Gaussian processes (week 11)

Solution

1 White Gaussian noise.

A Independent values. The values \( W(t_1) \) and \( W(t_2) \) (realizations of White Gaussian noise process) at different times \( t_1 \neq t_2 \) are independent if they are not correlated, that is if their autocorrelation function is 0. The autocorrelation function is:

\[
R_W(t_1, t_2) = \sigma^2 \delta(t_1 - t_2)
\]

where \( \sigma(t) \) represents a function that is infinite at \( t = 0 \) and zero everywhere else. This means that for different times \( t_1 \) and \( t_2 \), \( R_W(t_1, t_2) = \sigma^2 \delta(t_1 - t_2) = \sigma^2 \cdot 0 = 0 \). Therefore, the values are not correlated and are independent.

B Integral of WGN. [Refer to slide 13, and slides 40-41 of markov_gaussian_stationary_processes.] Integration over time is a linear functional. When the process \( X(t) \) is defined as:

\[
X(t) = \int_0^t W(u)du
\]

it is Gaussian because it is a linear functional of a Gaussian process, thus it is also a Gaussian process. Thus \( X(t) \) is a Gaussian process.

The mean and autocorrelation functions of \( X(t) \), given that \( \mu_W(t) = 0 \) and \( R_W(t_1, t_2) = \sigma^2 \delta(t_1 - t_2) \) are:

\[
\mu_X(t) = \mathbb{E} \left[ \int_0^t W(u)du \right] = \int_0^t \mathbb{E}[W(u)]du = \int_0^t \mu_W(t)du = 0
\]

\[
R_X(t_1, t_2) = \mathbb{E} \left[ \left( \int_0^{t_1} W(u_1)du_1 \right) \left( \int_0^{t_2} W(u_2)du_2 \right) \right]
\]

\[
= \mathbb{E} \left[ \int_0^{t_1} \int_0^{t_2} W(u_1)W(u_2)du_1du_2 \right]
\]

\[
= \int_0^{t_1} \int_0^{t_2} \mathbb{E}[W(u_1)W(u_2)]du_1du_2
\]

where, \( \mathbb{E}[W(u_2)W(u_1)] = R_W(u_1, u_2) = \sigma^2 \delta(u_1 - u_2) \)

\[
= \int_0^{t_1} \int_0^{t_2} \sigma^2 \delta(u_1 - u_2)du_1du_2
\]

**definition of \( \delta(u) \):**

\[
\begin{align*}
\delta(u) \quad \Rightarrow & = \begin{cases} 
\int_0^{t_1} \sigma^2 du_1 = \sigma^2 t_1 & \text{when } t_1 < t_2 \\
\int_0^{t_2} \sigma^2 du_2 = \sigma^2 t_2 & \text{when } t_2 < t_1 \\
= \sigma^2 \min(t_1, t_2) & 
\end{cases}
\end{align*}
\]
Now, knowing that $X(t)$ is a Gaussian process, at any given time its value is distributed according to a Gaussian distribution, i.e. $X(t) \sim \mathcal{N}(\mu(t), \sqrt{R_X(t,t)})$. Hence:

$$P\{X(t) > a\} = 1 - P\{X(t) \leq a\} = 1 - \Phi\left(\frac{a}{\sigma \sqrt{t}}\right)$$

where $\Phi(.)$ is the cdf of a standard normal random variable.

C Discrete time representation of WGN. For discrete time process $W_h(n)$, where

$$W_h(n) = \int_{nh}^{(n+1)h} W(u)du$$

the mean value function is found by:

$$\mu_{W_h}(n) = E[W_h(n)] = \mathbb{E}\left[\int_{nh}^{(n+1)h} W(u) du\right] = \int_{nh}^{(n+1)h} \mathbb{E}[W(u) du] = \int_{nh}^{(n+1)h} 0. dt = 0.$$

The autocorrelation function, $R_{W_h}(n_1, n_2)$, is found by:

$$R_{W_h}(n_1, n_2) = E[W_h(n_1, n_2)] = \mathbb{E}\left[\int_{n_1h}^{(n_1+1)h} W(u) du \int_{n_2h}^{(n_2+1)h} W(v) dv\right]$$

$$= \int_{n_1h}^{(n_1+1)h} \int_{n_2h}^{(n_2+1)h} \mathbb{E}[W(u)W(v) du dv] = \int_{n_1h}^{(n_1+1)h} \int_{n_2h}^{(n_2+1)h} \sigma^2 \delta(u-v) du dv = \begin{cases} 0 & n_1 \neq n_2 \\ \sigma^2 h & n_1 = n_2 \end{cases}$$

D Simulation of process $X(t)$. The discrete time simulation, $X_h(n)$ of the process $X(t)$ for parameters $h = 0.01$, $\sigma^2 = 1$, and $t_{max} = 10$ follows. Based on previous calculation, the mean, $\mu_{X_h(n)} = 0$, and the standard deviation, $SD = \sqrt{R_{X_h}(n)(t,t)} = \sqrt{\sigma^2 h}$ were used.

```matlab
clear all; close all; clc;
h=0.01; sigma=1; t_MAX=10;
W_vector=normrnd(0,sigma^2*sqrt(h),1,t_MAX/h);
X_vector=cumsum(W_vector);
plot(h:h:t_MAX,X_vector,'r','Linewidth',1);
xlabel('time');title(['Weiner Process Simulated, h=',num2str(h)]);
grid on; axis([0 t_MAX -5 5])
```
Fig. 1. Part D: a sample path of the simulated Weiner process $X(t)$ using its discrete approximation.