Continuous time Markov chains (week 8)

Solutions

1 Insurance cash flow.

A CTMC states.
Because \(c\) and \(d\) are assumed to be integers, and the premiums are each 1, the cash flow \(X(t)\) of the insurance company can be any integer between 0 and \(X_{\text{max}}\).

B Transition times out of given state When \(X(t) = x\) and \(x\) is in range (D), the transition probability out of state \(x\), \(T_x\), is exponential because it is the probability of a dividend, claim, or premium being paid, which are all exponentially distributed:

\[
T_p \sim \exp(\lambda) = \exp(N) \\
T_c \sim \exp(\alpha) = \exp(rN) \\
T_d \sim \exp(\beta)
\]

The transition will occur whenever one of these happens (whichever is first), therefore:

\[
T_x = \min(T_p, T_c, T_d)
\]

The transition has not occurred when \((T_x > t)\), so it follows that none of the transitions have occurred. Because each event is independent:

\[
P(T_x > t) = P(T_p > t)P(T_c > t)P(T_d > t) \\
P(T_x > t) = e^{-\lambda t}e^{-\alpha t}e^{-\beta t} \\
P(T_x > t) = e^{-(\lambda+\alpha+\beta)t}
\]

The cdf of \(T_x\) is \(e^{-(\lambda+\alpha+\beta)t}\), therefore it is exponentially distributed with parameter \(\nu_x = (\lambda + \alpha + \beta)\). By the same logic, the parameters for \(T_x\) in the other ranges are:

<table>
<thead>
<tr>
<th>Range</th>
<th>Possible events</th>
<th>Parameter (\nu_x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>premium</td>
<td>(\lambda)</td>
</tr>
<tr>
<td>B</td>
<td>premium, claim paid at (X(t))</td>
<td>(\lambda + \alpha)</td>
</tr>
<tr>
<td>C</td>
<td>premium, claim</td>
<td>(\lambda + \alpha)</td>
</tr>
<tr>
<td>E</td>
<td>claim, dividend</td>
<td>(\alpha + \beta)</td>
</tr>
</tbody>
</table>

C Possible states going out of \(X(t) = x\) Given that \(X(t) = x\), where \(x\) is in range (D), the possible states after a transition out of \(x\) are:

\[
x \rightarrow x + 1 \quad \text{(a premium is paid)} \\
x \rightarrow x - c \quad \text{(a claim is paid)} \\
x \rightarrow x - d \quad \text{(a dividend is paid)}
\]
The possible states for each range are:

<table>
<thead>
<tr>
<th>Range</th>
<th>Possible events</th>
<th>Possible states out of $X(t) = x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>premium</td>
<td>$1$</td>
</tr>
<tr>
<td>B</td>
<td>premium, claim paid at $X(t)$</td>
<td>$x + 1, 0$</td>
</tr>
<tr>
<td>C</td>
<td>premium, claim</td>
<td>$x + 1, x - c$</td>
</tr>
<tr>
<td>E</td>
<td>claim, dividend</td>
<td>$x - c, x - d$</td>
</tr>
</tbody>
</table>

### D Transition probabilities

(Reference: slide 11 of continuous_time_markov_chains.) When $x$ is in range (D), the transition probabilities from state $x$ to $j$, $P_{xj}$, for each possible state out of $x$, $j$, are:

$$P_{x,x+1} = \frac{\lambda}{\lambda + \alpha + \beta}$$

$$P_{x,x-c} = \frac{\alpha}{\lambda + \alpha + \beta}$$

$$P_{x,x-d} = \frac{\beta}{\lambda + \alpha + \beta}$$

This is because we know that a transition happens at time $t$, and the three possible events that could have caused it: a premium payment, a claim, or a dividend payment. The probability that one of these events happens (for example a premium payment), given a transition happens is:

$$P\{\text{premium payment} | \text{a transition happens}\} = \frac{P\{\text{premium payment}\} \cap P\{\text{transition happens}\}}{P\{\text{transition happens}\}} = \frac{\lambda}{\lambda + \alpha + \beta}$$

The same reasoning applies to each other possible event. The transition probabilities into each possible state when $x$ is in each other range are:

