We have considered the problem of diffusion (e.g. heat conduction) in a finite-length rod \((L)\) with boundary conditions \(u(0,t) = u_o\) and \(u(L,t) = u_L\) and initial condition \(u(x,0) = 0\). The separation of variables solution involves a superposition of products of functions that exponentially decay in time and functions that are trigonometric in space \(x\). As we also noted, the long-time or steady-state solution is simply linear in \(x\). What is not so clear is the nature of the short-time behavior, even if \(u_L = 0\) so that the major affect at short times of the disturbance at \(x = 0\) is only felt near that end. In that case, since the end \(x = L\) has little influence on the short-time solution, we can consider a so-called boundary layer formulation of the problem. Such a formulation is based upon the supposition that the short-time behavior near \(x = 0\) would be the same even if \(L \to \infty\) (which can be proven \textit{a posteriori} or, e.g., using integral transform techniques). The boundary layer problem is posed and solved below. Recall that using Matlab finite element analysis (PDE Toolbox), we did get a good sense of the overall time-dependent behavior, but we will see that powerful results can be obtained analytically for the short-time response.

Consider diffusion in a semi-infinite rod \((L \to \infty)\) governed by the following problem:

\[
\begin{align*}
u_{tt} &= k \nu_{xx} ; \quad 0 < x < \infty , \quad t > 0 \\
u(x,0) &= 0 ; \quad 0 \leq x < \infty \\
u(0,t) &= u_o ; \quad t > 0 \\
u(x \to \infty,t) &= 0 ; \quad t > 0
\end{align*}
\]

Note that for finite times the boundary condition \((1d)\) at \(x \to \infty\) is consistent with the initial condition \((1b)\), i.e. \(u(x,0) = 0\) everywhere. Also note the fact that the disturbance \(u_o\) is applied at the other end, \(x = 0\), which is infinitely far away. Therefore, the solution to \((1a-d)\) should provide useful information about the short-time diffusion near the end of a finite rod. This semi-infinite rod problem has one fewer variable than the finite rod problem (which includes \(L\)) and, in fact, has no physical length scale. This can be exploited using ideas of dimensional analysis.

We begin by first identifying the independent variables that enter \((1a-d)\). They are:

\[
x, t, k, u_o
\]
From the PDE (1a) we can infer that the diffusivity (or thermal conductivity) $k$ has dimensions of length-squared divided by time. For this discussion, we will assume that $u$ and $u_0$ have dimension of temperature.

On “dimensional grounds” the solution $u$ can be represented as some quantity with the same dimensions as $u$, e.g. $u_0$, times a non-dimensional function of non-dimensional variables. Then, and in general only then, if the units adopted for the variables are changed, e.g. from centigrade to Fahrenheit, from meters to feet, and from seconds to hours, the value of $u$ will be correct in the new units. For this problem we write this statement as:

$$u(x,t) = u_0 F(\text{non-dimensional variables})$$

where $F$ is a non-dimensional function of all the non-dimensional variables that enter the problem. In problem (1a-d) we will see that there is only one non-dimensional variable.

For $t$ to enter as a non-dimensional variable it must multiply $k$, but the product $kt$ has dimensions of length-squared. Furthermore, $kt$ is the only variable besides $x$ containing length. Therefore, we are led to the conclusion that $x$ and $t$ can only enter the solution in terms of the single variable

$$\frac{x}{\sqrt{kt}}$$

(3)

For convenience we will choose the similarity variable

$$\eta = \frac{x}{2\sqrt{kt}}$$

(4)

and write

$$u(x,t) = u_0 F(\eta)$$

(5)

Show that: $u_\eta = -\frac{\eta}{2t} F'$, $u_x = \frac{1}{2\sqrt{kt}} F'$, $u_{xx} = \frac{1}{4kt} F''$ and, therefore, that the PDE (1a) is transformed to the ODE

$$F'' + 2\eta F' = 0$$

(6)
Show that: the first integral of (6) is

\[ F' = c_1 \exp(-\eta^2) \] (7)

and that

\[ F(\eta) = c_1 \int_{\eta}^{\infty} e^{-s^2} \, ds + c_2 \] (8)

The integration constants \( c_1 \) and \( c_2 \) are determined from the initial condition (1b), re-written as \( F(\eta \to \infty) = 0 \), which leads to (show)

\[ F(\eta) = c_1 \int_{-\infty}^{\eta} e^{-s^2} \, ds \] (9)

and the boundary condition (1c), which with (5) and (9) requires that

\[ -c_1 \int_{0}^{\infty} e^{-s^2} \, ds = -c_1 \sqrt{\pi}/2 \quad \text{or} \quad c_1 = -2/\sqrt{\pi} \] (10)

