how many ways can we pick five cards from an ordinary 52-card deck?
(b) In how many ways can we pick five cards from an ordinary 52-card deck such that all cards are of the same suit?
(c) In how many ways can we pick five cards from an ordinary 52-card deck such that there are three cards of one denomination and two cards of a second denomination?

For all questions above, you can ignore the order in which the cards are picked from the deck, i.e., it does not matter the order in which the cards are picked from the deck. Also, provide a justification for your answers.

Guidelines:
Each item is worth eight points. There is no partial credit for this question, as the answers should be very straightforward. If you misunderstood the statement, then you should read it again to convince yourself you can understand the statement now. If you did understand the statement, but you did not address the question correctly, you should review the material for this question.
Problem 2 [16 points]

Find the least positive integer $n$ that satisfies the linear congruence $3^{56} \equiv n \pmod{7}$. Justify your answer.

Guidelines:
This question is worth 16 points. Partial credit was given for answers that contained minor mistakes only. The maximum amount of points given as partial credit was 12 points. To reach twelve points, you must get the RIGHT answer 2, but your solution has minor problems to justify your answer correctly. If you got the right answer, but your justification was completely wrong, you did not get any points as partial credit. If you got the wrong answer, the maximum amount of points you can have is 10 points. To reach 10 points in this situation, you have to address part of the problem correctly.
Problem 3 [10 points]
Use induction to show that $6 \cdot 7^n - 2 \cdot 3^n$ is divisible by 4, for any integer $n$, with $n \geq 0$.

Guidelines:
This is worth 10 points. Partial credit was only given for solution that contained the right “wording” of a proof by induction. If this is the case, extra points were deducted due to the following: (1) A mistake in the inductive step caused a deduction of 2 points and the lack of development of the inductive step caused a deduction of 3 points; (2) A mistake in the base case (e.g., base case $n = 1$) caused a deduction of 1 point; (3) A mistake in the induction hypothesis caused a single deduction of 9 points. Note that cases (1) and (2) can be combined, but case (3) is a single deduction.
Problem 4 [25 points]

How many distinct binary numbers with 30 digits are there such that each number has exactly five 0’s and no two 0’s are consecutive? Justify your answer. Recall that any digit in a binary number is either 0 or 1.

Guidelines:
This question is worth 25 points. Partial credit was given. If your solution provided a justification and this justification was totally correct, you can get up to 13 points. This total amount of partial credit was given only if your justification was totally correct, but your solution did not contain the right calculations based on the justification given. To be more precise, your justification must demonstrate that you had the right understanding of the statement of the problem. If your justification is not totally correct, your solution can get up to 8 points. Since most of the students understood the statement of the problem, let us focus on the extra deductions from the 13 points. Extra deductions were carried out in a case by case basis according to the calculations. In most of the cases, the calculations did not reflect the correct counting, and points were deducted according how close the solution was to the right answer.
Problem 5 [25 points]

Let \( X = \{x_1, x_2, x_3, x_4\} \) be any set with four distinct elements. Then, we define a 3-cut of \( X \) as an ordered family of three subsets \( X_1, X_2, \) and \( X_3 \) of \( X \) such that \( X_1 \cup X_2 \cup X_3 = X \) and \( X_1 \cap X_2 = \emptyset, X_1 \cap X_3 = \emptyset, \) and \( X_2 \cap X_3 = \emptyset \). For example,

\[- X_1 = \emptyset, X_2 = \{x_1, x_4\}, \text{ and } X_3 = \{x_2, x_3\},\]
\[- X_1 = \emptyset, X_2 = \emptyset, \text{ and } X_3 = \{x_1, x_2, x_3, x_4\},\]
\[- X_1 = \{x_1, x_3\}, X_2 = \{x_2\}, \text{ and } X_3 = \{x_4\}, \text{ and}\]
\[- X_1 = \{x_1, x_3\}, X_2 = \{x_4\}, \text{ and } X_3 = \{x_2\},\]

are four distinct 3-cuts of the set \( X \). First, note that one or two of the sets in a 3-cut can be empty. Second, note that the last two 3-cuts are distinct just because the sets \( X_2 \) (resp. \( X_3 \)) of each set 3-cut are not the same, even though the two 3-cuts have the same subsets of \( X \). In other words, the order of the subsets of a 3-cut matters.

So, how many distinct 3-cuts of \( X \) are there? Justify your answer. Do not try to write down all possible 3-cuts of \( X \). There are many of them! Furthermore, this is not an acceptable justification for your answer.

Guidelines:

The same as the previous question. This question is worth 25 points. Partial credit was given. If your solution provided a justification and this justification was totally correct, you can get up to 13 points. This total amount of partial credit was given only if your justification was totally correct, but your solution did not contain the right calculations based on the justification given. To be more precise, your justification must demonstrate that you had the right understanding of the statement of the problem. If your justification is not totally correct, your solution can get up to 8 points. Since most of the students understood the statement of the problem, let us focus on the extra deductions from the 13 points. Extra deductions were carried out in a case by case basis according to the calculations. In most of the cases, the calculations did not reflect the correct counting, and points were deducted according how close the solution was to the right answer.