<table>
<thead>
<tr>
<th>Range</th>
<th>State $j$ at time $t + 1$</th>
<th>Transition probabilities, $P_{x,j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>$\frac{\lambda}{\lambda + \alpha} = 1$</td>
</tr>
<tr>
<td>B</td>
<td>$x + 1$</td>
<td>$\frac{\lambda}{\lambda + \alpha}$</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>$\frac{\alpha}{\lambda + \alpha}$</td>
</tr>
<tr>
<td>C</td>
<td>$x + 1$</td>
<td>$\frac{\lambda}{\lambda + \alpha}$</td>
</tr>
<tr>
<td>C</td>
<td>$x - c$</td>
<td>$\frac{\lambda}{\lambda + \alpha}$</td>
</tr>
<tr>
<td>E</td>
<td>$x - c$</td>
<td>$\frac{\alpha}{\alpha + \beta}$</td>
</tr>
<tr>
<td>E</td>
<td>$x - d$</td>
<td>$\frac{\beta}{\alpha + \beta}$</td>
</tr>
</tbody>
</table>

### E System simulation

Two methods are provided below. Please notice the nuances, as each of them is a different interpretation of the CTMC:

```matlab
% following the method explained in the HW-8
function [X,T]=cashflow1(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max)
index=1;
X(index)=X_0;
T(index)=0;
while T(index)<T_max
    x=X(index);
```
if x==0 % only premium is possible
    tau=exprnd(1/lambda);
    T(index+1)=T(index)+tau;
    X(index+1)=x+1;
elseif 0<x && x<c % premium, claim payed at X(t) not c
    tau=exprnd(1/(lambda+alpha));
    T(index+1)=T(index)+tau;
    u=rand;
    if u<(lambda/(lambda+alpha)) % premium
        X(index+1)=x+1;
    else % claim
        X(index+1)=0;
    end
elseif c<=x && x<X_r % premium, claim
    tau=exprnd(1/(lambda+alpha));
    T(index+1)=T(index)+tau;
    u=rand;
    if u<(lambda/(lambda+alpha)) % premium
        X(index+1)=x+1;
    else
        X(index+1)=x-c;
    end
elseif X_r<=x && x<X_max % premium, claim, dividend
    tau=exprnd(1/(lambda+alpha+beta));
    T(index+1)=T(index)+tau;
    u=rand;
    if u<(lambda/(lambda+alpha+beta)) % premium
        X(index+1)=X(index)+1;
    elseif u<((lambda+alpha)/(lambda+alpha+beta)) % claim
        X(index+1)=X(index)-c;
    else % dividend
        X(index+1)=X(index)-d;
    end
elseif x==X_max % claim, dividend
    tau=exprnd(1/(alpha+beta));
    T(index+1)=T(index)+tau;
    u=rand;
    if u<(alpha/(lambda+alpha)) % claim
        X(index+1)=x-c;
    else % dividend
        X(index+1)=x-d;
    end
else
    disp(’Out Of Range’)
    break
end
index=index+1;
end
Method 2:

% An alternative method, alarm clock interpretation
function [X,T]=cashflow2(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max)
index=1;
X(index)=X_0;
T(index)=0;
while T(index)<T_max
    x=X(index);
    if x==0 %only premium is possible
        t_premium=exprnd(1/lambda);
        T(index+1)=T(index)+t_premium;
        X(index+1)=x+1;
    elseif 0<x && x<c %premium, claim payed at X(t) not c
        t_premium=exprnd(1/lambda);
        t_claim=exprnd(1/alpha);
        T(index+1)=T(index)+min(t_premium,t_claim);
        X(index+1)=x+1*(t_premium<t_claim)-x*(t_premium>t_claim);
    elseif c<=x && x<X_r %premium, claim
        t_premium=exprnd(1/lambda);
        t_claim=exprnd(1/alpha);
        T(index+1)=T(index)+min(t_premium,t_claim);
        X(index+1)=x+1*(t_premium<t_claim)-c*(t_premium>t_claim);
    elseif X_r<=x && x<X_max %premium, claim, dividend
        t_premium=exprnd(1/lambda);
        t_claim=exprnd(1/alpha);
        t_dividend=exprnd(1/beta);
        [t_min,I]=min([t_premium,t_claim,t_dividend]);
        T(index+1)=T(index)+t_min;
        X(index+1)=x+(1:3==I)*[1;-c;-d];
    elseif x==X_max % claim, dividend
        t_claim=exprnd(1/alpha);
        t_dividend=exprnd(1/beta);
        [t_min,I]=min([t_claim,t_dividend]);
        T(index+1)=T(index)+t_min;
        X(index+1)=x+(1:2==I)*[-c;-d];
    else
        disp('Out Of Range')
        break
    end
    index=index+1;
end
end

