Arbitrages, and pricing of stock options

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Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
Arbitrage

- Bet on different events with each outcome paying a random return
- **Arbitrage**: It is possible to devise a betting strategy that guarantees a positive return no matter the combined outcome of the events
- Arbitrages often involve operating in two different markets
Example

- Booker 1  ⇒  Phillies win pay 1.5:1, Phillies loose pay 3:1
  - Bet $x$ on Phillies and $y$ against Phillies. Guaranteed Earnings?
    
    Phillies win:  $0.5x - y > 0 \Rightarrow x > 2y$
    Phillies lose:  $-x + 2y > 0 \Rightarrow x < 2y$

  - Arbitrage not possible. Notice that $1/(1.5) + 1/3 = 1$

- Booker 2  ⇒  Phillies win pay 1.4:1, Phillies loose pay 3.1:1
  - Bet $x$ on Phillies and $y$ against Phillies. Guaranteed Earnings?
    
    Phillies win:  $0.4x - y > 0 \Rightarrow x > 2.5y$
    Phillies lose:  $-x + 2.1y > 0 \Rightarrow x < 2.1y$

  - Arbitrage not possible. Notice that $1/(1.4) + 1/(3.1) > 1$
First condition on Booker 1 and second on Booker 2 are compatible

Bet $x$ on Phillies on Booker 1, $y$ against Phillies on Booker 2

Guaranteed earnings possible. Make $y = 1,000$, $x = 2,066$

Phillies win: $0.5(2,066) - 1,000 = 33$
Phillies loose: $-2066 + 2.1(1000) = 34$

Notice that $1/(1.5) + 1/(3.1) < 1$

If you plan on doing this, do it on, e.g., currency exchange markets
Let events on which bets are posted be \( k = 1, 2, \ldots, K \)

Let \( j = 1, 2, \ldots, J \) index possible joint outcomes

- Joint realizations, also called “world realization”, or “world outcome”

If world outcome is \( j \), event \( k \) yields return \( r_{jk} \) per unit invested (bet)

Do not define probability \( p_j \) of outcome \( j \)

Invest (bet) \( x_k \) in outcome \( k \) \( \Rightarrow \) return for world \( j \) is \( x_k r_{jk} \)

Bets \( x_k \) can be positive (\( x_k > 0 \)) or negative (\( x_k < 0 \))

\( \Rightarrow \) Positive = regular bet. Negative = short bet

Total return \( \Rightarrow \sum_{k=1}^{K} x_k r_{jk} = \mathbf{x}^T \mathbf{r}_j \)

Vectors of returns for outcome \( j \) \( \Rightarrow \mathbf{r}_j := [r_{j1}, \ldots, r_{jK}]^T \) (given)

Vector of bets \( \Rightarrow \mathbf{x}_j := [x_{j1}, \ldots, x_{jK}]^T \) (controlled by gambler)
Arbitrage (clearly defined now)

- Arbitrage is possible if there exists investment strategy $x$ such that
  \[ x^T r_j > 0, \quad \text{for all } j = 1, \ldots, J \]

- Equivalently, arbitrage is possible if
  \[ \max_x \left( \min_j (x^T r_j) \right) > 0 \]

- Portfolio $x$ and returns $r_j$ are vectors in $\mathbb{R}^K$
- Earnings $x^T r_j$ are the inner product of $x$ and $r_j$

- Earnings are positive if angle between $x$ and $r_j$ is less than $\pi/2$ ($90^\circ$)
When is arbitrage possible?

- There is a line that leaves all $r_j$ vectors to one side
  
- There is not a line that leaves all $r_j$ vectors to one side
  
- Arbitrage possible
  
- Prob. vector $p = [p_1, \ldots, p_J]^T$ on world outcomes such that
  
  $$E_p(r) = \sum_{j=1}^{J} p_j r_j = 0$$
  
  does not exist
  
- Arbitrage not possible
  
- There is prob. vector $p = [p_1, \ldots, p_J]^T$ on world outcomes such that
  
  $$E_p(r) = \sum_{j=1}^{J} p_j r_j = 0$$
  
  Think of $p_j$ as scaling factor
Arbitrage theorem

Have “proved” following result, called arbitrage theorem

Theorem

Given vectors of returns $r_j$, associated with random outcome $j = 1, \ldots, J$ an arbitrage is not possible if and only if there exist a probability vector $p$ such that $E_p(r) = 0$. Equivalently,

$$
\max_x \left( \min_j (x^T r_j) \right) \leq 0 \iff \sum_{j=1}^J p_j r_j = 0
$$

