The following problems on Sturm-Liouville problems are taken from Haberman, pp. 162-163:

5.3.5 a,b,c (part b should read “n-1 zeros on 0<x<L”)

5.3.6

5.3.9

Since there is an exam on March 6th, you only need to hand in the above 3 problems. Nevertheless, you should work on the next problem, which we will discuss in class.

Consider the following problem, which is derived from the problem of vibrations of a cantilever beam and has a wide range of applications including atomic-force microscopes. If the beam deflection is denoted \( v(x,t) \), the spatial problem associated with the function \( u(x) \) defined below corresponds to harmonic oscillations, i.e. \( v(x,t) = u(x) \cos(\omega t) \), where \( \omega \) is the natural frequency of vibration (i.e., it is real).

Consider:

\[
EI \frac{d^4 u}{dx^4} - \rho \omega^2 u = 0
\]

\[
u = 0 \text{ and } u' = 0 \quad \text{at} \quad x = 0
\]

\[
u'' = 0 \text{ and } u''' = 0 \quad \text{at} \quad x = L
\]

where \( E \) (Young’s modulus), \( I \) (moment of inertia of the cross section), and \( \rho \) (density) are all constants (which of course can be lumped together but are written separately to emphasize their physical interpretation). Derive the characteristic (transcendental) equation for the eigenvalues (frequencies) which are conveniently expressed in terms of:

\[
\mu_n = \left( \frac{\rho \omega_n^2}{EI} \right)^{1/4}
\]

Discuss how you would determine the corresponding eigenfunctions.