Basic Principles of Counting

Many problems can be solved simply by counting the number of different ways that a certain event can occur.

Here, we are concerned with doing the counting in an efficient way.

We start our journey by studying two simple and extremely useful principles of counting: the addition principle and the multiplication principle.

Next, we study more sophisticated ways of counting: permutations and combinations.

If time permits, we will also study a counting method based on deriving and solving recursion equations.
Basic Principles of Counting

Consider the following problems:

- A women’s group consists of twelve women, each of whom has three children. If one woman and one of her children are to be chosen, how many different choices are possible?

- A school planning committee consists of 3 freshmen, 4 sophomores, 4 juniors, and 5 seniors. If a subcommittee is to be chosen that consists of one person from each class, how many choices are possible?

Both problems can be solved by using a basic principle of counting. Can you guess it?
Basic Principles of Counting

**Multiplication Principle:** Suppose that an experiment consists of two steps. If the first step can result in any of $m$ possible outcomes and if, for each outcome of the first step, there are $n$ possible outcomes of the second step, then there are a total of $m \cdot n$ possible outcomes of the experiment.

We can think of the first problem we saw before as an experiment with two steps: (1) Choose a woman from a group of 12 women and (2) choose one children out of the three children of the woman chosen in step (1).

So, if a women’s group consists of twelve women, each of whom has three children, then there are $12 \cdot 3 = 36$ different ways of choosing one woman from the group and one of her children.
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**Generalized Multiplication Principle:** Suppose that an experiment consists of \( r \) steps. If the first step can result in any of \( n_1 \) possible outcomes, and if for each outcome of the first step, there are \( n_2 \) possible outcomes of the second step, and if for each possible outcomes of the first two steps there are \( n_3 \) possible outcomes of the third step, and in general, no matter how the outcomes of the preceding steps, there are \( n_k \) possible outcomes of the \( k \)-th step, with \( 1 \leq k \leq r \), then there are a total of \( \prod_{k=1}^{r} n_k = n_1 \cdot n_2 \cdot \ldots \cdot n_r \) possible outcomes of the experiment.

By using the above principle, what is the answer for the second problem we saw before?
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More counting problems:

• How many different license plates of three letters from the English alphabet, A to Z, followed by four digits from 0 to 9 can be made? What if no repetition of letters or numbers is allowed?

• Let $A$ be a set with 8 elements and $B$ be a set with 6 elements. Find the number of functions from $A$ into $B$.

• How many strings of 0’s and 1’s of length 10 are there?

• How many strings of 0’s and 1’s of length 8 that end with 10 are there?
Consider the following problems:

- How many different *ordered* arrangement of the letters $a$, $b$, and $c$ are possible?

- An advanced mathematics class consists of six women and four men. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score. So,

  - How many different rankings are possible?

  - If the women are ranked just among themselves and the men among themselves, how many different rankings are possible?
**Permutations**

Let $S$ be a set with $n$ distinct elements such that $n > 0$. Let $r$ be any integer such that $0 < r \leq n$. Then, an $r$-permutation of $S$ is an ordered arrangement of $r$ elements of $S$.

We denote the number of $r$-permutations of $S$ by $P(n, r)$.

By using the generalized multiplication principle, we can show that

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1) = \prod_{i=0}^{r-1} (n - i).$$

Note that $P(n, n) = n!$.

What is the answer for the first problem in the previous slide? If you think of $S$ as $\{a, b, c\}$ then the problem asked us for the number of 3-permutations of $S$, $P(3, 3)$. 

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Permutations

What about the second problem?

If we consider $S$ the set of men and women in the class, then we can write $S$ as $\{w_1, w_2, w_3, w_4, w_5, w_6, m_1, m_2, m_3, m_4\}$, where $w_i$ and $m_j$ represent the $i$-th woman and the $j$-th man, for $1 \leq i \leq 6$ and $1 \leq j \leq 4$.

So, what is the number of all possible rankings in class?

What is a ranking? It is any ordered arrangement of ALL elements in $S$.

For instance, $w_6, m_2, w_3, w_4, m_3, m_1, w_1, w_2, m_4, w_5$. So, we were actually asked for the number of 10-permutations of $S$: $P(10, 10) = 10! = 3,628,800$.
What about the second part of the problem?

Let $S_1$ be the set of women in the class, and let $S_2$ be the set of men in the class. Then, we can write $S_1$ as $\{w_1, w_2, w_3, w_4, w_5, w_5\}$, and $S_2$ as $\{m_1, m_2, m_3, m_4\}$.