Prob. vector $p$ is NOT the prob. distribution of events $j = 1, \ldots, J$
Consider a stock price $X(nh)$ that follows a geometric random walk

$$X((n+1)h) = X(nh)e^{\sigma \sqrt{h}Y_n}$$

where $Y_n$ is a binary random variable with probability distribution

$$P[Y_n = 1] = \frac{1}{2} \left( 1 + \frac{\mu}{\sigma} \sqrt{h} \right), \quad P[Y_n = -1] = \frac{1}{2} \left( 1 - \frac{\mu}{\sigma} \sqrt{h} \right)$$

Recall that as $h \to 0$, $X(nh)$ becomes geometric Brownian motion.

Are there arbitrage opportunities in the price of the stock?

⇒ Too general, let us consider a narrower problem.
Consider the following investment strategy (stock flip):

**Buy:** Buy $1 in stock at time 0 for price $X(0)$ per unit of stock

**Sell:** Sell stock at time $h$ for price $X(h)$ for unit of stock

- Cost of transaction is $1$. Units of stock purchased are $1/X(0)$
- Cash after selling stock is $X(h)/X(0)$
- Return on investment is $X(h)/X(0) - 1$

There are two possible outcomes for the price of the stock at time $h$

- As per model we may have $Y_0 = 1$ or $Y_0 = -1$ respectively yielding
  \[ X(h) = X(0)e^{\sigma\sqrt{h}}, \quad X(h) = X(0)e^{-\sigma\sqrt{h}} \]

Possible returns are therefore

\[ r_1 = \frac{X(0)e^{\sigma\sqrt{h}}}{X(0)} - 1 = e^{\sigma\sqrt{h}} - 1, \quad r_2 = \frac{X(0)e^{-\sigma\sqrt{h}}}{X(0)} - 1 = e^{-\sigma\sqrt{h}} - 1 \]
Present value of returns

- One dollar at time $h$ is not the same as 1 dollar at time 0
- Interest rate of a risk-free investment is $\alpha$ continuously compounded
- In practice, $\alpha$ is the money market rate
- Prices have to be compared at their present value

- The present value of $X(h)$ at time 0 is $X(h)e^{-\alpha h}$
- Then, return on investment is $e^{-\alpha h}X(h)/X(0) - 1$
- Present value of possible returns (whether $Y_0 = 1$ or $Y_0 = -1$) are

$$r_1 = \frac{e^{-\alpha h}X(0)e^{\sigma \sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{\sigma \sqrt{h}} - 1,$$

$$r_2 = \frac{e^{-\alpha h}X(0)e^{-\sigma \sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{-\sigma \sqrt{h}} - 1$$
Arbitrage not possible if and only if there exists $0 \leq q \leq 1$ such that

$$qr_1 + (1 - q)r_2 = 0$$

Arbitrage theorem in 1 dimension (only one bet, buy stock)

Substituting $r_1$ and $r_2$ for their respective values

$$q \left(e^{-\alpha h} e^{\sigma \sqrt{h}} - 1\right) + (1 - q) \left(e^{-\alpha h} e^{-\sigma \sqrt{h}} - 1\right) = 0$$

Can be easily solved for $q$. Expanding product and reordering terms

$$qe^{-\alpha h} e^{\sigma \sqrt{h}} + (1 - q)e^{-\alpha h} e^{-\sigma \sqrt{h}} = 1$$

Multiplying by $e^{\alpha h}$ and grouping terms with a $q$ factor

$$q \left(e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}\right) = e^{\alpha h} - e^{-\sigma \sqrt{h}}$$
Solving for $q$ finally yields \[ q = \frac{e^{\alpha h} - e^{-\sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}} \]

For small $h$ we have $e^{\alpha h} \approx 1 + \alpha h$ and $e^{\pm \sigma \sqrt{h}} \approx 1 \pm \sigma \sqrt{h} + \sigma^2 h/2$

