Name: ____________________________________________________________

Student ID (8 digits): ______________________________________________

Email: ____________________________________________________________

Signature: __________________________________________________________

PLEASE, READ THE FOLLOWING INSTRUCTIONS:

• This is a closed-book, closed-device exam: You may not make use of any lecture notes, books or electronic devices (e.g., calculators).

• You have 80 minutes to answer all of the questions. The entire exam is worth 100 points. The point value of each question is given.

• Partial credit will be given. Full credit will be given only in the case where the correct answer has been properly justified with complete explanations. Do not spend disproportionate time on any one question.

• Write your answers in the spaces provided: You must turn in this printed form. The back side of each page may be used as scratch pad.

• All writings must be neat, well-organized, and include sufficient explanations in the delineation of the solutions. Messy, poorly organized, or illegible material will be returned ungraded.

• Questions during the exam should be about the wording of the exam only. If you have a question, raise your hand and we will come to you.

• Please turn your exam in at the end of the class.

• Good luck!

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Score</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TOTAL: ____________________________________________________________
Problem 1 [20 points]

The power set $\mathcal{P}(A)$ of any finite set $A$ with $n$ elements has cardinality $2^n$, where $n$ is a nonnegative integer.

Now, suppose that you are given a set $X$ with exactly three elements. What is the cardinality of the set $\mathcal{P}(\mathcal{P}(X))$, where $\mathcal{P}(\mathcal{P}(X))$ is the power set of the power set of $X$? Justify your answer.
Problem 2 [20 points]

Let $A$ and $B$ be two finite sets such that $A$ has cardinality $m$ and $B$ has cardinality $n$, where both $m$ and $n$ are nonnegative integers. If there exists a function $f : A \rightarrow B$ from $A$ to $B$ such that $f$ is surjective, what can you say about $m$ and $n$? What if $f$ is injective but not surjective? What if $f$ is a bijection? Justify your answer.
Problem 3 [20 points]

We are now asked to show that the following statement is true: For any integer \( n \), we have that \( n \) is odd if and only if \( 5n + 4 \) is odd. To prove the validity of this statement, you must show that the implication is valid in both directions, i.e., you must show that

(a) If \( n \) is odd then \( 5n + 4 \) is odd.
(b) If \( 5n + 4 \) is odd then \( n \) is odd.

Hint: For item (a), it is easier to use direct proof. For item (b), you can use the fact that (b) is equivalent to “if \( n \) is even then \( 5n + 4 \) must be even”.

Problem 4 [20 points]

You are now asked to prove or disprove four given statements. If you think a statement is true, you must provide a proof for asserting its validity. Otherwise, you must disprove the statement by providing a counterexample.

1. Let $a$ and $b$ be any two irrational numbers. Then, the product $a \cdot b$ is always an irrational number. Note that $a$ and $b$ do not have to be distinct numbers.

2. Let $g: \mathbb{Z} \rightarrow \mathbb{R}$ be a function from the set of integers to the set of real numbers such that $g(x) = \lceil \frac{x}{2} \rceil$, for all $x \in \mathbb{Z}$. Then, the function $g$ is a surjection.

3. Let $f: \mathbb{Z} \rightarrow \mathbb{R}$ be a function from the set of integers to the set of real numbers such that $f(x) = \lfloor \frac{x}{2} \rfloor$, for all $x \in \mathbb{Z}$. Then, the function $f$ is an injection.

4. Let $a$, $b$, and $c$ be three integers such that $c \mid (a \cdot b)$. Then, we have that $c$ must divide either $a$ or $b$ or both of them.
Problem 5 [20 points]

Prove that $7n + 1$ and $15n + 2$ are relatively prime, for any integer $n$.

*Hint:* Use direct proof, and start your proof by defining $d = \gcd(7n + 1, 15n + 2)$. Your goal is to show that $d = 1$. You are allowed to use the following fact: “For any three integers $a$, $b$, and $c$, with $a \neq 0$, if $a \mid b$ and $a \mid (b + c)$ then $a \mid c$”. By using this fact (perhaps, more than once) and the fact that $15n + 2 = 2 \cdot (7n + 1) + n$, work out your solution. You do not need any theorem seen in class to complete the proof. However, you are allowed to use any tool you know to be correct. If you do use, do not forget to mention it explicitly, so that I will be able to understand your claims.