Recitation 5
Problem 1
Show that if $L_1$ and $L_2$ are regular languages then so is $L_1 \cap L_2$; that is, show that regular languages are closed under intersection.

Solution:
Let $D_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $D_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ be deterministic finite automata such that $L(D_1) = L_1$ and $L(D_2) = L_2$. Our goal is to build a deterministic finite automata $D = (Q, \Sigma, \delta, q_0, F)$, specified in terms of $D_1$ and $D_2$, such that $L(D) = L_1 \cap L_2$. The main idea of our construction is to build a DFA that simulates the execution of $D_1$ and $D_2$ on the same input string. This is done by interpreting each state of $D$ as a pair $(p, q)$, where $p$ is a state of $D_1$ and $q$ is a state of $D_2$. So, in the same spirit of the subset construction, we define the set $Q$ of states of $D$ as $Q = Q_1 \times Q_2$. Note that each pair $(p, q) \in Q$ is the "denomination" of a state of $D$. The initial state of $D$ is the pair (state) $(q_{01}, q_{02})$ of $Q$. The final set $F$ of states of $D$ consists of all pairs (states) $(p, q)$ of $Q$ such that $p \in F_1$ and $q \in F_2$. Finally, the transition function $\delta : Q \times \Sigma \rightarrow Q$ of $D$ is defined as $\delta((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$, for all $(p, q) \in Q$ and for all $a \in \Sigma$.

For an example of the above construction, let $D_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$, where $Q_1 = \{s_1, s_2\}$, $\Sigma = \{0, 1\}$, $q_{01} = s_1$, $F_1 = \{s_2\}$, and $\delta_1 : Q_1 \times \Sigma \rightarrow Q_1$ is

\[
\begin{array}{c|cc}
\delta_1 & 0 & 1 \\
\hline
s_1 & s_2 & s_1 \\
s_2 & s_2 & s_2 \\
\end{array}
\]

and let $D_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$, where $Q_2 = \{r_1, r_2\}$, $\Sigma = \{0, 1\}$, $q_{02} = r_1$, $F_1 = \{r_2\}$, and $\delta_2 : Q_2 \times \Sigma \rightarrow Q_2$ is

\[
\begin{array}{c|cc}
\delta_2 & 0 & 1 \\
\hline
r_1 & r_1 & r_2 \\
r_2 & r_2 & r_2 \\
\end{array}
\]

Now, using our construction, we obtain $D = (Q, \Sigma, \delta, q_0, F)$ such that $Q = \{(s_1, r_1), (s_1, r_2), (s_2, r_1), (s_2, r_2)\}$, $\Sigma = \{0, 1\}$, $q_0 = (s_1, r_1)$, $F = \{(s_2, r_2)\}$, and $\delta : Q \times \Sigma \rightarrow Q$ is given by the table below:

\[
\begin{array}{c|cc}
\delta & 0 & 1 \\
\hline
(s_1, r_1) & (s_2, r_1) & (s_1, r_2) \\
(s_1, r_2) & (s_2, r_2) & (s_1, r_2) \\
(s_2, r_1) & (s_2, r_1) & (s_2, r_2) \\
(s_2, r_2) & (s_2, r_2) & (s_2, r_2) \\
\end{array}
\]

Note that $L(D_1)$ is the language of all strings over $\{0, 1\}$ that contain at least one 0 and $L(D_2)$ is the language of all strings over $\{0, 1\}$ that contain at least one 1. So, $L(D_1) \cap L(D_2)$ is the language of all strings over $\{0, 1\}$ that contain at least one 0 and at least one 1. By examining $D$, we can see that $L(D)$ is precisely $L(D_1) \cap L(D_2)$.

Let us now prove that our DFA $D$ is such that $L(D) = L(D_1) \cap L(D_2)$ for any two DFAs $D_1$ and $D_2$. We want to show that

$\bar{\delta}(q_0, w) \in F \text{ if and only if } \bar{\delta}_1(q_{01}, w) \in F_1 \text{ and } \bar{\delta}_2(q_{02}, w) \in F_2$
for all \(w \in \Sigma^*\). In order to show that the above claim is true, we first prove by induction on the length \(|w|\) of \(w\) that
\[
\tilde{\delta}((q_0, q_{o2}), w) = (\tilde{\delta}_1(q_0, w), \tilde{\delta}_2(q_{o2}, w))
\]
for all \(w \in \Sigma^*\).

**Base Case \(|w| = 0\)**

If \(|w| = 0\) then \(w = \epsilon\), and therefore \(\tilde{\delta}((q_0, q_{o2}), w) = \tilde{\delta}((q_0, q_{o2}), \epsilon) = (q_0, q_{o2})\). Since \(\tilde{\delta}_1(q_0, \epsilon) = q_0\) and \(\tilde{\delta}_2(q_{o2}, \epsilon) = q_{o2}\), we have that
\[
\tilde{\delta}((q_0, q_{o2}), w) = \tilde{\delta}((q_0, q_{o2}), \epsilon) = (q_0, q_{o2}) = (\tilde{\delta}_1(q_0, \epsilon), \tilde{\delta}_2(q_{o2}, \epsilon)) = (\tilde{\delta}_1(q_0, w), \tilde{\delta}_2(q_{o2}, w)).
\]
So, our claim holds for \(|w| = 0\).

**Induction hypothesis \(|w| = k\)**

Assume that
\[
\tilde{\delta}((q_0, q_{o2}), w) = (\tilde{\delta}_1(q_0, w), \tilde{\delta}_2(q_{o2}, w))
\]
for every \(w \in \Sigma^*\), with \(|w| = k\), where \(k\) is a nonnegative integer.