Thus, the value of $q$ as $h \to 0$ may be approximated as

\[ q \approx \frac{1 + \alpha h - (1 - \sigma \sqrt{h} + \sigma^2 h/2)}{1 + \sigma \sqrt{h} - (1 - \sigma \sqrt{h})} = \frac{\sigma \sqrt{h} + (\alpha - \sigma^2/2) h}{2\sigma \sqrt{h}} \]

\[ = \frac{1}{2} \left(1 + \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h}\right) \]

Approximation proves that at least for small $h$ $0 < q < 1 \Rightarrow$ Arbitrage not possible

Also, suspiciously similar to probabilities of geometric random walk \Rightarrow Fundamental observation as we’ll see next
Risk neutral measure

Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
No arbitrage condition on geometric random walk

- Stock prices $X(t)$ follow geometric random walk (drift $\mu$, variance $\sigma^2$)
- Risk free investment has return $\alpha$ (cost of money, money market)
- Arbitrage is not possible in stock flips if there is $0 \leq q \leq 1$ such that

$$q = \frac{e^{\alpha h} - e^{-\sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}}$$

- Notice that $q$ satisfies the equation (which we’ll use later on)

$$qe^{\sigma \sqrt{h}} + (1 - q)e^{-\sigma \sqrt{h}} = e^{\alpha h}$$

- Can we have arbitrage using a more complex set of possible bets?
Consider the following general investment strategy:

**Observe:** Observe the stock price at times $h, 2h, \ldots, nh$

**Compare:** Is $X(h) = x_1, X(2h) = x_2, \ldots, X(nh) = x_n$?

**Buy:** If above answer is yes, buy stock at price $X(nh)$

**Sell:** Sell stock at time $mh$ for price $X(mh)$

Possible bets are the observed values of the stock $x_1, x_2, \ldots, x_l$

$\Rightarrow$ There are $2^n$ possible bets

Possible outcomes are value at time $mh$ and observed values

$\Rightarrow$ There are $2^m$ possible outcomes
Explanation of general investment strategy

- Bet 1 = \( n \) price increases, bet 2 = price increases in 1, \ldots, \( n-1 \) and price decrease in \( n \) ...
- For each bet we have \( 2^{m-n} \) possible outcomes: \( m-n \) price increases, price increases in \( n+1, \ldots, m-1 \) and price decrease in \( m \) ...

<table>
<thead>
<tr>
<th></th>
<th>( X(h) )</th>
<th>( X(2h) )</th>
<th>( X(3h) )</th>
<th>( X(nh) )</th>
<th>( X((n+1)h) )</th>
<th>( X((n+2)h) )</th>
<th>( X(mh) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>bet 1</td>
<td>( e^{\sigma \sqrt{h}} )</td>
<td>( e^{2\sigma \sqrt{h}} )</td>
<td>( e^{3\sigma \sqrt{h}} )</td>
<td>( e^{n\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{2\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{m\sigma \sqrt{h}} )</td>
</tr>
<tr>
<td>bet 2</td>
<td>( e^{\sigma \sqrt{h}} )</td>
<td>( e^{2\sigma \sqrt{h}} )</td>
<td>( e^{3\sigma \sqrt{h}} )</td>
<td>( e^{(n-2)\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{2\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{(m-2)\sigma \sqrt{h}} )</td>
</tr>
<tr>
<td>bet ( 2^n )</td>
<td>( e^{-\sigma \sqrt{h}} )</td>
<td>( e^{-2\sigma \sqrt{h}} )</td>
<td>( e^{-3\sigma \sqrt{h}} )</td>
<td>( e^{-n\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{-\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{-2\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{-m\sigma \sqrt{h}} )</td>
</tr>
</tbody>
</table>

- Figure assumes \( X(0) = 1 \) for simplicity

Outcomes per each bet
Explanation of general investment strategy

- Define the prob. distribution \( q \) over possible outcomes as follows
- Start with a sequence of independent identically distributed \( Y_n \)
- Each element \( Y_n \) is a binary random variable with probabilities

\[
P[Y_n = 1] = q, \quad P[Y_n = -1] = 1 - q
\]

