The asymptotic behavior \((r \to 0)\) of solutions \(u(r, \theta)\) to Laplace’s equation near a vertex depends on the vertex angle and the boundary conditions (BCs) along the segments intersecting the vertex. Depending on the angle and BCs, there can be a singularity in gradients (unbounded) even though \(u(r, \theta)\) is everywhere bounded. The first two problems reveal aspects of such behavior.

1) Consider the problem for \(u(r, \theta)\) on a circular sector of opening angle \(\beta\) defined by:

\[
\nabla^2 u = 0; \quad 0 < r < a, \quad 0 < \theta < \beta
\]

\[u(r, 0) = u_o, \quad u(r, \beta) = u_\beta, \quad u(a, \theta) = f(\theta)
\]

\[u(r, 0) = u_o\]

Let \(u(r, \theta) = v(r, \theta) + w(r, \theta)\), where both \(v\) and \(w\) satisfy Laplace’s equation on the circular segment with:

\[v(r, 0) = u_o, \quad v(r, \beta) = u_\beta, \quad v(a, \theta) = g(\theta)\]

\[w(r, 0) = 0, \quad w(r, \beta) = 0, \quad w(a, \theta) = f(\theta) - g(\theta)\]

a) Find \(v(r, \theta)\) and \(g(\theta)\) if \(v\) is separable in \(r\) and \(\theta\). Even though \(v\) is everywhere bounded, i) show that the gradient of \(v\) (flux) is unbounded (singular) as \(r \to 0\) and ii) show that the integrated flux \(Q\) is unbounded along \(\theta = 0\) and \(\theta = \beta\), where along each radial segment of the boundary

\[Q \approx \frac{1}{a} \int_0^a \frac{1}{r} \frac{\partial v}{\partial \theta} \, dr\]

b) Use the method of separation of variable in \(r\) and \(\theta\) to find \(w(r, \theta)\). If \(w\) (and \(Q\)) is required to be bounded everywhere, what is the leading term in the series solution for \(w\) as \(r \to 0\)? Note that the leading term depends on \(\beta\). With \(w \approx r^\delta\) as \(r \to 0\), plot \(\delta\) versus \(\beta\).
2) Consider the following variant of problem 1b):

\[ \nabla^2 u = 0 ; \quad 0 < r < a , \quad 0 < \theta < \beta \]

\[ \frac{\partial u}{\partial \theta} = 0 \text{ on } \theta = 0 \]

\[ u(r, \beta) = 0 \]

\[ u(a, \theta) = f(\theta) \]

Use the method of separation of variable in \( r \) and \( \theta \) to find \( u(r, \theta) \). If \( u \) (and \( Q \)) is required to be bounded everywhere, what is the leading term in the series solution for \( u \) as \( r \to 0 \)? Note again that the leading term depends on \( \beta \). With \( u \propto r^\delta \) as \( r \to 0 \), plot \( \delta \) versus \( \beta \).

Although problems 1) and 2) were formulated for a circular sector (i.e., with boundary segment \( r = a \)), even for non-circular domains containing a vertex the leading behavior near the vertex \( (r = 0) \) only depends on the opening angle \( \beta \) and the BCs along the radial segments intersecting the vertex.

**Also do the following problems from the Haberman text:**

2.5.3b (p.83)

2.5.4 (p.83)