PLEASE, READ THE FOLLOWING INSTRUCTIONS:

• This is a closed-book, closed-device exam: You may not make use of any lecture notes, books or electronic devices (e.g., calculators).

• You have 80 minutes to answer all of the questions. The entire exam is worth 100 points. The point value of each question is given.

• Partial credit will be given. Full credit will be given only in the case where the correct answer has been properly justified with complete explanations. Do not spend disproportionate time on any one question.

• Write your answers in the spaces provided: You must turn in this printed form. The back side of each page may be used as scratch pad.

• All writings must be neat, well-organized, and include sufficient explanations in the delineation of the solutions. Messy, poorly organized, or illegible material will be returned ungraded.

• Questions during the exam should be about the wording of the exam only. If you have a question, raise your hand and we will come to you.

• Please turn your exam in at the end of the class.

• Good luck!

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TOTAL: ____________________________
Problem 1 [20 points]

Assume that the finite state diagram given below is the one of a nondeterministic finite automaton (NFA) \( N = (Q, \Sigma, \delta, q_0, F) \) whose the set \( \Sigma \) of input symbols is \{a, b, c\}. Then, specify the parts \( Q, \delta, q_0, \) and \( F \) of the NFA \( N \).

Solution:
\( Q = \{p, q, r, t\}, q_0 = p, F = \{t\}, \) and \( \delta : Q \times \Sigma \rightarrow \mathcal{P}(Q) \) is defined as

\[
\begin{array}{c|ccc}
\delta & a & b & c \\
p & \{q\} & \emptyset & \emptyset \\
q & \{r\} & \emptyset & \emptyset \\
r & \{t\} & \{p\} & \emptyset \\
t & \emptyset & \{t\} & \emptyset \\
\end{array}
\]
Problem 2 [20 points]

Let $N = (Q, \Sigma, \delta, q_0, F)$ be a nondeterministic finite automaton (NFA) such that

$$Q = \{s_1, s_2, s_3, s_4\}, \quad \Sigma = \{a, b\}, \quad q_0 = s_1, \quad F = \{s_1\},$$

and $\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q)$ is given by

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<tr>
<td>$s_1$</td>
<td>${s_1, s_2, s_3, s_4}$</td>
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<td>$s_2$</td>
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<td>${s_1}$</td>
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<td>$s_3$</td>
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<td>$s_4$</td>
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Then,

(a) Draw a finite state diagram for $N$.
(b) Give a string beginning with $a$ that is not accepted by $N$.

Solution:

(a) A finite state diagram for the NFA $N$:

(b) There are several correct answers for this problem, as there are several strings that are not accepted by $N$. For instance, any string of the form $a^m b^n$, where $m \geq 1$ and $n \geq 4$ is a correct answer.
Problem 3 [20 points]

Match each NFA with an equivalent regular expression, i.e., match each NFA with a regular expression that describes the language accepted by the NFA.

(i) $\epsilon + 0(01^*1 + 00)^*01^*$
(ii) $\epsilon + 0(10^*1 + 10)^*10^*$
(iii) $\epsilon + 0(10^*1 + 00)^*0$
(iv) $\epsilon + 0(01^*1 + 00)^*0$
(v) $\epsilon + 0(10^*1 + 10)^*1$

Solution:

(a) matches (i)
(b) matches (iii)
(c) matches (v)
(d) matches (iv)
(e) matches (ii)
Problem 4 [20 points]

Draw a finite state diagram of a nondeterministic finite automaton (NFA) \( N \), with or without empty transitions, such that \( N \) has exactly four states, and \( L(N) = L(R) \), where \( R \) is the following regular expression:

\[(01 + 011 + 0111)^*\]

Solution:
Problem 5 [20 points]

We saw in class that the Pumping Lemma for regular languages can be written as

“If $L$ is a regular language over an alphabet $\Sigma$ then the property $P$ is true”,

where $P$ is the statement

“There is a nonnegative integer $p$ such that for all strings $w \in L$ with $|w| \geq p$, there are strings $x, y, z$ in $\Sigma^*$ such that $w = xyz$, $|y| > 0$, $|xy| \leq p$, and $xyz \in L$ for all nonnegative integers $i$”.