- Joint prob. distribution \( q \) on \( X(h), X(2h), \ldots, X((n + m)h) \) outcomes obtained through transformation

\[
X((n + 1)h) = X(nh)e^{\sigma \sqrt{h}Y_n}
\]

- Notice once more that this is NOT the prob. distribution of \( X(nh) \)
- Will show that expected value of earnings with respect to \( q \) is null

\[
\Rightarrow \text{Thus, arbitrages are not possible}
\]
Consider a time 0 unit investment in given arbitrary outcome

Stock units purchased depend on the price $X(nh)$ at buying time

\[
\text{Units bought} = \frac{1}{X(nh)e^{-\alpha nh}}
\]

Have corrected $X(nh)$ to express it in time 0 values

Cash after selling stock given by price $X(mh)$ at sell time $m + n$

Expressed in time 0 values

\[
\text{Cash after sell} = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}}
\]

Return is then

\[
\Rightarrow r(X(h), \ldots, X(mh)) = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1
\]

Depends on $X(mh)$ and $X(nh)$ only
Expected return with respect to measure $q$

Consider expected value of all possible returns with respect to $q$

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_q \left[ \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \right]$$

Condition on observed values $X(h), \ldots, X(nh)$

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_{q(1:n)} \left[ \mathbb{E}_{q(n+1:m)} \left[ \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \mid X(h), \ldots, X(nh) \right] \right]$$

In innermost expectation $X(nh)$ is given. Furthermore, process $X(t)$ is Markov, thus conditioning on $X(h), \ldots, X((n - 1)h)$ is irrelevant. Thus

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_{q(1:n)} \left[ \frac{\mathbb{E}_{q(n+1:m)} \left[ X(mh) \mid X(nh) \right] e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \right]$$
Expected value of future values (measure $q$)

- Need to find expectation of future value $\mathbb{E}_{q(n+1:m)} [X(mh) \mid X(nh)]$.
- From recursive relation for $X(nh)$ in terms of $Y_n$ sequence:

$$X(mh) = X((m - 1)h)e^{\sigma\sqrt{h}Y_{m-1}}$$

$$= X((m - 2)h)e^{\sigma\sqrt{h}Y_{m-1}}e^{\sigma\sqrt{h}Y_{m-2}}$$

$$\vdots$$

$$= X(nh)e^{\sigma\sqrt{h}Y_{m-1}}e^{\sigma\sqrt{h}Y_{m-2}}\ldots e^{\sigma\sqrt{h}Y_{n+1}}$$

- All the $Y_n$ are independent. Then, upon taking expected value:

$$\mathbb{E}_{q(n+1:m)} [X(mh) \mid X(nh)] = X(nh)\mathbb{E} [e^{\sigma\sqrt{h}Y_{m-1}}] \mathbb{E} [e^{\sigma\sqrt{h}Y_{m-2}}] \ldots \mathbb{E} [e^{\sigma\sqrt{h}Y_{n+1}}]$$

- Need to determine expectation of relative price increase $\mathbb{E} [e^{\sigma\sqrt{h}Y_n}]$.
The expected value of the relative price increase $E\left[ e^{\sigma \sqrt{h} Y_n} \right]$ is

$$E\left[ e^{\sigma \sqrt{h} Y_n} \right] = e^{\sigma \sqrt{h}} \Pr [Y_n = 1] + e^{-\sigma \sqrt{h}} \Pr [Y_n = -1]$$

According to definition of measure $q$, it holds

$$\Pr [Y_n = 1] = q, \quad \Pr [Y_n = -1] = 1 - q$$

Substituting in expression for $E\left[ e^{\sigma \sqrt{h} Y_n} \right]$:

$$E\left[ e^{\sigma \sqrt{h} Y_n} \right] = e^{\sigma \sqrt{h}} q + e^{-\sigma \sqrt{h}} (1 - q) = e^{\alpha h}$$

where last equality follows from definition of probability $q$.