Here, we run the code (call one of the functions) for the following values of the parameters: $X_0 = 200$, number of clients $N = 200$, risk $r = 4\%$, dividends payed quarterly; claim and dividend costs
c = 20 and d = 30; capital thresholds, \( X_r = 200 \), and \( X_{max} = 300 \); and maximum amount of time \( T_{max} = 5 \) years. The result is depicted in fig. 1(a) and a zoomed one in fig. 1(b).

```matlab
clc; clear all; close all;
X_0=200;
N=200;
r=0.04;
lambda=N;
alpha=r*N;
beta=4;
X_r=200;
X_max=300;
T_max=5;
d=30;
c=20;

[X,t]=cashflow2(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max);

% Plotting the results
hold on
grid on
xlabel('time','Fontsize',14)
ylabel('Cash Level','Fontsize',14)
title('(a sample) Evolution of Cash Level over 5 Years','Fontsize',14)
axis([0 5 0 310])
stairs(t,X,'Linewidth',2,'Color','r');
```

The Kolomogorovs forward equation.
(Reference: slide 54 of continuous_time_markov_chains.) The transition rate \( q_{xy} \) is found by the equation \( q_{xy} = \nu_x P_{xy} \), where \( \nu_x \) is the rate of transition out of state \( x \), and \( P_{xy} \) is the probability of transitioning from state \( x \) into state \( y \). Therefore, the transition rates for each scenario are:

**Range (A), state \( x \) at time \( t = 0 \):**

<table>
<thead>
<tr>
<th>state ( y ) at ( t + 1 )</th>
<th>( \nu_x ) * ( P_{xy} )</th>
<th>( q_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>( \lambda ) * ( 1 )</td>
<td>( \lambda )</td>
</tr>
</tbody>
</table>

**Range (B), \( 0 < x < c \):**

<table>
<thead>
<tr>
<th>state ( y ) at ( t + 1 )</th>
<th>( \nu_x ) * ( P_{xy} )</th>
<th>( q_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + 1 )</td>
<td>( (\lambda + \alpha) * \frac{\lambda}{\lambda + \alpha} )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( (\lambda + \alpha) * \frac{\alpha}{\lambda + \alpha} )</td>
<td>( \alpha )</td>
</tr>
</tbody>
</table>

**Range (C), \( c \leq x < X_r \):**

<table>
<thead>
<tr>
<th>state ( y ) at ( t + 1 )</th>
<th>( \nu_x ) * ( P_{xy} )</th>
<th>( q_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + 1 )</td>
<td>( (\lambda + \alpha) * \frac{\lambda}{\lambda + \alpha} )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>( x - c )</td>
<td>( (\lambda + \alpha) * \frac{\alpha}{\lambda + \alpha} )</td>
<td>( \alpha )</td>
</tr>
</tbody>
</table>
Range (D), \( X_r \leq x < X_{\text{max}} \):

<table>
<thead>
<tr>
<th>state ( y ) at ( t + 1 )</th>
<th>( \nu_x * P_{xy} = )</th>
<th>( q_{xy} = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + 1 )</td>
<td>( (\lambda + \alpha + \beta) * \frac{\lambda}{\alpha + \beta} )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>( x - c )</td>
<td>( (\lambda + \alpha + \beta) * \frac{\alpha}{\alpha + \beta} )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>( x - d )</td>
<td>( (\lambda + \alpha + \beta) * \frac{\beta}{\alpha + \beta} )</td>
<td>( \beta )</td>
</tr>
</tbody>
</table>

Range (E), \( x = X_{\text{max}} \):

<table>
<thead>
<tr>
<th>state ( y ) at ( t + 1 )</th>
<th>( \nu_x * P_{xy} = )</th>
<th>( q_{xy} = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x - c )</td>
<td>( (\alpha + \beta) * \frac{\alpha}{\alpha + \beta} )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>( x - d )</td>
<td>( (\alpha + \beta) * \frac{\beta}{\alpha + \beta} )</td>
<td>( \beta )</td>
</tr>
</tbody>
</table>