The language $L_1$ such that

$L_1 = \{a^ib^jc^k \mid \text{for some nonnegative integers } i, j, k \text{ such that if } i = 1 \text{ then } j = k\}$

is not regular. However, $L_1$ satisfies the above property $P$ of the Pumping Lemma for regular languages.

Now, consider the following problems:

(a) Show that $L_1$ satisfies the above property $P$ of the Pumping Lemma for regular languages. 

Hint: Note that you can choose an appropriate numerical value for $p$ in order to prove this fact. For instance, any $p \geq 2$ will do. Note also that every string $w \in L_1$ such that $|w| \geq p$ can only be of one of the following three forms: (1) $w = b^mc^n$, (2) $w = ab^me^n$, and (3) $w = a^ib^mc^n$, with $l \geq 2$, where $l, m,$, and $n$ are nonnegative integers. For each of these forms, you must find appropriate strings $x, y, \text{ and } z$ satisfying property $P$. Recall that the strings $x, y, \text{ and } z$ do not have to be the same for every $w \in L_1$ with $|w| \geq p$. Actually, strings of the form (2) may require a choice of $x, y, \text{ and } z$ that is different from the ones for strings of the form (2) and (3).

(b) Based on the fact that “$L_1$ is not regular and $L_1$ satisfies the above property $P$ of the Pumping Lemma for regular languages”, explain why the contrapositive of the Pumping Lemma cannot be used to show that $L_1$ is not regular.

Solution:

(a) To show that $L_1$ satisfies the property $P$ of the Pumping Lemma (as described in the statement of the problem), you first must pick an integer $p \geq 0$. Let $p = 2$. Now, we must must show that, for every string $w \in L_1$ such that $|w| \geq p$, we can find strings $x, y, \text{ and } z$ such that $w = xyz$, $|y| > 0$, $|xy| \leq p$, and $xyz \in L_1$ for every nonnegative integer $i \geq 0$. By examining $L_1$, we note that every string of $L_1$ is of the form $a^ib^mc^n$, for some nonnegative integers $l, m,$, and $n$, subject to the restriction that $m = n$ if $l = 1$. For every string $w \in L_1$ such that $|w| \geq 1$, we distinguish the following three cases for $w$: (1) $w = b^mc^n$, (2) $w = ab^me^n$, and (3) $w = a^ib^mc^n$, with $l \geq 2$. For strings of the form (1), we choose $x = \epsilon$, $y = b$, and $z = b^{m-1}c^n$, if $m \geq 1$, and we choose $x = \epsilon$, $y = c$, and $z = c^{n-1}$, if $m = 0$. For strings of the form (2), we choose $x = \epsilon$, $y = a$, and $z = b^mc^n$. For strings of the form (3), we choose $x = \epsilon$, $y = aa$, and $z = a^{l-2}b^mc^n$ if $l = 3$, and we choose we choose $x = a$, $y = a$, and $z = ab^mc^n$ if $l = 3$. In these three cases, we have $|y| > 0$, and $|xy| \leq 2$. It re mains to show that $xy^iz \in L_1$ for every integer $i \geq 0$ in the three cases. But, this is immediately verified, as $xy^iz = b^i(c^{n-1})c^n$ or $xy^iz = c^{i(n-1)}$ in case (1); $xy^iz = a^ib^mc^n$ in case (2); and $xy^iz = a^{2i+l-2}b^mc^n$ or $xy^iz = a^{i+2}b^mc^n$ in case (3). So, $L_1$ satisfies the above property $P$ of the Pumping Lemma for regular languages.
(b) Since $P$ holds for $L_1$, we have that the negation of $P$ will not hold for $L_1$. Hence, the hypothesis of the contrapositive of the Pumping Lemma does not hold for $L_1$, and consequently we cannot use it to show that $L_1$ is not regular. This tells us that the Pumping Lemma is not “invincible”, and for some languages like $L_1$, we must resort to other tools to prove that the language is not regular.