Reweave the quilt $\Rightarrow$ use expected relative price increase to compute expected future value to find expected return.
Reweave the quilt

- Substitute expected relative price increase into expression for expected future value

\[ E_{q(n+1:m)} [X(mh) | X(nh)] = X(nh) e^{\alpha h} e^{\alpha h} \ldots e^{\alpha h} = X(nh) e^{\alpha (m-n)h} \]

- Substitute result into expression for expected return

\[ E_q [r(X(h), \ldots, X(mh))] = E_{q(1:n)} \left[ \frac{X(nh) e^{\alpha (m-n)h} e^{-\alpha mh}}{X(nh) e^{-\alpha nh}} - 1 \right] \]

- Exponentials cancel each other, finally yielding

\[ E_q [r(X(h), \ldots, X(mh))] = E_{q(1:n)} [1 - 1] = 0 \]

- Arbitrage not possible in any trading strategy if \( 0 \leq q \leq \) exists
If prices follow geometric Brownian motion

Stock prices follow a geometric Brownian motion, i.e.,

\[ X(t) = X(0)e^{Y(t)} \]

with \( Y(t) \) Brownian motion with drift \( \mu \) and variance \( \sigma^2 \)

What is the no arbitrage condition?

Approximate geometric Brownian motion by geometric random walk

No arbitrage measure \( q \) exists for geometric random walk

This requires \( h \) sufficiently small

Notice that prob. distribution \( q = q(h) \) is a function of \( h \)

Approximation arbitrarily accurate by letting \( h \to 0 \)

Existence of the prob. distribution \( q := \lim_{h \to 0} q(h) \) proves that arbitrages are not possible in stock trading
Recall that as $h \to 0 \Rightarrow q \approx \frac{1}{2} \left(1 + \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h}\right)$

And consequently $\Rightarrow (1 - q) = \frac{1}{2} \left(1 - \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h}\right)$

Thus, measure $q := \lim_{h \to 0} q(h)$ is geometric Brownian motion

\[ \Rightarrow \text{Variance} \Rightarrow \sigma^2 \text{ (same as stock price)} \]

\[ \Rightarrow \text{Drift} \Rightarrow \alpha - \sigma^2/2 \]

Measure showing arbitrage not possible is a geometric random walk

Which is also the way stock prices evolve

Furthermore, the variance is the same as that of stock prices

The drifts are different $\Rightarrow \mu$ for stocks and $\alpha - \sigma^2/2$ for no arbitrage
Expected investment growth

- Compute expected return on an investment on stock $X(t)$
- Buy 1 share of stock at time 0. Cash invested $\Rightarrow X(0)$
- Sell stock at time $t$. Cash after sell $\Rightarrow X(t)$
- Expected value of cash after sell given $X(0)$ is

$$
\mathbb{E} [X(t) \mid X(0)] = X(0)e^{(\mu + \sigma^2/2)t}
$$

- Alternatively, invest $X(0)$ risk free in the money market
- Guaranteed cash at time $t$ is $X(0)e^{\alpha t}$
- **Invest in stock only if** $\mu + \sigma^2/2 > \alpha \Rightarrow \text{risk premium}$
Compute expected return as if $q$ were the actual distribution
▶ And recall that $q$ is NOT the actual distribution
▶ As before, cash invested is $X(0)$ and cash after sale is $X(t)$
▶ Expected cash value is different because prob. distribution is different

$$E_q [X(t) \mid X(0)] = X(0)e^{(\alpha - \sigma^2/2 + \sigma^2/2)t} = X(0)e^{\alpha t}$$

▶ Same return as risk free investment regardless of parameters’ values
▶ Measure $q$ is called risk neutral measure
▶ Risky stock investments yield same return as risk free investments
▶ “Alternate universe” in which investors do not demand risk premiums
▶ Pricing of derivatives, e.g., options, is always based on expected returns with respect to risk neutral valuation (pricing in alternate universe)
▶ Basis for Black-Scholes. More later
Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
Options

- An option is a contract to buy shares of a stock at a future time.
- Strike time $t = \text{Convened time for stock purchase}$.
- Strike price $K = \text{Price at which stock is purchased at strike time}$.
- At time $t$, option holder may decide to:
  - Buy a stock at strike price $K = \text{exercise the option}$.
  - Do not exercise the option.
- May buy option at time 0 for price $c$.
- How do we determine the option’s worth, i.e., price $c$, at time 0?
- Answer given by Black-Scholes formula for option pricing.
Stock price model