The self-similar solution to the original problem (1a-d) given by (5) and (9) with (10) is written in terms of the complementary error function denoted “erfc” or the error function “erf” as

\[
  u(x,t) = u_o \, F(\eta) = \frac{2u_o}{\sqrt{\pi}} \int_{\eta}^{\infty} e^{-s^2} \, ds \\
  = \frac{2u_o}{\sqrt{\pi}} \text{erf}(\eta) \equiv u_o \left[ 1 - \text{erf}(\eta) \right] \\
  = u_o \, \text{erfc} \left( \frac{x}{2\sqrt{kt}} \right) 
\]

(11)

Note that (show)

\[ \text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_{0}^{\eta} e^{-s^2} \, ds ; \quad \text{erf}(0) = 0 \] and \( \text{erf}(\infty) = 1 \) (12)

---

1 To evaluate the integral \( I = \int_{0}^{\infty} e^{-s^2} \, ds \), consider \( I^2 = \int_{0}^{\infty} \int_{0}^{\infty} e^{-(s^2+v^2)} \, ds \, dv \) which, in polar variables, can be expressed as \( I^2 = \int_{0}^{\pi/2} \int_{0}^{\infty} e^{-r^2} \, r \, dr \, d\theta = \frac{\pi}{4} \).
Properties, plots and tables of error functions can be found in *Handbook of Mathematical Functions*, by Abramowitz and Stegun, Chapter 7. (Also see Matlab and Maple functions erf and erfc.) The series for the error function is:

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}
\]

\[
= \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{n=0}^{\infty} \frac{2^n z^{2n+1}}{1 \cdot 3 \cdots (2n+1)}
\]

and the asymptotic behavior for large \( z \) is:

\[
-\text{erf}(z) = \text{erfc}(z) = \frac{1}{\sqrt{\pi}} \frac{e^{-z^2}}{z} \left[ 1 + O(z^{-2}) \right] \quad \text{as} \quad z \to \infty
\]

One can interpret the solution (11) in terms of the similarity variable \( \eta = x / (2\sqrt{k t}) \), in the following way: the behavior at time \( t = t_1 \) at a given \( x = x_1 \) is identical to the behavior at time \( t = t_2 \) for a value of \( x = x_2 = x_1 \sqrt{t_2/t_1} \) since the value of \( \eta \) is the same. Furthermore, since the complementary error function is a monotonically decaying function with \( \text{erfc}(0) = 1 \) and \( \text{erfc}(\infty) = 0 \) and given this scaling it is natural to associate \( \sqrt{k t} \) with a diffusion length. For example, if the diffusion length is defined to be the distance from the end \( x = 0 \) where \( u \) equals half the concentration at that end, i.e., \( u = u_0 / 2 \), this diffusion length is \( x \approx 0.48 \sqrt{k t} \). Below is a plot of \( \text{erfc}(x/2\sqrt{k t}) \) versus \( x \) for values \( 2\sqrt{k t} = 1, 5, 10, 20, 50, 100, 200, 500, 1000, 10000 \).
The small and large $x$ behavior of this solution can be characterized by the following asymptotic behaviors. From the series representation (13) one can show that for finite times the self-similar solution (11) to problem (1) near the disturbed end $x = 0$ has the behavior

$$u(x,t) = u_o \left[ 1 - \left( \frac{2}{\sqrt{\pi}} \right) \left( \frac{x}{2\sqrt{kt}} \right) \right] \quad \text{for} \quad x << \sqrt{kt}, \quad (15)$$

where the next term that has been neglected is order $(x/\sqrt{kt})^3$. For a fixed time, this small-$x$ asymptotic represents a decay in $u$ away from the disturbed end $x = 0$. From the asymptotic representation (14), for finite times and $x >> \sqrt{kt}$ the solution can be approximated as:

$$u(x,t) = u_o \frac{1}{\sqrt{\pi}} \left( \frac{2\sqrt{kt}}{x} \right) \exp \left( -\frac{x^2}{4kt} \right) \quad \text{for} \quad x >> \sqrt{kt} \quad (16)$$

For a fixed time, this large-$x$ asymptotic represents a strong decay in $x$ to $u = 0$. 