So, the number of all possible rankings involving women only is the number of 6-permutations of $S_1$: $P(6, 6) = 6!$, and the number of all possible rankings involving men only is the number of 4-permutations of $S_2$: $P(4, 4) = 4!$.

Now, how many different ways do we have to rank women among themselves and men among themselves?

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Permutations

We can think of this as an “experiment” with two steps.

First, we rank the women among themselves. How many possible rankings are there? \( P(6, 6) = 6! \)

Then, we rank the men among themselves. How many possible rankings are there? \( P(4, 4) = 4! \)

By the principle of multiplication, there are \( P(6, 6) \cdot P(4, 4) = 6! \cdot 4! = 17,280 \).
More problems on permutations:

- How many different ways can the letters of the word MONDAY be arranged?

- How many of them start with M and do not end with Y?

- How many even numbers greater than 500, with three distinct digits, can be formed using the digits 3, 4, 5, 6, and 7?

- Find the number of seating arrangements in a row of eight students so that two particular students will not sit side by side?
Combinations

Consider the following problems:

- A committee of four is to be chosen from a group of 16 people. How many different committees are possible?

- From a group of six women and five men, how many different committees consisting of three women and two men can be formed? How many can be formed if two of the women do not want to serve together?

What do you see in the above problems that make them different from the ones we have seen so far?
Combinations

Let $S$ be a set with with $n > 0$ elements. Let $r$ be an integer such that $0 \leq r \leq n$.

The number of subsets that contain $r$ elements of $S$ is \[ \frac{n!}{(n - r)!r!} \].

So, if we think of the group of 16 people in the first problem of the previous slide as the set $S = \{p_1, p_2, \ldots, p_{16}\}$, then the number of different committees consisting of four people is exactly equal to the number of subsets of four elements of $S$:

\[ \frac{n!}{(n - r)!r!} = \frac{16!}{(16 - 4)!4!} = \frac{16 \cdot 15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = 1820. \]

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Combinations

What about the second problem? Again, we can use the principle of multiplication.

First, we choose 3 women out of 6:

\[
\frac{6!}{(6-3)!3!} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20.
\]

Next, we choose 2 men out of 5:

\[
\frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2 \cdot 1} = 10.
\]

So, the number of distinct ways of choosing 3 women out of 6 AND 2 men out of 5 is 20 \cdot 10 = 200.
What if two women do not want to serve together?

Let us denote the women by \( w_1, w_2, w_3, w_4, w_5 \) and \( w_6 \).

Without loss of generality, assume that \( w_1 \) and \( w_2 \) are the ones that do not want to serve together.

We can think of the possible committees as committees that do not contain either \( w_1 \) or \( w_2 \) or contain only one of them.

So, the number we want is the sum of the number of these possible committees.
Combinations

The number of committees that do not contain either $w_1$ or $w_2$ is the number of committees of three women from the group consisting of $w_3$, $w_4$, $w_5$ and $w_6$. That is,

\[
\frac{4!}{(4-3)!3!} = \frac{4!}{1!3!} = 4.
\]

The number of committees that contain only one of $w_1$ and $w_2$ is equal to the number of ways of choosing one of $w_1$ and $w_2$ times the number of ways of choosing two of $w_3$, $w_4$, $w_5$ and $w_6$:

\[
\frac{2!}{(2-1)!1!} \cdot \frac{4!}{(4-2)!2!} = \frac{2!}{1!1!} \cdot \frac{4!}{2!2!} = 2 \cdot 6 = 12.
\]

By the multiplication principle, the number of committees that we can form is $(4 + 12) \cdot 10 = 160$. 

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Combinations

We say that an $r$-combination of the elements of $S$ or a combination of the elements of $S$ taken $r$ at a time is a subset $A$ of $S$ such that $A$ contains $r$ elements of $S$.

As we saw before, the number of $r$-combinations of $S$ is exactly the number of subsets of $r$ elements of $S$, i.e.,

\[
\frac{n!}{(n - r)!r!}.
\]

We denote the above number by \( \binom{n}{r} \) or \( C(n, r) \).

Why is that the number of $r$-combinations of $S$ is \( \frac{n!}{(n - r)!r!} \)?

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Combinations

More problems on combinations:

- In how many ways can a soccer team of 11 players be selected from a group of 20 players?

- If \( C(16, r) = C(16, r + 2) \) then find \( r \).

- A committee of six is to be made from four students and eight teachers. In how many ways can this be done:
  - If the committee contains exactly three students?
  - If the committee contains at least three students?