- Let $e^{\alpha t}$ be the compounding of a risk free investment
- Let $X(t)$ be the stock’s price at time $t$
- Price modeled as geometric Brownian motion, drift $\mu$, variance $\sigma^2$
- Risk neutral measure $q$ is also a geometric Brownian motion
  \[ \Rightarrow \text{Variance } \sigma^2 \text{ and drift } \alpha - \sigma^2/2 \]
Return of option investment

- At time $t$, the option’s worth depends on the stock’s price $X(t)$
- If stock’s price smaller or equal than strike price $\Rightarrow X(t) \leq K$
  $\Rightarrow$ Option is worthless (better to buy stock at current price)
- Since had paid $c$ for the option at time 0, lost $c$ on this investment
  $\Rightarrow$ return on investment is $r = -c$
- If stock’s price larger than strike price $\Rightarrow X(t) > K$
  $\Rightarrow$ Exercise option and realize a gain of $X(t) - K$
- To obtain return express as time 0 values and subtract $c$
  \[ r = e^{-\alpha t}(X(t) - K) - c \]
- May combine both in single equation $\Rightarrow r = e^{-\alpha t}(X(t) - K)^+ - c$
- $(\cdot)^+$ denotes projection on positive reals
Consider mixed positions on stocks and options
Is there a position guaranteeing positive return, i.e., an arbitrage?
Assume expected return under risk neutral measure is nonzero

\[
\mathbb{E}_q [r] = \mathbb{E}_q \left[ e^{-\alpha t} (X(t) - K)^+ - c \right] \neq 0
\]

Then, an arbitrage is possible according to arbitrage theorem
If expected return under risk neutral measure is zero

\[
\mathbb{E}_q [r] = \mathbb{E}_q \left[ e^{-\alpha t} (X(t) - K)^+ - c \right] = 0
\]

Then, no arbitrage is possible according to arbitrage theorem
Select options price \( c \) to prevent arbitrage opportunities
To have no arbitrage, must select option’s price \( c \) so that

\[
\mathbb{E}_q \left[ e^{-\alpha t} (X(t) - K)^+ - c \right] = 0
\]

where expectation is with respect to risk neutral measure.

From above condition, the no-arbitrage price of the option is

\[
\begin{align*}
c & = e^{-\alpha t} \mathbb{E}_q \left[ (X(t) - K)^+ \right] \\
& = e^{-\alpha t} \mathbb{E}_q \left[ (X(t) - K)^+ \right]
\end{align*}
\]

Source of Black-Scholes formula for option valuation.

Rest of derivation is just evaluation of expected value.

Same argument used to price any derivative of the stock’s price.
Use fact that prices are a geometric random walk

- Let us evaluate expectation to compute option’s price $c$
- Prices follow a geometric random walk $\Rightarrow X(t) = X_0 e^{Y(t)}$
- $X_0 =$ price at time 0,
- $Y(t)$ random walk with drift parameter $\mu$ and variance parameter $\sigma^2$
- Can rewrite no arbitrage condition as

$$c = e^{-\alpha t} \mathbb{E}_q \left[ (X_0 e^{Y(t)} - K)^+ \right]$$

- $Y(t)$ random walk. Then, in particular, $Y(t) \sim \mathcal{N}(\mu t, t\sigma^2)$

$$c = e^{-\alpha t} \frac{1}{\sqrt{2\pi t\sigma^2}} \int_{-\infty}^{\infty} (X_0 e^y - K)^+ e^{-(y-\mu t)^2/(2t\sigma^2)} dy$$
Evaluation of the integral

- Note that \((X_0 e^{Y(t)} - K)^+ = 0\) for all values \(Y(t) \leq \log(K/X_0)\)
- Because integrand is null for \(Y(t) \leq \log(K/X_0)\) can write

\[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi t\sigma^2}} \int_{\log(K/X_0)}^{\infty} (X_0 e^{y} - K) e^{-\left(y - \mu t\right)^2/(2t\sigma^2)} \, dy
\]

- Change of variables \(z = (y - \mu t)/\sqrt{t\sigma^2}\). Associated replacements