**Inductive step \(|w| = k + 1\)**

Let \(w\) be any string over \(\Sigma\) such that \(|w| = k + 1\). Since \(k \geq 0\), we can write \(w\) as \(xa\), where \(x \in \Sigma^k\) and \(a \in \Sigma\). By definition of \(\tilde{\delta}\),
\[
\tilde{\delta}((q_0, q_{o2}), w) = \tilde{\delta}((q_0, q_{o2}), xa) = \delta(\tilde{\delta}((q_0, q_{o2}), x), a).
\]
Let \(\tilde{\delta}((q_0, q_{o2}), x) = (p, q) \in Q\). Since \(|x| = k\), from the induction hypothesis, we must have that \(p = \tilde{\delta}_1(q_0, x)\) and \(q = \tilde{\delta}_2(q_{o2}, x)\). Hence,
\[
\tilde{\delta}((q_0, q_{o2}), w) = \tilde{\delta}((q_0, q_{o2}), xa) = \delta(\tilde{\delta}((q_0, q_{o2}), x), a) = \delta((\tilde{\delta}_1(q_0, x), \tilde{\delta}_2(q_{o2}, x)), a).
\]
But, from the definition of \(\delta\), we also know that \(\delta((p, q), a)\) is equal to \((\tilde{\delta}_1(p, a), \tilde{\delta}_2(q, a))\). So,
\[
\tilde{\delta}((q_0, q_{o2}), w) = \delta((\tilde{\delta}_1(q_0, x), \tilde{\delta}_2(q_{o2}, x)), a) = (\tilde{\delta}_1(\tilde{\delta}_1(q_0, x), a), \tilde{\delta}_2(\tilde{\delta}_2(q_{o2}, x), a)).
\]
From the definitions of \(\tilde{\delta}_1\) and \(\tilde{\delta}_2\), we know that
\[
\delta_1(\tilde{\delta}_1(q_0, x), a) = \tilde{\delta}_1(q_0, xa) = \tilde{\delta}_1(q_0, w) \quad \text{and} \quad \delta_2(\tilde{\delta}_2(q_{o2}, x), a) = \tilde{\delta}_2(q_{o2}, xa) = \tilde{\delta}_2(q_{o2}, w),
\]
which implies that
\[
\tilde{\delta}((q_0, q_{o2}), w) = (\tilde{\delta}_1(q_0, w), \tilde{\delta}_2(q_{o2}, w))
\]
for all \(w \in \Sigma^{k+1}\). From the first principle of induction, we have that our claim holds for every \(w \in \Sigma^*\).

The construction above is very important in automata theory and it is called the **cross product construction**.
Problem 2

Consider the problem of converting an $\epsilon$-NFA into an equivalent DFA. According to what you have seen in class, such a conversion must be done in two steps: (1) we convert the $\epsilon$-NFA into an NFA by eliminating $\epsilon$ transitions, and (2) we use the subset construction to obtain the DFA equivalent to the NFA obtained in (1). It turns out that this two-step process can be replaced with a one-step process by carefully modifying the subset construction, so that we obtain a DFA from an equivalent $\epsilon$-NFA. How can we modify the subset construction to perform this conversion? Justify your answer.

Solution:

This modification of the subset construction is actually much simpler than one could expect. All we have to do is to define the initial state and the transition function of the DFA in such a way that we take into account the $\epsilon$-transitions.

Recall that if $N = (Q, \Sigma, \delta, q_0, F)$ is an NFA without $\epsilon$-transitions, the transition function $\delta_D : Q_D \times \Sigma \rightarrow Q_D$ of the DFA $D = (Q_D, \Sigma, \delta_D, q_{0_D}, F_D)$ resulting from the subset construction is defined as

$$\delta_D(S, a) = \bigcup_{q \in S} \delta(q, a)$$

for every $S \in \mathcal{P}(Q) = Q_D$ and every $a \in \Sigma$.

Now, assume that $N = (Q, \Sigma, \delta, q_0, F)$ is an NFA with $\epsilon$-transitions. The definition of the DFA $D = (Q_D, \Sigma, \delta_D, q_{0_D}, F_D)$ is the same as the one in the subset construction seen in class, except that we define $q_{0_D} = \epsilon$-CLOSURE($q_0$), and we define the transition function $\delta_D : Q_D \times \Sigma \rightarrow Q_D$ as

$$\delta_D(S, a) = \epsilon$\text{-CLOSURE} \left( \bigcup_{q \in S} \delta(q, a) \right),$$

for every $S \in \mathcal{P}(Q) = Q_D$ and every $a \in \Sigma$.

To make sure you understood the modification, apply it to the $\epsilon$-NFA $N = (Q, \Sigma, \delta, q_0, F)$, where $Q = \{s_0, s_1, s_2, s_3\}$, $\Sigma = \{a, b, c\}$, $q_0 = s_0$, $F = \{s_3\}$, and $\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow \mathcal{P}(Q)$ is the transition function such that

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$\epsilon$</th>
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<tr>
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<td>${s_2}$</td>
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</tr>
</tbody>
</table>

This is the $\epsilon$-NFA of Problem 1 in Homework 4.

How about a proof for convince yourself that the above modification works? In Recitation 3, we gave a proof for the correctness of the subset construction. A good exercise is to try to modify that proof to accomodate the modification we proposed here. You can start thinking about this proof, as it may be in your next homework.