Using the above transition rates, we can write Kolmogorov’s forward equations for each range. The equation takes the form:

\[
\frac{\partial P_{xy}(t)}{\partial t} = P'_{xy}(t) = \sum_{k=0, k \neq y}^{\infty} q_{ky}P_{xk}(t) - \nu_yP_{xy}(t)
\]

For Range (A), \( Y = 0 \):

\[
P'_{xY} = \alpha \sum_{k=1}^{c} P_{xk} - \lambda P_{xY}
\]

For Range (B), \( 0 < Y < c \):

\[
P'_{xY} = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} - (\lambda + \alpha)P_{xY}
\]

For Range (C), \( c \leq Y < X_r \):

- When \( Y < X_r - d \) (a dividend was not paid to move from state \( x \) to state \( y \) because the initial \( x \) was not greater than \( X_r \)):

\[
P'_{xY} = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} - (\lambda + \alpha)P_{xY}
\]

- When \( Y \geq X_r - d \) (\( Y \) is close enough to the threshold \( X_r \) that the transition into \( Y \) could have occurred due to a dividend payment):

\[
P'_{xY} = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} + \beta P_{x,Y+d} - (\lambda + \alpha)P_{xY}
\]

For Range (D), \( X_r \leq Y \leq X_{\text{max}} \):

- When \( Y \leq \text{min}(c,d) \) (the transition could have been a result of a claim payment, premium payment, or dividend payment):

\[
P'_{xY} = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} + \beta P_{x,Y+d} - (\lambda + \alpha + \beta)P_{xY}
\]

- When \( Y > \text{min}(c,d) \) (the transition could only have been a result of a premium payment):

\[
P'_{xY} = \lambda P_{x,Y-1} - (\lambda + \alpha + \beta)P_{xY}
\]
• When $c < d$ and $d < Y \leq c$ (the transition could have been a result of a claim payment or a premium payment, but not a dividend payment):

$$P_{xY}' = \lambda P_{x,Y-1} + \alpha P_{x,Y+c} - (\lambda + \alpha + \beta)P_{xY}$$

• When $d < c$ and $c < Y \leq d$ (the transition could have been a result of a dividend payment or a premium payment, but not a claim payment):

$$P_{xY}' = \lambda P_{x,Y-1} + \beta P_{x,Y+d} - (\lambda + \alpha + \beta)P_{xY}$$

For Range (E), $Y = X_{\text{max}}$

$$P_{xY}' = \lambda P_{x,Y-1} - (\alpha + \beta)P_{xY}$$

G  Kolmogorov's backward equation.
(Reference: slide 56 of continuous_time_markov_chains.) Kolmogorov’s backward equation is:

$$\frac{\partial P_{xy}(t)}{\partial t} = P_{xy}'(t) = \sum_{k=0, k \neq x}^{\infty} q_{xk} P_{ky}(t) - \nu_x P_{xy}(t)$$

For Range (A), $X = 0$ (the only possible state $y$ is $y = 1$):

$$P_{Xy}' = \lambda P_{X+1,y} - P_{X,y}$$

For Range (B), $0 < X < c$:

$$P_{Xy}' = \lambda P_{X+1,y} + \alpha P_{0,y} - (\lambda + \alpha)P_{X,y}$$

For Range (C), $c \leq X < X_r$:

$$P_{Xy}' = \lambda P_{X+1,y} + \alpha P_{X-c,y} - (\lambda + \alpha)P_{X,y}$$

For Range (D), $X_r \leq X < X_{\text{max}}$:

$$P_{Xy}' = \lambda P_{X+1,y} + \alpha P_{X-c,y} + \beta P_{X-d,y} - (\lambda + \alpha + \beta)P_{X,y}$$

For Range (E), $X = X_{\text{max}}$:

$$P_{Xy}' = \alpha P_{X-c,y} + \beta P_{X-d,y} - (\alpha + \beta)P_{X,y}$$

H  Solution of Kolmogorov equations  A function is written to construct the matrix $R$ such that the forward equation is represented by $\dot{P} = RP$. Then using matlab’s exponential of a matrix, find the solution. The code and the result follows (fig. (2)).

function [R]=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max)