  Variable: \(y \Rightarrow \sqrt{t\sigma^2} z + \mu t\)

  Differential: \(dy \Rightarrow \sqrt{t\sigma^2} \, dz\)

  Integration limit: \(\log(K/X_0) \Rightarrow a := \frac{\log(K/X_0) - \mu t}{\sqrt{t\sigma^2}}\)

- Option price then given by

\[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} \left(X_0 e^{\sqrt{t\sigma^2} z + \mu t} - K\right) e^{-z^2/2} \, dz
\]
Separate in two integrals $c = e^{-\alpha t}(l_1 - l_2)$ where

$$l_1 := \frac{1}{\sqrt{2\pi}} \int_a^{\infty} X_0 e^{\sqrt{t\sigma^2}z + \mu t} e^{-z^2/2} dz$$

$$l_2 := \frac{K}{\sqrt{2\pi}} \int_a^{\infty} e^{-z^2/2} dz$$

Gaussian Q function (ccdf of normal RV with mean 0 and variance 1)

$$Q(x) := \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-z^2/2} dz$$

Comparing last two equations we have $l_2 = KQ(a)$

$l_1$ requires some more work
Evaluation of the integral (continued)

- Reorder terms in integral $I_2$

$$I_1 := \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} X_0 e^{\sqrt{t\sigma^2}z + \mu t} e^{-z^2/2} \, dz = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_{a}^{\infty} e^{\sqrt{t\sigma^2}z - z^2/2} \, dz$$

- The exponent can be written as a square minus a “constant” (no $z$)

$$-\left(z - \sqrt{t\sigma^2}\right)^2 / 2 + t\sigma^2 / 2 = -z^2/2 + \sqrt{t\sigma^2}z - t\sigma^2/2 + t\sigma^2/2$$

- Substituting the latter into $I_1$ yields

$$I_1 = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-\left(z - \sqrt{t\sigma^2}\right)^2 / 2 + t\sigma^2 / 2} \, dz = \frac{X_0 e^{\mu t + t\sigma^2 / 2}}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-\left(z - \sqrt{t\sigma^2}\right)^2 / 2} \, dz$$
Evaluation of the integral (continued)

- Change of variables \( u = z - \sqrt{t\sigma^2} \Rightarrow du = dz \) and integration limit
  \[
a \Rightarrow b := a - \sqrt{t\sigma^2} = \frac{\log(K/X_0) - \mu t}{\sqrt{t\sigma^2}} - \sqrt{t\sigma^2}
\]

- Implementing change of variables in \( I_1 \)
  \[
  I_1 = \frac{X_0 e^{\mu t + t\sigma^2/2}}{\sqrt{2\pi}} \int_b^\infty e^{u^2/2} \, du = X_0 e^{\mu t + t\sigma^2/2} Q(b)
  \]

- Putting together results for \( I_1 \) and \( I_2 \)
  \[
  c = e^{-\alpha t} (I_1 - I_2) = e^{-\alpha t} X_0 e^{\mu t + t\sigma^2/2} Q(b) - e^{-\alpha t} KQ(a)
  \]

- For non-arbitrage stock prices \( \Rightarrow \alpha = \mu + \sigma^2/2 \)

- Substitute to obtain Black-Scholes formula
Black-Scholes formula for option pricing

\[ c = X_0 Q(b) - e^{-\alpha t} K Q(a) \]

Where: \( a \) is given by
\[ a = \frac{\log(K/X_0) - \mu t}{\sqrt{t \sigma^2}} \]
and \( b = a - \sqrt{t \sigma^2} \).

Note further that \( \mu = \alpha - \sigma^2/2 \). Can then write \( a \) as
\[ a = \frac{\log(K/X_0) - (\alpha - \sigma^2/2) t}{\sqrt{t \sigma^2}} \]

- \( X_0 \) is the stock price at time 0, \( c \) is the option cost at time 0,
- \( K \) is the option's strike price, \( t \) is the option's strike time,
- \( \alpha \) is the benchmark risk-free rate of return (cost of money),
- \( \sigma^2 \) is the volatility of the stock,
- Black-Scholes formula independent of stock's mean tendency \( \mu \).