Name: ____________________________________________
Student ID (8 digits): ________________________________
Email: ____________________________________________
Signature: __________________________________________

PLEASE, READ THE FOLLOWING INSTRUCTIONS:

• Fill out this form with your name, student ID, email, and signature and return it as the cover page of your homework.

• Turn in your homework at the beginning of your class on the due date described at the top of this page.

• Late assignments will be penalized 25% and will not be accepted after 1:30PM of the day following the due date.

• Late assignments must be turned in to Janean Williams in room 308, 3rd floor, Levine Building.

• All writings must be neat, well-organized, and include sufficient explanations in the delineation of the solutions.

• Full credit will be given only in the case where the correct answer has been properly justified with complete explanations.

• Good luck!

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TOTAL: ____________________________________________
Problem 1 [25 points]

\[ L = \{ x \in \{a, b\}^* \mid x = ww, \text{ for some } w \in \{a, b\}^* \} = \{\epsilon, aa, bb, aaaa, abab, baba, bbbb, \ldots\} \]

is not a regular language.

Solution:

We show that \( L \) is not regular by using the contrapositive form of the Pumping Lemma. Given any nonnegative integer \( p \), consider any integer \( n \) such that \( n > p \), and let \( w \) be the string \( 0^n1^n0^n1^n \). Note that \( w \in L \) and \( |w| \geq p \). For all strings \( x, y \) and \( z \) from \( \{a, b\}^* \) such that \( xyz = w \), \( |y| > 0 \), and \( |xy| \leq p \), we want to show that \( xy^iz \not\in L \), for some \( i \geq 0 \). We now claim that \( i = 2n + 1 \) does the job. Since \( |xy| \leq p \) and \( p < n \), we must have \( x = 0^k \), \( y = 0^l \), and \( z = 0^n1^n0^n1^n \), with \( k + l + m \leq p \). Since \( |y| > 0 \) and \( l = |y| \), we must have \( 0 < l \leq p \). So, \( xy^{2n+1}z = 0^k0^{2nl+l}0^n1^n0^n1^n = 0^{2nl+n}1^n0^n1^n \). Since \( n > p \) and \( l > 0 \), we must have that \( 2nl + n = (2 + l)n > 3n \), which means that \( 0^{2nl+n}1^n0^n1^n \not\in L \). To see why, we consider two cases: (1) \( 2nl + n + 3n \) is odd, and (2) \( 2nl + n + 3n \) is even. If (1) holds then \( 0^{2nl+n}1^n0^n1^n \not\in L \), as every string in \( L \) must have even length. If (2) holds then we can break \( 0^{2nl+n}1^n0^n1^n \) into two strings with the same length, say \( u \) and \( v \). Since \( 2nl + n \geq 3n \), the string \( u \) must be of the form \( 0^{(2nl+4n)/2} \), and therefore \( u \) and \( v \) cannot be equal. Hence, \( 0^{2nl+n}1^n0^n1^n \not\in L \), and \( L \) is not a regular language.
Problem 2 [25 points]

Show that the language $L$ such that

$$L = \{ w \in \{a\}^* \mid |w| = j^2, \text{ for some integer } j \} = \{ \epsilon, a, aaaa, aaaaaaa, \ldots \}$$

is not a regular language.

Solution:

We show that $L$ is not regular by using the contrapositive form of the Pumping Lemma. Given any nonnegative integer $p$, let $n$ be any integer greater than $p$, and let $w$ be the string $a^n$. Note that $w \in L$ and $|w| \geq p$. For all strings $x$, $y$ and $z$ from $\{a\}^*$ such that $xyz = w$, $|y| > 0$, and $|xy| \leq p$, we want to show that $xy^iz \notin L$, for some $i \geq 0$. We now claim that $i = 2$ does the job. Since $|xy| \leq p$ and $p < n$, we must have $x = a^k$, $y = a^l$, and $z = a^m$, with $k + l + m = n^2$ and $k + l \leq p$. Since $|y| > 0$ and $l = |y|$ and $k + l < p$, we also have that $l > 0$ and $l \leq p$. So, $xy^2z = a^k a^{2l} a^m = a^{n^2 + l}$. Since $l > 0$, we have that $n^2 + l > n^2$. Since $l \leq p$ and $p < n$, we have that $n^2 + l \leq n^2 + p < n^2 + n < (n + 1)^2$, which is the smallest perfect square greater than $n^2$. So, $n^2 < |xy^2z| < (n + 1)^2$, which implies that $xy^2z \notin L$, and consequently $L$ is not a regular language.
Problem 3 [25 points]

Show that the language \( L \) such that 

\[
L = \{ w \in \{a, b\}^* \mid |w|_a \geq |w|_b \} = \{ \epsilon, a, aab, aba, baa, \ldots \}
\]

is not a regular language.