R=zeros(X_max+1); % initialization

% Range A:
R(1,1)=-lambda;

% For Range (A), X = 0 (the only possible state y is y = 1):
% R(1,1) = -lambda;

% For Range (B), 0 < X < c:
% R(2,1) = lambda;
% R(2,2) = lambda + alpha;
% R(2,3) = -lambda - alpha - beta;

% For Range (C), c <= X < X_r:
% R(3,1) = lambda;
% R(3,2) = lambda + alpha;
% R(3,3) = -lambda - alpha - beta;

% For Range (D), X_r <= X < X_max:
% R(4,1) = lambda;
% R(4,2) = lambda + alpha;
% R(4,3) = lambda + alpha + beta;
% R(4,4) = -lambda - alpha - beta;

% For Range (E), X = X_max:
% R(5,1) = alpha;
% R(5,2) = beta;
% R(5,3) = -alpha - beta;

end
R(1, 2:c+1)=alpha;

% Range B:
for i=2:c
    R(i, i-1)=lambda;
    R(i, i+c)=alpha;
    R(i, i)=-(lambda+alpha);
end

% Range C-1:
for i=c+1:X_r-d
    R(i, i-1)=lambda;
    R(i, i+c)=alpha;
    R(i, i)=-(lambda+alpha);
end

% Range C_2:
for i=X_r-d+1:X_r
    R(i, i-1)=lambda;
    R(i, i+c)=alpha;
    R(i, i+d)=beta;
    R(i, i)=-(lambda+alpha);
end

% Range D_1:
for i=X_r+1:X_max-d+1
    R(i, i-1)=lambda;
    R(i, i+c)=alpha;
    R(i, i+d)=beta;
    R(i, i)=-(lambda+alpha+beta);
end

% Range D_2:
for i=X_max-d+2:X_max-c+1
    R(i, i-1)=lambda;
    R(i, i+c)=alpha;
    R(i, i)=-(lambda+alpha+beta);
end

% Range D_3:
for i=X_max-c+2:X_max
    R(i, i-1)=lambda;
    R(i, i)=-(lambda+alpha+beta);
end

% Range E:
R(X_max+1, X_max)=lambda;
R(X_max+1, X_max+1)=- (alpha+beta);
The execution code:

```matlab
R=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max);
p0=zeros(X_max+1,1);
p0(X_0+1,1)=1;
T=0:0.25:5;
figure
hold on
xlabel('X','Fontsize',14)
ylabel('pmf','Fontsize',14)
title('pmf of the states between 0 and 5 over quarterly intervals','Fontsize',14)
axis([0 300 0 0.016])
for t=T
    pmf=expm(R.*t)*p0;
    plot(0:X_max,pmf,'r')
end
```

1. **Probability of paying dividends.**

A dividend can be paid only when $X_r \leq x \leq X_{max}$ (when $x$ is in range (D) or (E)). While in this interval the expected number of occurrences per year is given by the parameter $\nu_x$ because number of occurrences is Poisson with $\nu_x$. This is approximately $(\lambda + \alpha + \beta) = 212$ (note: approximation ignores that the parameter changes for $x = X_{max}$). We can divide this number by four to find the expected number of events for each quarter, which is 53. For a given quarter, the probability that at least one dividend is paid is:

$$P(\text{at least one dividend paid}) = 1 - P(\text{no dividends paid}) = 1 - P(\text{all events are premium or claim payments})$$

$$= 1 - P(\text{single event is premium or claim payment})^{53}$$

$$= 1 - \left( \frac{\lambda + \alpha}{\lambda + \alpha + \beta} \right)^{53}$$

$$= 1 - \left( \frac{208}{212} \right)^{53}$$

$$\approx 0.64$$

Thus, the probability of a dividend is approximately 0.6 times the probability that $X(t) \geq X_r$, which is achieved by summing up the pmf's from $X_r$ to $X_{max}$. The result is depicted in fig. [3]
Fig. 1. Evolution of the Cash Flow in part E for 5 years (subfig. (a)). Subfig. (b) shows a zoomed portion of the path to clarify the details: small steps are due to premiums.

Fig. 2. Part H: pmf of the states between 0 and 5 over quarterly intervals using solution of Kolmogorov’s Forward equation.
Fig. 3. Part I: Approximate probability of giving a dividend in each of the quarters.