1. (20 points) Orthogonality of the eigenfunctions of the Sturm Liouville ODE:

\[
\frac{d}{dx} \left[ p(x) \frac{d\phi(x)}{dx} \right] + q(x) \phi(x) + \lambda \sigma(x) \phi(x) = 0, \quad a < x < b
\]

is guaranteed if:

\[
\left[ p(x) \left( \phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx} \right) \right]_a^b = 0 \tag{1a}
\]

If \( p(x) = \text{constant} \), determine the relationship between \( \alpha, \beta, \gamma \), and \( \delta \) such that the boundary conditions:

\[
\phi(b) + \alpha \phi(a) + \beta \frac{d\phi(a)}{dx} = 0
\]

\[
\frac{d\phi(b)}{dx} + \gamma \phi(a) + \delta \frac{d\phi(a)}{dx} = 0 \tag{1b}
\]

also imply orthogonality. Note: \( \frac{d\phi(a)}{dx} \equiv \frac{d\phi(x)}{dx} \bigg|_{x=a} \), etc. Hint: use (1b) to express values of \( \phi \) and \( \frac{d\phi}{dx} \) at \( x = b \) in equation (1a) in terms of their values at \( x = a \). Note that (1b) are a generalization of symmetry conditions.

2. (20 points) Consider the sine series for the function \( f(x) \) where \( f(0) = 0 \) and \( f(L) = 0 \) but \( f \) has a jump discontinuity at \( x_o \), i.e. \( f(x^-_0) = \alpha \) and \( f(x^+_0) = \beta \):

\[
f(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right)
\]

Determine the coefficients \( A_o \) and \( A_n \) of the cosine series of the function \( f'(x) \) in terms of \( B_n \), \( \alpha \), and \( \beta \) where:

\[
f'(x) = A_o + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{L} \right)
\]
3. (30 points) Consider Laplace's equation on a circular annulus \( a < r < b \) and \( 0 \leq \theta < 2\pi \):

\[
\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0
\]

a) Find the (axisymmetric) solution corresponding to the boundary conditions:

\[
u(a, \theta) = u_a, \quad u(b, \theta) = u_b
\]

b) Find the separable solution corresponding to the boundary conditions:

\[
u(a, \theta) = 0, \quad u(b, \theta) = \cos(\theta)
\]

4. (30 points) Consider the problem of diffusion in a rod of length \( L \) where the concentration is proportional to the flux at the left end and the right end is held at fixed concentration:

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad ; \quad 0 < x < L , \quad t > 0
\]

\[
\left. \left( u + \alpha \frac{\partial u}{\partial x} \right) \right|_{x=0}^{t>0} = 0 , \quad u(L,t) = 0
\]

\[
u(x,0) = f(x)
\]

Construct the solution for any initial concentration \( f(x) \) using the method of separation of variables. Determine the transcendental equation that determines the eigenvalues, determine the eigenfunctions, and write down the Fourier coefficients entering the infinite series solution.
Some Useful Results

- **Fourier Series**

Fourier sine series:

\[ g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \text{ where } b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \, dx \]

Fourier cosine series:

\[ g(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \text{ where } a_0 = \frac{1}{L} \int_0^L g(x) \, dx \text{ and } a_n = \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi x}{L}\right) \, dx \]

A (non-standard) Fourier sine series:

\[ g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n-1}{2L}\pi x\right) \text{ where } b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{2n-1}{2L}\pi x\right) \, dx \]