Solution:

We show that \( L \) is not regular by using the contrapositive form of the Pumping Lemma. Given any nonnegative integer \( p \), let \( n \) be any integer greater than \( p \), and let \( w \) be the string \( a^n b^n \). Note that \( w \in L \) (as \( |w|_a = |w|_b \) and \( |w| \geq p \)). For all strings \( x, y \) and \( z \) from \( \{a\}^* \) such that \( xyz = w \), \(|y| > 0\), and \(|xy| \leq p\), we want to show that \( xy^i z \not\in L \), for some \( i \geq 0 \). We now claim that \( i = 0 \) does the job. Since \(|xy| \leq p \) and \( p < n \), we must have \( x = a^k \), \( y = a^l \), and \( z = a^m b^n \), with \( k + l + m = n \) and \( k + l \leq p \). Since \(|y| > 0 \) and \( l = |y| \) and \( k + l < p \), we also have that \( l > 0 \) and \( l \leq p \). So, \( xy^0 z = a^k a^m b^n = a^{k+m} b^n = a^{n-l} b^n \). Since \( l > 0 \) and \( l \leq p \) and \( p < n \), we must have that \( 0 < n - l < n \). So, \( xy^0 z = a^{n-l} b^n \not\in L \), as \(|w|_a = n - l < n = |w|_b \). So, \( L \) is not a regular language.
Problem 4 [25 points]

Let \( D = (Q, \Sigma, \delta, q_0, F) \) be a deterministic finite automaton. Then, show that \( L(D) \) is infinite if and only if there exists a string \( w \) in \( L(D) \) such that \( |Q| \leq |w| < 2|Q| \), where \( |w| \) is the length of \( w \) and \( |Q| \) is the number of states of \( Q \).

Solution:

First, assume that there exists a string \( w \) in \( L(D) \) such that \( |Q| \leq |w| < 2|Q| \). Since \( L(D) \) is a regular language, we can invoke the Pumping Lemma to conclude that \( L(D) \) is infinite. That is, according to the Pumping Lemma, if \( |w| \geq |Q| \) then there are strings \( x, y, z \) in \( \Sigma^* \), with \( w = xyz \), \( y \neq \epsilon \), and \( |xy| \leq |Q| \), such that \( xy^iz \in L(D) \) for all \( i \geq 0 \). So, \( L(D) \) is indeed infinite. Conversely, if \( L(D) \) is infinite then there exists a string \( w \in L(D) \) such that \( |w| \geq |Q| \). Otherwise, \( L(D) \) would be finite. If \( |w| < 2|Q| \), we are done. If \( L(D) \) has no string of length between \( |Q| \) and \( 2|Q| - 1 \), assume that \( w \) is the shortest string of \( L(D) \) whose length is equal to or greater than \( 2|Q| \). Again, according to the Pumping Lemma, we can write \( w \) as \( w = xyz \), with \( x, y, z \in \Sigma^* \), \( y \neq \epsilon \), and \( |xy| \leq |Q| \), such that \( xy^iz \in L(D) \), for all \( i \geq 0 \). This means that \( xy^0z = xz \in L(D) \). But, since \( |y| > 0 \) and \( |xy| \leq |Q| \), we must have that \( |xy^0z| < |w| \) and \( |xy^0z| \geq |w| - |Q| \geq |Q| \). So, either \( |xy^0z| \) satisfies \( |Q| \leq |xy^0z| < 2|Q| \) or \( 2|Q| \leq |xy^0z| < |w| \). The former case contradicts the assumption that \( L(D) \) has no string whose length is between \( |Q| \) and \( 2|Q| - 1 \), and the latter case contradicts the fact that \( w \) is the shortest string whose length is equal to or greater than \( 2|Q| \). So, \( L(D) \) must contain a string \( w \) whose length is between \( |Q| \) and \( 2|Q| - 1 \).