Recitation 1

Friday, January 14, 2005
Problem 1. Determine \( \{1\} \times \{1, 2\} \times \{1, 2, 3\} \).

Solution:

Given sets \( A_1, A_2, \ldots, A_k \), recall that \( A_1 \times A_2 \times \cdots A_k \) is the set of all \( k \)-tuples \((a_1, a_2, \ldots, a_k)\) such that \( a_i \in A_i \) for \( i = 1, 2, \ldots, k \). So,

\[
{1} \times {1, 2} \times {1, 2, 3} = \{(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 1), (1, 2, 2), (1, 2, 3)\}.
\]
Problem 2. Let \( S = \{a, b, c, d\} \). List all partitions of \( S \) with exactly two elements.

Solution:

By definition, a partition of a nonempty set \( S \) is a set of nonempty subsets of \( S \) such that each element of \( S \) belongs to exactly one set of the partition. The problem asks us to list all partitions of \( S \) with exactly two elements, i.e., with exactly two subsets of \( S \), say \( S_1 \) and \( S_2 \). What are all possibilities for \( S_1 \) and \( S_2 \)? In the following we enumerate all of them:

- \( S_1 = \{a\} \) and \( S_2 = \{b, c, d\} \).
- \( S_1 = \{b\} \) and \( S_2 = \{a, c, d\} \).
- \( S_1 = \{c\} \) and \( S_2 = \{a, b, d\} \).
- \( S_1 = \{d\} \) and \( S_2 = \{a, b, c\} \).
- \( S_1 = \{a, b\} \) and \( S_2 = \{c, d\} \).
- \( S_1 = \{a, c\} \) and \( S_2 = \{b, d\} \).
- \( S_1 = \{a, d\} \) and \( S_2 = \{b, c\} \).

So, \( S \) has exactly seven partitions \( \Pi = \{S_1, S_2\} \) with exactly two elements, \( S_1 \) and \( S_2 \), which are listed above.
Problem 3. Let \( n \) be a nonzero integer. Show that the binary relation \( R \) on \( \mathbb{Z} \) such that

\[
R = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y \pmod{n}\}.
\]

Solution:

To prove that \( R \) is an equivalence relation, we must show that \( R \) is (1) reflexive, (2) symmetric, and (3) transitive.

(1) To prove that \( R \) is reflexive, we must show that \( xRx \) for every \( x \in \mathbb{Z} \).

So, let \( x \) be any element of \( \mathbb{Z} \). Since \( n \mid (x - x) \) for every \( x \in \mathbb{Z} \), we have that \( x \equiv x \pmod{n} \), and therefore \( xRx \).

(2) To prove that \( R \) is symmetric, we must show that \( yRx \) whenever \( xRy \). So, let \((x, y)\) be any pair of \( R \). By definition of \( R \), if \( xRy \) then \( x \equiv y \pmod{n} \). But, since \( x \equiv y \pmod{n} \) implies \( y \equiv x \pmod{n} \), we also have that \( yRx \).

(3) To prove that \( R \) is transitive, we must show that \( xRz \) whenever \( xRy \) and \( yRz \). So, let \((x, y)\) and \((y, z)\) be any two pairs of \( R \). By definition of \( R \), if \( xRy \) then \( x \equiv y \pmod{n} \). Similarly, if \( yRz \) then \( y \equiv z \pmod{n} \). But, since \( x \equiv y \pmod{n} \) and \( y \equiv z \pmod{n} \) implies \( x \equiv z \pmod{n} \), we also have \( xRz \).
Problem 4. Is there a *simple* graph of order 3 such that every two vertices are adjacent and every two edges are adjacent? Does such a graph of order 4 exist?

Solution:

For the first question, the answer is “yes”. Let $G_1$ be a graph with vertex set $V(G_1) = \{v_1, v_2, v_3\}$ and edge set $E(G_1) = \{v_1v_2, v_1v_3, v_2v_3\}$. Graph $G_1$ has order 3, and every edge of $G_1$ has exactly two adjacent edges. For the second question, the answer is “no”. Why?

Aiming at a contradiction, assume that such a graph exist. So, let $G$ be a graph of order 4 satisfying the property that every two vertices are adjacent and every two edges are adjacent. Since $G$ has four vertices and every two vertices are adjacent, there must be an edge connecting any two vertices of $G$. Let $V(G)$ be the set $\{u, v, w, z\}$. Consider the vertices $u$ and $v$. Since they are adjacent, the set $E(G)$ must contain the edge $uv$. Now, consider the vertices $w$ and $z$. Since they are adjacent, the set $E(G)$ must also contain the edge $wz$. However, these two edges are not adjacent, as they have no vertex in common. But, this contradicts the fact that every two edges of $G$ are adjacent. So, such a graph $G$ cannot exist.
Problem 5. Show that, for any simple graph $G$, the sum of the degree of all vertices of $G$ is twice the number of edges of $G$.

Solution:

By definition, the degree of a vertex is the number of edges the vertex is incident to. Since there are exactly two vertices incident to each edge of a simple graph, by adding the degree of the vertices of a graph, we are counting each edge “twice”; that is, if $u$ and $v$ are two vertices of $G$ and $G$ contains the edge $uv$, the sum $d(u) + d(v)$ counts $uv$ twice, once for each of the two vertices incident to $uv$. So, the sum

$$
\sum_{v \in V(G)} d(v)
$$

counts each edge of $G$ twice, and therefore $\sum_{v \in V(G)} d(v)$ is equal to $2 \cdot |E(G)|$, where $|E(G)|$ is the cardinality of $E(G)$. 